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Residues and Principal Values on Complex Spaces

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Let ξ and θ be smooth differential forms with compact support, of dimensions $2n$ and $2n-1$, respectively, defined on an open set W in $\mathbb{C}^n$, and let φ be any holomorphic function defined on W. We prove in this paper that the limits

$$\lim_{\delta\to 0}\int_{|\varphi|>\delta}\frac{\xi}{\varphi} \quad \text{and} \quad \lim_{\delta\to 0}\int_{|\varphi|=\delta}\frac{\theta}{\varphi} \tag{1}$$

exist, where $|\varphi|$ denotes the absolute value of φ, and relate them to the topology of the variety $\varphi=0$ and its complement in W.

The limits exist even when W is an open set in a paracompact and reduced complex space X of pure dimension n, although in this case the domains $W(|\varphi|>\delta)$ and $W(|\varphi|=\delta)$ are only semianalytic, in general, and integration on them means integration on their regular points, in the sense explained in [2] and [7]. No assumptions are needed about the singular sets of X or Y.

We prove the existence of these limits in the last two sections of the paper (Section 6 and 7), supposing first that W is a manifold and $\varphi=0$ has only normal crossings, and using then resolution of singularities to handle the general case (cf. Theorem 7.1).

Suppose now that Y is a closed 1-codimensional subspace of the complex space X, locally defined by one equation. In Section 5 we use the results just described to define the *residue* $\operatorname{Res}[\tilde{\omega}]$ and the *principal value* $\operatorname{PV}[\tilde{\omega}]$ of a meromorphic p-differential form $\tilde{\omega}$ on X, having its poles on Y: they are currents on X of dimension $2n-p-1$ and $2n-p$, respectively, and $\operatorname{Res}[\omega]$ has support on Y.

We proceed first locally. Consider an open set W in X, on which $\tilde{\omega}$ is representable as a quotient ω/φ^q (q integer $\geqq 0$), where ω is a holomorphic p-form (in the sense of Grauert-Grothendieck) on W and $\varphi\in\Gamma(W,\mathcal{O}_X)$ is a holomorphic equation for Y in W. Then

$$\operatorname{Res}\left[\frac{\omega}{\varphi^q}\right](\alpha)=\lim_{\delta\to 0}\mathrm{I}[W(=\delta)]\left(\frac{\omega\wedge\alpha}{\varphi^q}\right)$$

and

$$\operatorname{PV}\left[\frac{\omega}{\varphi^q}\right](\beta)=\lim_{\delta\to 0}\mathrm{I}[W(>\delta)]\left(\frac{\omega\wedge\beta}{\varphi^q}\right), \tag{2}$$

for all smooth forms α and β with compact support defined on W, of dimensions $2n-p-1$ and $2n-p$, respectively, and where the notations $I[W(>\delta)]$, etc., denote integration on the corresponding semianalytic domain.

Moreover, $\operatorname{Res}\left[\frac{\omega}{\varphi^q}\right]$ and $\operatorname{PV}\left[\frac{\omega}{\varphi^q}\right]$ do not depend on the particular representation ω/φ^q of $\tilde{\omega}$ on W, so that these local currents can be patched together to globally defined currents $\operatorname{Res}[\tilde{\omega}]$ and $\operatorname{PV}[\tilde{\omega}]$ of the space X.

In Section 5 we study the relationship of Res and PV with the standard cohomology and homology sequences associated with the couple (Y, X). To this purpose, we consider the exact sequence

$$0 \to \Omega_X^{\cdot} \to \Omega_X^{\cdot}(*Y) \to Q_X^{\cdot} \to 0,$$

where $\Omega_X^{\cdot}$ and $\Omega_X^{\cdot}(*Y)$ are the sheaves' complexes of holomorphic forms on X and of meromorphic forms on X with poles on Y, respectively, and $Q_X^{\cdot}$ is the quotient complex, and the exact sequence

$$0 \to {}'\mathscr{D}_{\cdot Y^\infty} \to {}'\mathscr{D}_{\cdot X} \to {}'\mathscr{D}_{\cdot X/Y^\infty} \to 0,$$

where ${}'\mathscr{D}_{\cdot X}$ is the sheaves' complex of currents on X, ${}'\mathscr{D}_{\cdot Y^\infty}$ is the subcomplex of those currents with support on Y and ${}'\mathscr{D}_{\cdot X/Y^\infty}$ is the quotient complex.

We show that PV and Res define complexes' homomorphisms

$$\operatorname{PV}: \Omega_X^p(*Y) \to {}'\mathscr{D}_{2n-p, X/Y^\infty}$$

and

$$\operatorname{Res}: Q_X^p \to {}'\mathscr{D}_{2n-p-1, Y^\infty}$$

which, together with the standard map

$$\operatorname{V}: \Omega_X^p \to {}'\mathscr{D}_{2n-p, X}$$

constructed by integration, provide a commutative diagram of hypercohomology

$$\begin{array}{ccccccc}
\cdots \longrightarrow H^p(X;\Omega_X^{\cdot}) & \longrightarrow & H^p(X;\Omega_X^{\cdot}(*Y)) & \longrightarrow & H^p(X;Q_X^{\cdot}) & \longrightarrow \cdots \\
\downarrow \mathrm{V} & & \downarrow \mathrm{PV} & & \downarrow \mathrm{Res} & \\
\cdots \longrightarrow H_{2n-p}\Gamma(X, {}'\mathscr{D}_{\cdot X}) & \longrightarrow & H_{2n-p}\Gamma(X, {}'\mathscr{D}_{\cdot X/Y^\infty}) & \longrightarrow & H_{2n-p-1}\Gamma(X, {}'\mathscr{D}_{\cdot Y^\infty}) & \longrightarrow \cdots .
\end{array} \tag{3}$$

The hyperhomologies of the bottom line have been replaced here by homologies of global sections; they are isomorphic, since the different sheaves of currents involved are fine.

The topological meaning of V, PV, and Res is expressed by Theorem 5.1, which states the existence of a homomorphism from diagram (3) to the commutative topological diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow H^p(X;\mathbb{C}) & \longrightarrow & H^p(U;\mathbb{C}) & \longrightarrow & H_Y^{p+1}(X;\mathbb{C}) & \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow H_{2n-p}(X;\mathbb{C}) & \longrightarrow & H_{2n-p}(U;\mathbb{C}) & \longrightarrow & H_{2n-p-1}(Y;\mathbb{C}) & \longrightarrow \cdots,
\end{array} \tag{4}$$

in which the top line is the classical sequence of cohomology with closed supports (and supports in Y) associated to the couple (Y, X), and where the bottom line is the exact sequence of Borel-Moore homology (closed supports) associated to (Y, X). The vertical maps are constructed by cap-product with the fundamental class of X.

In the case that X and Y are manifolds, this homomorphism between (3) and (4) reduces to an isomorphism (cf. 5.8), from which Leray-Norget's theory of residues can be deduced (cf. 5.9). In the general case, only the splitting of the homomorphism at the V-level can be asserted.

The explicit construction of the homomorphism between diagrams (3) and (4) is given in the first sections of the paper. Most of the material in these sections is general, but we find its inclusion necessary for the proofs of commutativity in Section 5.

Some details, however, cannot be found in the literature, as the existence of a splitting of De Rham homology similar to that described in [2] for cohomology, and the compatibility of this splitting with cup product.

We want to remark that the local definitions of Res and PV on manifolds have been first proposed by L. Bungart in 1967. Our proof of their existence is an improvement of a proof also offerted by Bungart (unpublished). Particular cases of these results can be found in Dolbeault's thesis [12], F. Norguet [8] and L. Schwartz [11].

P. Dolbeault has also constructed recently a canonical current extending a meromorphic form on a manifold (cf. [13], [14], [15]), which can be shown to be equal to the principal value defined here. Both treatments rely on Hironaka's resolution of singularities, but we think they are different enough as to justify its separate publication. Results of the present work have been exposed in the Congress of Several Complex Variables at the University of Maryland [7a].

1. Cup Product and Hypercohomology

All sheaves considered in this section are sheaves of $\mathbb{C}$-vector spaces. Tensor products are always defined over $\mathbb{C}$, and all double complexes are considered with their total differentials and graduations.

Let $\mathscr{B}^{\cdot} = (\mathscr{B}^q : q \in \mathbf{Z})$ be a complex of sheaves on a topological space X, such that $\mathscr{B}^p = 0$ if $p < 0$, and let φ be a family of supports in X. The *hypercohomology* $\boldsymbol{H}^{\cdot}_{\varphi}(X; \mathscr{B}^{\cdot})$ of $\mathscr{B}^{\cdot}$ on X with supports in φ is the cohomology of the complex $\Gamma_{\varphi}\mathscr{C}^{\cdot}(X; \mathscr{B}^{\cdot})$ of global sections. $\mathscr{C}^{\cdot}(X; \mathscr{B}^q)$ denotes here the canonical flabby resolution of $\mathscr{B}^q$ (cf. [3], II.2).

To define hypercohomology one can also use, instead of $\mathscr{C}^{\cdot}(X;\ \ )$, any functorial exact resolution of sheaves on X by φ-acyclic sheaves; f.i., the canonical simplicial resolution $\mathscr{A} \to \mathscr{F}^{\cdot}(X; \mathscr{A})$ of Godement (cf. [4], II.2).

We recall that, if $\mathscr{B}$ is a sheaf on another space Y, there is a canonical product

$$\asymp : \mathscr{F}^{\cdot}(X; \mathscr{A}) \hat{\otimes} \mathscr{F}^{\cdot}(X; \mathscr{B}) \to \mathscr{F}^{\cdot}(X \times Y; \mathscr{A} \hat{\otimes} \mathscr{B}),$$

and that

$$\mathscr{F}^0(X; \mathscr{A}) = \mathscr{C}^0(X; \mathscr{A}), \quad \mathscr{F}^n(X; \mathscr{A}) = \mathscr{C}^0(X; \mathscr{F}^{n-1}(X; \mathscr{A})).$$

1.1. Consider two complexes of sheaves $\mathscr{L}^{\cdot}$ and $\mathscr{N}^{\cdot}$ on the topological spaces X and Y, respectively, and such that $\mathscr{L}^p = \mathscr{N}^p = 0$ if $p < 0$, together with augmentations $\mathscr{A} \to \mathscr{L}^0$ and $\mathscr{B} \to \mathscr{N}^0$. The products

$$(-)^{qr} \times : \mathscr{F}^p(X; \mathscr{L}^q) \hat{\otimes} \mathscr{F}^r(Y; \mathscr{N}^s) \to \mathscr{F}^{p+r}(X \times Y; \mathscr{L}^q \hat{\otimes} \mathscr{N}^s), \quad (q, r \in \mathbf{Z})$$

induce a differential homomorphism $\times$ such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}^{\cdot}(X; \mathscr{L}^{\cdot}) \hat{\otimes} \mathscr{F}^{\cdot}(Y; \mathscr{N}^{\cdot}) & \xrightarrow{\times} & \mathscr{F}^{\cdot}(X \times Y; \mathscr{L}^{\cdot} \hat{\otimes} \mathscr{N}^{\cdot}) \\ \uparrow & & \uparrow \\ \mathscr{F}^{\cdot}(X; \mathscr{A}) \hat{\otimes} \mathscr{F}^{\cdot}(Y; \mathscr{B}) & \xrightarrow[\times]{} & \mathscr{F}^{\cdot}(X \times Y; \mathscr{A} \hat{\otimes} \mathscr{B}). \end{array} \tag{1}$$

Let φ and ψ be families of supports in X and Y, and apply to (1) the functor $\Gamma_\varphi \times \Gamma_\psi$. Then one obtains in the cohomology level a commutative diagram

$$\begin{array}{ccc} H^p_\varphi(X; \mathscr{L}^{\cdot}) \otimes H^q_\psi(Y; \mathscr{N}^{\cdot}) & \xrightarrow{\times} & H^{p+q}_{\varphi \times \psi}(X \times Y; \mathscr{L}^{\cdot} \hat{\otimes} \mathscr{N}^{\cdot}) \\ \uparrow & & \uparrow \\ H^p_\varphi(X; \mathscr{A}) \otimes H^q_\psi(Y; \mathscr{B}) & \xrightarrow[\times]{} & H^{p+q}_{\varphi \times \psi}(X \times Y; \mathscr{A} \hat{\otimes} \mathscr{B}) \end{array}. \tag{2}$$

1.2. Suppose now that Y is a closed subspace of X, that $\mathscr{U}$ and $\mathscr{V}$ are sheaves on X and that φ is a family of supports in X. There exists a canonical homomorphism

$$\Delta' : \Gamma_{Y \times \varphi}(X \times Y); \ \mathscr{U} \hat{\otimes} \mathscr{V}) \to \Gamma_{\varphi|Y}(X; \mathscr{U} \otimes \mathscr{V}),$$

where Y also denotes the family of closed sets in Y and $\varphi|Y$ the family of closed sets in Y that belong to φ. In fact, take a section γ in $\Gamma_{Y \times \varphi}(X \times Y; \mathscr{U} \hat{\otimes} \mathscr{V})$; it is immediate that the function $\Delta'_\gamma(x) = 0$ if $x \in X - Y$, and $\Delta'_\gamma(x) = \gamma(x) \in \mathscr{U}_x \otimes \mathscr{V}_x$ if $x \in Y$, is continuous, so that $\Delta'_\gamma \in \Gamma_{\varphi|Y}(X; \mathscr{U} \otimes \mathscr{V})$.

One also obtains a homomorphism

$$\cup' : \Gamma_Y(X; \mathscr{U}) \otimes \Gamma_\varphi(Y; \mathscr{V}) \to \Gamma_{\varphi|Y}(X; \mathscr{U} \otimes \mathscr{V})$$

composing Δ' with the canonical map

$$\Gamma_Y(X; \mathscr{U}) \otimes \Gamma_\varphi(Y; \mathscr{V}) \to \Gamma_{Y \times \varphi}(X \times Y; \mathscr{U} \otimes \mathscr{V}).$$

1.3. Let $\mathscr{L}^{\cdot}$ and $\mathscr{N}^{\cdot}$ be complexes of sheaves on X which are zero in negative degrees, together with augmentations $\mathscr{A} \to \mathscr{L}^{\cdot}$ and $\mathscr{B} \to \mathscr{N}^{\cdot}$. Let Y be closed in X and φ a family of supports in X. The maps Δ', $\cup'$ and $\times$ give us a commutative diagram of differential homomorphisms

$$\begin{array}{ccc} \Gamma_Y(X; \mathscr{F}^{\cdot}(\mathscr{L}^{\cdot})) \otimes \Gamma_\varphi(Y; \mathscr{F}^{\cdot}(\mathscr{N}^{\cdot})) & \searrow^{\cup'} & \\ \downarrow & & \\ \Gamma_{Y \times \varphi}(X \times Y; \mathscr{F}^{\cdot}(\mathscr{L}^{\cdot}) \hat{\otimes} \mathscr{F}^{\cdot}(\mathscr{N}^{\cdot})) & \xrightarrow[\Delta']{} & \Gamma_{\varphi|Y}(X; \mathscr{F}^{\cdot}(\mathscr{L}^{\cdot}) \otimes \mathscr{F}^{\cdot}(\mathscr{N}^{\cdot})) \\ \downarrow & & \downarrow \\ \Gamma_{Y \times \varphi}(X \times Y; \mathscr{F}^{\cdot}(\mathscr{L}^{\cdot} \hat{\otimes} \mathscr{N}^{\cdot})) & \longrightarrow & \Gamma_{\varphi|Y}(X; \mathscr{F}^{\cdot}(\mathscr{L}^{\cdot} \otimes \mathscr{N}^{\cdot})); \end{array} \tag{3}$$

in fact, only the bottom arrow needs to be constructed, and this can be done by induction on $\mathscr{F}^n(X) = \mathscr{C}^0(X; \mathscr{F}^{n-1})$.

Passing to cohomology, we get commutative diagrams

$$\begin{array}{ccc} \boldsymbol{H}^{\cdot}_{Y}(X;\mathscr{L}^{\cdot})\otimes \boldsymbol{H}^{\cdot}_{\varphi}(Y;\mathscr{N}^{\cdot}) & & \\ \downarrow & \searrow & \\ \boldsymbol{H}^{\cdot}_{Y\times\varphi}(X\times Y;\mathscr{L}^{\cdot}\hat{\otimes}\mathscr{N}^{\cdot}) & \rightarrow & \boldsymbol{H}^{\cdot}_{\varphi|Y}(X;\mathscr{L}^{\cdot}\otimes\mathscr{N}^{\cdot}) \end{array} \tag{4}$$

and

$$\begin{array}{ccc} H^{\cdot}_{Y}(X;\mathscr{A})\otimes H^{\cdot}_{\varphi}(Y;\mathscr{B}) & & \\ \downarrow & \overset{\cup'}{\searrow} & \\ H^{\cdot}_{Y\times\varphi}(X\times Y;\mathscr{A}\hat{\otimes}\mathscr{B}) & \rightarrow & H^{\cdot}_{\varphi|Y}(X;\mathscr{A}\otimes\mathscr{B}) \end{array} \tag{5}$$

which are compatible in an obvious sense.

1.4. In the situation of 1.1, suppose moreover that X is a complex analytic space, that $Y=X$ and $\mathscr{L}^{\cdot}=\mathscr{N}^{\cdot}=\mathscr{E}^{\cdot}_X$ is the sheaf of germs of smooth differential forms on X, with complex coefficients ([2], 3.4), with its augmentation $\mathbb{C}\rightarrow\mathscr{E}^{\cdot}_X$. We have in this case a product $\wedge:\mathscr{E}^{\cdot}_X\hat{\otimes}\mathscr{E}^{\cdot}_X\rightarrow\mathscr{E}^{\cdot}_{X\times X}$ which, after followed by the diagonal map in (2), gives us diagram

$$\begin{array}{ccc} \boldsymbol{H}^{p}_{\varphi}(X;\mathscr{E}^{\cdot}_X)\otimes \boldsymbol{H}^{q}_{\psi}(X;\mathscr{E}^{\cdot}_X) & \xrightarrow{\cup} & \boldsymbol{H}^{p+q}_{\varphi\cap\psi}(X;\mathscr{E}^{\cdot}_X) \\ \uparrow & & \uparrow \\ H^{p}_{\varphi}(X;\mathbb{C})\otimes H^{q}_{\psi}(X;\mathbb{C}) & \xrightarrow[\cup]{} & H^{p+q}_{\varphi\cap\psi}(X;\mathbb{C})\,. \end{array} \tag{6}$$

In the situation of 1.3, a similar argument provides us with a commutative diagram

$$\begin{array}{ccc} \boldsymbol{H}^{\cdot}_{Y}(X;\mathscr{E}^{\cdot}_X)\otimes \boldsymbol{H}^{\cdot}_{\varphi}(Y;\mathscr{E}^{\cdot}_X) & \xrightarrow{\cup} & \boldsymbol{H}^{\cdot}_{\varphi|Y}(X;\mathscr{E}^{\cdot}_X) \\ \uparrow & & \uparrow \\ H^{\cdot}_{Y}(X;\mathbb{C})\otimes H^{\cdot}_{\varphi}(Y;\mathbb{C}) & \xrightarrow[\cup']{} & H^{\cdot}_{\varphi|Y}(X;\mathbb{C}) \end{array} \tag{7}$$

that can be factored through the product $X\times Y$. The bottom product, followed by the restriction $H^{\cdot}_{\varphi|Y}(X;\mathbb{C})\rightarrow H^{\cdot}_{\varphi|Y}(Y;\mathbb{C})$, gives rise to the standard cup product.

1.5. Let Y be closed in X, $U=X-Y$ and denote by $j:U\rightarrow X$ the inclusion. Consider a flabby resolution $\mathscr{S}^{\cdot}_X$ of $\mathbb{C}$ on X and the exact sequence of flabby sheaves

$$0\rightarrow\mathscr{S}^{\cdot}_{X,U}\rightarrow\mathscr{S}^{\cdot}_X\rightarrow j\mathscr{S}^{\cdot}_U\rightarrow 0\,,$$

where $\mathscr{S}^{\cdot}_U=\mathscr{S}^{\cdot}_X|U$, and $j\mathscr{S}^{\cdot}_U$ denotes the direct image of $\mathscr{S}^{\cdot}_U$ by j.

In the associated exact sequence of sections over X one has $\Gamma(X;j\mathscr{S}^{\cdot}_U)=\Gamma(U;\mathscr{S}^{\cdot}_X)$ and $\Gamma(X;\mathscr{S}^{\cdot}_{X,U})=\Gamma_Y(X;\mathscr{S}^{\cdot}_X)$. Passing to cohomology one obtains the long exact sequence

$$\cdots\rightarrow H^p(X;\mathbb{C})\rightarrow H^p(U;\mathbb{C})\rightarrow H^{p+1}_Y(X;\mathbb{C})\rightarrow\cdots\,. \tag{8}$$

There is a product between this sequence and the usual exact sequence with compact supports

$$\cdots\rightarrow H^q_c(U;\mathbb{C}^{\cdot})\rightarrow H^q_c(X;\mathbb{C})\rightarrow H^q_c(Y;\mathbb{C})\rightarrow\cdots\,, \tag{9}$$

by means of cup product and the product defined in 1.3(5), and which is described in the following commutative diagram:

$$\begin{array}{ccccccc}
H^p(X) & \to & H^p(U) & \to & H_Y^{p+1}(X) & \to & H^{p+1}(X) \\
\otimes & & \oplus & & \otimes & & \otimes \\
H_c^{q+1}(X) & \leftarrow & H_c^{q+1}(U) & \leftarrow & H_c^q(Y) & \leftarrow & H_c^q(X) \\
\downarrow{\scriptstyle\cup} & & \downarrow{\scriptstyle\cup} & & \downarrow{\scriptstyle\cup'} & & \downarrow{\scriptstyle\cup} \\
H_c^{p+q+1}(X) & \leftarrow & H_c^{p+q+1}(U) & & H_{c|Y}^{p+q+1}(X) & \to & H_c^{p+q+1}(X)\,.
\end{array} \tag{10}$$

1.6. In the conditions of 1.5, suppose that X is locally compact. The Borel-Moore homology groups $H_\cdot(X;\mathbb{C})$ with closed supports and coefficients in $\mathbb{C}$ are canonically dual to $H_c^\cdot(X;\mathbb{C})$ (cf. [3], V.3(2))

$$H_m(X;\mathbb{C}) \approx \mathrm{Hom}(H_c^m(X,\mathbb{C}),\mathbb{C})\,. \tag{11}$$

So, each fixed homology class in $H_{p+q+1}(X;\mathbb{C})$ allows to replace (10) by a commutative diagram of pairings (all groups with coefficients in $\mathbb{C}$)

$$\begin{array}{ccccc}
H^p(X) & \to & H^p(U) & \to & H_Y^{p+1}(X) \\
\otimes & & \otimes & & \otimes \\
H_c^{q+1}(X) & \leftarrow & H_c^{q+1}(U) & \leftarrow & H_c^q(Y) \\
& \searrow & \downarrow & \swarrow & \\
& & \mathbb{C} & &
\end{array} \tag{12}$$

This method defines, in fact, a cap product that makes commutative the following diagram

$$\begin{array}{ccccccc}
\cdots\to H_m(X)\otimes H^p(X) & \to & H_m(X)\otimes H^p(U) & \to & H_m(X)\otimes H_Y^{p+1}(X) & \to\cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\cdots\to \mathrm{Hom}(H_c^{m-p}(X),\mathbb{C}) & \to & \mathrm{Hom}(H_c^{m-p}(U),\mathbb{C}) & \to & \mathrm{Hom}(H_c^{m-p-1}(Y),\mathbb{C}) & \to\cdots\,.
\end{array} \tag{13}$$

1.7. The case that will be of importance to us is when X is a complex analytic space of dimension n, Y is a closed 1-codimensional subvariety and we choose the fundamental homology class $[X]\in H_{2n}(X,\mathbb{C})$ of X. Then (13) yields, modulo the isomorphism in (11), the commutative diagram

$$\begin{array}{ccccccc}
\to & H^p(X;\mathbb{C}) & \to & H^p(U;\mathbb{C}) & \to & H_Y^{p+1}(X;\mathbb{C}) & \to \\
& \downarrow{\scriptstyle\cap(X)} & & \downarrow{\scriptstyle\cap(U)} & & \downarrow{\scriptstyle\cap(Y)} & \\
\to & H_{2n-p}(X;\mathbb{C}) & \to & H_{2n-p}(U;\mathbb{C}) & \to & H_{2n-p-1}(Y;\mathbb{C}) & \to\,.
\end{array}$$

If X is a manifold (has only simple points), $\cap(X)$ and $\cap(U)$ are both isomorphisms (inverse to Poincaré dualities' isomorphisms, cf. [3], V.10.2), so that $\cap(Y)$ is also an isomorphism.

2. De Rham Cohomology of a Complex Space

The purpose of this section is to construct the exact sequence 1(14), in the condition stated there, by means of analytic differential forms on X with polar singularities on Y.

2.1. Let X be a complex space of dimension n, Y a closed analytic subspace of X locally defined by one equation and $i: Y \to X$ the inclusion. Let $\Omega_X^{\cdot}$ be the complex of sheaves of (Grauert-Grothendieck) analytic differential forms on X (cf. [2], 3.2), and $\Omega_X^{\cdot}(*Y)$ be the complex of *analytic differential forms on X with poles on Y* ([5], p. 97). We suppose that $U = X - Y$ is dense in X, and denote by $j: U \to X$ the inclusion.

The complex $\Omega_X^{\cdot}((*Y)$ is defined as the sheaf on X such that, for any open W in X, $\Gamma(W; \Omega_X^{\cdot}(*Y))$ is the subcomplex of those sections $\tilde{\omega}$ in the complex $\Gamma(W - Y; \Omega_X^{\cdot})$ with the property: for any point $x \in W \cap Y$, there exists a neighborhood $W_x \subset W$ of x and an equation $\varphi \in \Gamma(W_x; \mathcal{O}_X)$ of Y on W_x such that $\varphi^p \tilde{\omega} | W_x - Y = \omega | W_x - Y$ for some $\omega \in \Gamma(W_x; \Omega_X^{\cdot})$, and some integer $p \geqq 0$.

The couple (φ^p, ω) will be called a *representation* of $\tilde{\omega}$ on W_x, and the notation $\tilde{\omega} = \dfrac{\omega}{\varphi^p}$ on W_x will be frequently used. $\Omega_X^{\cdot}(*Y)$ is a subcomplex of the direct image sheaf $j(\Omega_U^{\cdot})$ of $\Omega_U^{\cdot} = \Omega_X^{\cdot} | U$.

In a similar way, using the complex $\mathscr{E}_X^{\cdot}$ of smooth differential forms on X (cf. [2], 3.4) in the place of $\Omega_X^{\cdot}$, one defines the complex $\mathscr{E}_X^{\cdot}(*Y)$ of semi-meromorphic forms on X with poles on Y; it is a subcomplex of $j(\mathscr{E}_U^{\cdot})$, and there is a canonical homomorphism $\Omega_X^{\cdot}(*Y) \to \mathscr{E}_X^{\cdot}(*Y)$.

It is clear that $\Omega_X^{\cdot}$ is a subcomplex of $\Omega_X^{\cdot}(*Y)$; we denote by $Q_X^{\cdot} = \Omega_X^{\cdot}(*Y)/\Omega_X^{\cdot}$ the quotient complex, and proceed to study the exact sequence

$$0 \to \Omega_X^{\cdot} \to \Omega_X^{\cdot}(*Y) \to Q_X^{\cdot} \to 0\,. \tag{1}$$

Consider the complex $\mathscr{S}_X$ of semianalytic cochains of X with coefficients in $\mathbb{C}$, which is a flabby resolution of $\mathbb{C}$ on X ([2], 2.10). Integration on the semianalytic chains of X and U provides homomorphisms $\Omega_X^{\cdot} \to \mathscr{S}_X^{\cdot}$ and $j(\Omega_U^{\cdot}) \to j(\mathscr{S}_U^{\cdot})$, where $\mathscr{S}_U^{\cdot} = \mathscr{S}_X^{\cdot} | U$ (cf. [2], 3.6), so that there is a commutative diagram of complexes

$$\begin{array}{ccc} \mathscr{S}_X^{\cdot} & \longrightarrow & j\mathscr{S}_U^{\cdot} \\ {\scriptstyle I}\uparrow & & \uparrow{\scriptstyle I(*Y)} \\ \Omega_X^{\cdot} & \longrightarrow & \Omega_X^{\cdot}(*Y). \end{array} \tag{2}$$

For any family of supports φ on X, $H_\varphi^{\cdot}(X; \mathscr{S}_X^{\cdot}) \simeq H_\varphi^{\cdot}(X, \mathbb{C})$, since $\mathscr{S}_X^{\cdot}$ is a resolution, and $H_\varphi^{\cdot}(X; j\mathscr{S}_U^{\cdot}) \simeq H^{\cdot}\Gamma_\varphi(X; j\mathscr{S}_U^{\cdot}) \simeq H^{\cdot}\Gamma_{\varphi \cap U}(U; \mathscr{S}_X^{\cdot}) = H_{\varphi \cap U}^{\cdot}(U; \mathbb{C})$, because $j\mathscr{S}_U^{\cdot}$ is flabby. Taking hypercohomology in (2) we obtain a commutative diagram

$$\begin{array}{ccc} H_\varphi^{\cdot}(X; \mathbb{C}) & \longrightarrow & H_{\varphi \cap U}^{\cdot}(U; \mathbb{C}) \\ {\scriptstyle I\varphi}\uparrow & & \uparrow{\scriptstyle I(*Y)} \\ \boldsymbol{H}_\varphi^{\cdot}(X; \Omega_X^{\cdot}) & \longrightarrow & \boldsymbol{H}_\varphi^{\cdot}(X; \Omega_X^{\cdot}(*Y))\,. \end{array} \tag{3}$$

We recall that I_φ is a right inverse to the canonical edge homomorphism $e: H_\varphi^{\cdot}(X;\mathbb{C}) \to H_\varphi^{\cdot}(X;\Omega_X^{\cdot})$, which gives a splitting of the exact sequence

$$0 \to H_\varphi^{\cdot}(X;\mathbb{C}) \xrightarrow{e} H_\varphi^{\cdot}(X;\Omega_X^{\cdot}) \xrightarrow{I_\varphi} H_\varphi^{\cdot}(X;\mathbb{C}) \to 0; \tag{4}$$

the homomorphism $\Omega_X^{\cdot} \to \mathscr{E}_X^{\cdot}$ maps (4) into the similar sequence for $\mathscr{E}_X^{\cdot}$, with compatibility of the splittings ([2], 3.11).

2.2. We now want consider the two exact sequences of hypercohomology associated to (1), with closed supports and with supports in Y; they are related by a commutative diagram

$$\begin{array}{ccccccc}
\cdots \to & H^p(X;\Omega_X^{\cdot}(*Y)) & \to & H^p(X;Q_X^{\cdot}) & \longrightarrow & H^{p+1}(X;\Omega_X^{\cdot}) & \to \cdots \\
& \uparrow & & \uparrow\scriptstyle{\approx} & \searrow^{-\eta} & \uparrow & \\
\cdots \to & H_Y^p(X;\Omega_X^{\cdot}(*Y)) & \to & H_Y^p(X;Q_X^{\cdot}) & \xrightarrow[\delta]{} & H_Y^{p+1}(X;\Omega_X^{\cdot}) & \to \cdots,
\end{array} \tag{5}$$

by means of which we identify η and $-\delta$; δ denotes the connexion homomorphism.

The complex $Q_X^{\cdot}$ has support on Y, so that the hypercohomologies of $Q^{\cdot}$ with closed supports and with supports in Y are isomorphic. Composition with $-\delta$, followed by $I_Y: H_Y^{p+1}(X;\Omega_X^{\cdot}) \to H_Y^{p+1}(X;\mathbb{C})$ (as in (4)), gives us a homomorphism $\mu = I_Y \circ \eta = -I_Y \circ \delta$

$$\begin{array}{ccc}
H_Y^{p+1}(X;\mathbb{C}) & \xleftarrow{I_Y} & \\
\uparrow{\scriptstyle\mu} & & H_Y^{p+1}(X;\Omega_X^{\cdot}) \\
H^p(X;Q_X^{\cdot}) & \xrightarrow{\eta} &
\end{array} \tag{6}$$

2.3. Theorem. *Let Y be a closed analytic subspace of the complex space X and suppose that Y is locally defined by one equation, and that $U = X - Y$ is dense in X. Then there is a commutative diagram*

$$\begin{array}{ccccccc}
\to & H^p(X;\mathbb{C}) & \to & H^p(U;\mathbb{C}) & \to & H_Y^{p+1}(X;\mathbb{C}) & \to \\
& \uparrow{\scriptstyle I} & & \uparrow{\scriptstyle I(*Y)} & & \uparrow{\scriptstyle\mu} & \\
\to & H^p(X;\Omega_X^{\cdot}) & \to & H^p(X;\Omega_X^{\cdot}(*Y)) & \to & H^p(X;Q_X^{\cdot}) & \to,
\end{array} \tag{7}$$

where I splits canonically. In the case X is reduced, and U is regular, μ also splits and $I(*Y)$ is an isomorphism.

Proof. Construct the top sequence in (7) with the complex of semianalytic cochains on X, as in 1.5. The homomorphisms I, $I(*Y)$ and μ have been constructed in 2.1 and 2.2.

The left square commutes by (3), for closed supports. Commutativity of the right square follows from (5) and the commutativity in the canonical diagram

$$\begin{array}{ccc}
H_Y^{p+1}(X;\mathbb{C}) & \to & H^{p+1}(X;\mathbb{C}) \\
\uparrow & & \uparrow \\
H_Y^{p+1}(X;\Omega_X^{\cdot}) & \to & H^{p+1}(X;\Omega_X^{\cdot}).
\end{array} \tag{8}$$

The commutativity of the middle square involves only diagram chasing and is left to the reader. I splits canonically by ([2], 3.11).

Suppose now that U is regular. By Grothendieck's Theorem 2 in [5], $I(*Y): \mathscr{H}^q(\Omega_X^{\cdot}(*Y)) \to \mathscr{H}^q(j\mathscr{S}_U^{\cdot}) \simeq R^q_{j*}(\mathbb{C}_U)$ is an isomorphism, and a standard spectral sequences argument assures that

$$H_\varphi^{\cdot}(X; \Omega_X^{\cdot}(*Y)) \to H_\varphi^{\cdot}(X; j\mathscr{S}_U^{\cdot}) \approx H_{\varphi \cap U}^{\cdot}(U; \mathbb{C})$$

is an isomorphism. In the case $\varphi =$ closed subsets of X, one concludes that $I(*Y)$ is an isomorphism. In the case $\varphi =$ closed subsets of Y, one deduces that $H_Y^{\cdot}(X; \Omega_X^{\cdot}(*Y)) = 0$. So, the connexion homomorphism for the sequence in (5) with supports in Y is an isomorphism, and η is an isomorphism. Then $\mu = I_Y \circ \eta$ splits canonically, because I_Y does ([2], 3.11).

2.4. Corollary. *In the conditions of Theorem 2.3, suppose that X is a manifold (more generally, that U is regular and $\Omega_X^{\cdot}$ is a resolution of $\mathbb{C}$ on X). Then I, $I(*Y)$ and μ in (7) are all isomorphisms.*

In fact, $I(*Y)$ is an isomorphism by Grothendieck's theorem, and I also is, since $\Omega_X^{\cdot}$ is a resolution, so that μ is an isomorphism, too.

3. Splitting and Cup Product

We prove now that the splitting of the De Rham cohomology of an analytic space proved in [2] is compatible with the cup product, and state some lemmas to be used later. We assume that X is a paracompact analytic space over $\mathbb{C}$, although the proofs also work for real analytic or semianalytic spaces. $\mathscr{E}_X^{\cdot}$ is the complex of smooth of differentiable forms of X, with coefficients in $\mathbb{C}$ ([2], 3.3), and $\mathscr{S}.$ is the complex of sheaves of semianalytic chains on X, with coefficients in $\mathbb{C}$.

3.1. For each open U in X, $S.(U) = \Gamma_c(U; \mathscr{S}.(U))$ denotes the complex of semianalytic chains in U with compact support (cf. [2], 2). As proved in [2], 2.8, $S.: U \to S.(U)$ is a torsion free quasi-coresolution of $\mathbb{C}$ on X by flabby cosheaves, in G. Bredon's terminology. This implies that

$$H.(S.(U)) \simeq H_{\cdot}^c(U; \mathbb{C}), \tag{1}$$

the Borel-Moore homology of U with compact supports.

Define now the complex of sheaves $\mathscr{S}_{X\odot X}^{\cdot}$ on $X \times X$ as the one generated by the presheaves

$$\mathscr{S}_{X\odot X}^m : U \times V \to \sum_{p+q=m} \operatorname{Hom}\left(S_p(U) \underset{\mathbb{C}}{\otimes} S_q(V), \mathbb{C}\right), \quad m \in \mathbf{Z},$$

where U and V are open in X, and the coboundaries are constructed as usual from the boundaries in $S.$. One deduces from (1) and the algebraic Künneth formula applied to the complex $\Sigma S.(U) \otimes S(V)$, that $\mathscr{S}_{X\odot X}^{\cdot}$ is a flabby resolution of $\mathbb{C}$ on $X \times X$.

We have also canonical homomorphisms of resolutions

$$\mathcal{S}^{\cdot}_X \hat{\otimes} \mathcal{S}^{\cdot}_X \to \mathcal{S}^{\cdot}_{X \odot X} \leftarrow \mathcal{S}^{\cdot}_{X \times X}, \tag{2}$$

where the right complex is that of the semianalytic cochains of $X \times X$, and the right arrow is induced by the products

$$S.(U) \otimes S.(U) \to S.(U \times U) \quad (U \text{ open in } X),$$

defined in [2], 2.9.

3.2. Proposition. *The following diagram commutes*

$$\begin{array}{ccc} \boldsymbol{H}^{\cdot}_{\varphi}(X;\mathscr{E}^{\cdot}_X) \otimes \boldsymbol{H}^{\cdot}_{\psi}(X;\mathscr{E}^{\cdot}_X) & \xrightarrow{\cup} & \boldsymbol{H}^{\cdot}_{\varphi\cap\psi}(X;\mathscr{E}^{\cdot}_X) \\ \downarrow{\scriptstyle I\otimes I} & & \downarrow{\scriptstyle I} \\ H^{\cdot}_{\varphi}(X;\mathbb{C}) \otimes H^{\cdot}_{\psi}(X;\mathbb{C}) & \xrightarrow[\cup]{} & H^{\cdot}_{\varphi\cap\psi}(X;\mathbb{C}). \end{array}$$

Proof. Cartesian product and integration give a commutative diagram

$$\begin{array}{ccc} \boldsymbol{H}^{\cdot}_{\varphi}(X;\mathscr{E}^{\cdot}_X) \otimes \boldsymbol{H}^{\cdot}_{\psi}(X;\mathscr{E}^{\cdot}_X) & \to & \boldsymbol{H}^{\cdot}_{\varphi}(X;\mathcal{S}^{\cdot}_X) \otimes \boldsymbol{H}_{\psi}(X;\mathcal{S}^{\cdot}_X) \\ \downarrow{\scriptstyle \times} & & \downarrow{\scriptstyle \times} \\ \boldsymbol{H}^{\cdot}_{\varphi\times\psi}(X\times X;\mathscr{E}^{\cdot}_X \hat{\otimes} \mathscr{E}^{\cdot}_X) & \to & \boldsymbol{H}^{\cdot}_{\varphi\times\psi}(X\times X;\mathcal{S}^{\cdot}_X \hat{\otimes} \mathcal{S}^{\cdot}_X), \end{array}$$

which we follow by the diagram of hypercohomologies that corresponds to the canonical diagram

$$\begin{array}{ccccc} \mathscr{E}^{\cdot}_X \hat{\otimes} \mathscr{E}^{\cdot}_X & \xrightarrow{\wedge} & \mathscr{E}^{\cdot}_{X\times X} & & \\ \downarrow & & \downarrow & \searrow & \\ \mathcal{S}^{\cdot}_X \hat{\otimes} \mathcal{S}^{\cdot}_X & \longrightarrow & \mathcal{S}^{\cdot}_{X\odot X} & \longleftarrow & \mathcal{S}^{\cdot}_{X\times X}. \end{array}$$

The three complexes of the bottom line are flabby resolutions of $\mathbb{C}$ on $X \times X$; hence, the hypercohomology homomorphisms that they induce are isomorphisms. Finally, the diagonal map $X \to X \times X$ provides a commutative diagram

$$\begin{array}{ccc} \boldsymbol{H}^{\cdot}_{\varphi\times\psi}(X\times X;\mathscr{E}^{\cdot}_{X\times X}) & \to & \boldsymbol{H}^{\cdot}_{\varphi\times\psi}(X\times X;\mathcal{S}^{\cdot}_{X\times X}) \\ \downarrow & & \downarrow \\ \boldsymbol{H}^{\cdot}_{\varphi\cap\psi}(X;\mathscr{E}^{\cdot}_X) & \longrightarrow & \boldsymbol{H}^{\cdot}_{\varphi\cap\psi}(X;\mathcal{S}^{\cdot}_X), \end{array}$$

which, joined to the others, gives the proposition.

We omit the proof of the following proposition, which is similar to 3.2.

3.3. Proposition. *Let Y be a closed analytic subspace of X, and φ a family of supports in X. Integration and cup product (as defined in 1.3 and 1.4) give a commutative diagram*

$$\begin{array}{ccc} \boldsymbol{H}^{\cdot}_Y(X;\mathscr{E}^{\cdot}_X) \otimes \boldsymbol{H}^{\cdot}_{\varphi}(Y;\mathscr{E}^{\cdot}_X) & \to & H^{\cdot}_Y(X;\mathbb{C}) \otimes H^{\cdot}_{\varphi}(Y;\mathbb{C}) \\ \downarrow{\scriptstyle \cup'} & & \downarrow{\scriptstyle \cup'} \\ \boldsymbol{H}^{\cdot}_{\varphi|Y}(X;\mathscr{E}^{\cdot}_X) & \longrightarrow & H^{\cdot}_{\varphi|Y}(X;\mathbb{C}). \end{array}$$

3.4. Proposition. *In the conditions of Proposition 3.3, suppose that X has dimension n. Then there exists a commutative diagram of canonical pairings*

$$\begin{array}{c} H^p_Y(X;\mathbb{C})\otimes H^{2n-p}_c(Y;\mathbb{C})\to H^p_Y(X;\mathscr{E}^{\cdot}_X)\otimes H^{2n-p}_c(Y;\mathscr{E}^{\cdot}_X)\to H^p_Y(X;\mathbb{C})\otimes H^{2n-p}_c(Y;\mathbb{C})\to 0 \\ \searrow \quad \downarrow \quad \swarrow \\ \mathbb{C} \end{array} \tag{3}$$

Proof. One gets sequence (3) by tensoring over $\mathbb{C}$ the exact sequence

$$0\to H^p_Y(X;\mathbb{C})\to H^p_Y(X;\mathscr{E}^{\cdot}_X)\to H^p_Y(X;\mathbb{C})\to 0 \tag{4}$$

with the similar one for $\mathscr{E}^{\cdot}_X$ on Y; both sequences split canonically, since the proof of the splitting for $\mathscr{E}^{\cdot}_X$ in [2], 3.11, applies also for $\mathscr{E}^{\cdot}_X|Y$ on Y. Hence, $(I\otimes I)\circ(e\otimes e)$ in (3) an isomorphism, and the sequence splits. To construct the pairings and prove commutativity, we consider the following diagram

$$\begin{array}{ccccc} H^p_Y(X;\mathbb{C})\otimes H^{2n-p}_c(Y;\mathbb{C}) & \to & H^p_Y(X;\mathscr{E}^{\cdot}_X)\otimes H^{2n-p}_c(Y;\mathscr{E}^{\cdot}_X) & \to & H^p_Y(X;\mathbb{C})\otimes H^{2n-p}_c(Y;\mathbb{C}) \\ \downarrow{\scriptstyle\cup'} & & \downarrow{\scriptstyle\cup'} & & \downarrow{\scriptstyle\cup'} \\ H^{2n}_{c|Y}(X;\mathbb{C}) & \longrightarrow & H^{2n}_{c|Y}(X;\mathscr{E}^{\cdot}_X) & \longrightarrow & H^{2n}_{c|Y}(X;\mathbb{C}). \end{array}$$

The left square commutes by 1.4(7), and the right one by 3.3. Inclusion of supports maps the bottom sequence homomorphically into

$$0\to H^{2n}_c(X;\mathbb{C})\xrightarrow{e} H^{2n}_c(X;\mathscr{E}^{\cdot}_X)\xrightarrow{I} H^{2n}_c(X;\mathbb{C})\to 0, \tag{5}$$

where we can identify $H^{2n}_c(X;\mathscr{E}^{\cdot}_X)\simeq H^{2n}(\Gamma_c(X,\mathscr{E}^{\cdot}_X))$, since $\mathscr{E}^{\cdot}_X$ is c-fine. Finally, integration of the terms in (5) over the fundamental class of X gives a diagram

$$\begin{array}{c} 0\to H^{2n}_c(X;\mathbb{C})\to H^{2n}_c(\Gamma_c(X;\mathscr{E}^{\cdot}_X))\to H^{2n}_c(X;\mathbb{C})\to 0 \\ \searrow \quad \downarrow \quad \swarrow \\ \mathbb{C} \end{array} \tag{6}$$

whose commutativity follows from next lemma.

3.5. Lemma. *Integration defines a commutative diagram of canonical pairings* $(q\in \boldsymbol{Z})$

$$\begin{array}{c} H_q(X;\mathbb{C})\otimes H^q_c(X;\mathbb{C})\to H_q(X;\mathbb{C})\otimes H^q(\Gamma_c(X,\mathscr{E}^{\cdot}_X))\to H_q(X;\mathbb{C})\otimes \mathrm{H}^q_c(X;\mathbb{C}) \\ \searrow \quad \downarrow \quad \swarrow \\ \mathbb{C} \end{array}$$

Proof. Consider the sheaves $\mathscr{S}_.$ and $\mathscr{S}^{\cdot}$ of semianalytic chains and cochains on X, with coefficients in $\mathbb{C}$, and denotes $S^{\cdot}(X)=\Gamma_c(X,\mathscr{S}^{\cdot})$. There is a pairing

$$v:\mathscr{S}_q(X)\otimes S^q(X)\to\mathbb{C} \tag{7}$$

defined as follows: choose $s \in \mathcal{S}_q(X)$ and $f \in S^q(X)$, and let K be the (compact) support of f; choose a relatively compact open set $U \supset K$, and call $V = X - K$. The sheaf $\mathcal{S}_q$ is soft, so that we can write $s = s_U + s_V$, with support $(s_U) \subset U$ and support $(s_V) \subset V$. Then the definition $v(s \otimes f) = f(s_U)$ doesn't depend on the elections, and is compatible with differentials. It induces a perfect pairing

$$v : H_q(X;\mathbb{C}) \otimes H_c^q(X;\mathbb{C}) \to \mathbb{C} \tag{8}$$

and an isomorphism

$$H_q(X;\mathbb{C}) \approx \mathrm{Hom}(H_c^q(X;\mathbb{C}), \mathbb{C}) \tag{9}$$

that coincides with that given by formula (2) in [3], V.3. Moreover, integration defines a pairing

$$\tau : \mathcal{S}_q(X) \otimes \Gamma_c(X, \mathcal{E}_X^\cdot) \to \mathbb{C}, \tag{10}$$

that is compatible with the pairing in (7) and the integration homomorphism $I : \mathcal{E}_X^\cdot \to \mathcal{S}^\cdot$. Passing to cohomology we obtain commutativity in the right side of the diagram in 3.5.

Consider finally $a \otimes b \in H_q(X;\mathbb{C}) \otimes H_c^q(X;\mathbb{C})$; $v(a \otimes b) = v(a \otimes Ie(b))$, because Ie is a canonical isomorphism, and $v(a \otimes Ie(b)) = \tau(a \otimes e(b))$ by what we proved first. This finishes the lemma, and the proof of 3.4.

The proof of the following proposition is similar to that of 3.4:

3.6. Proposition. *Suppose the analytic space X has dimension n. Then there is a commutative diagram of canonical pairings*

$$0 \to H^p(X;\mathbb{C}) \otimes H_c^{2n-p}(X;\mathbb{C}) \to H^p(X;\mathcal{E}_X^\cdot) \otimes H_c^{2n-p}(X;\mathcal{E}_X^\cdot) \to H^p(X;\mathbb{C}) \otimes H_c^{2n-p}(X;\mathbb{C}) -$$
$$\searrow \qquad \downarrow \qquad \swarrow$$
$$\mathbb{C}$$

where the top sequence is exact and splits canonically.

3.7. Corollary. *Let Y be a closed analytic subspace of the analytic space X; suppose that X has dimension n. Then the diagrams*

$$\begin{array}{ccc} H^p\Gamma(X, \mathcal{E}_X^\cdot) & \xrightarrow{I} & H^p(X;\mathbb{C}) \\ \downarrow{\scriptstyle \cap_s} & & \downarrow{\scriptstyle \cap} \\ [H^{2n-p}\Gamma_c(X, \mathcal{E}_X^\cdot)]^* & \xrightarrow{e^*} & [H_c^{2n-p}(X;\mathbb{C})]^*, \end{array} \tag{11}$$

and

$$\begin{array}{ccc} H_Y^p(X, \mathcal{E}_X^\cdot) & \xrightarrow{I} & H_Y^p(X;\mathbb{C}) \\ \downarrow{\scriptstyle \cap'_s} & & \downarrow{\scriptstyle \cap'} \\ [H^{2n-p}\Gamma_c(Y, \mathcal{E}_X^\cdot)]^* & \xrightarrow{e^*} & [H_Y^{2n-p}(X;\mathbb{C})]^* \end{array} \tag{12}$$

are commutative, where $[\cdot]^* \approx \mathrm{Hom}(\cdot, \mathbb{C})$, *and the vertical homomorphisms are deduced from the pairings in 3.6 and 3.4 (hypercohomology is replaced by cohomology, when possible, and both e* are dual to canonical edge's homomorphisms, cf. 2.1 (4)).*

Proof. Consider for instance diagram (11). Take $\omega \in H^p\Gamma(X, \mathscr{E}_X^{\cdot})$ and $\alpha \in H_c^{2n-p}(X, \mathbb{C})$; then $\cap \circ I(\omega)$ sends α into the evaluation on $[X]$ of $I(\omega) \cup \alpha$, and $e^* \circ \cap_s(\omega)$ sends α into the evaluation on $[X]$ of $\omega \cup e(\alpha)$, which is equal to the evaluation on $[X]$ of $I(\omega \cup e(\alpha))$, by 3.5; finally, $I(\omega \cup e(\alpha)) = I\omega \cup Ie(\alpha) = I\omega \cup \alpha$, by 3.2.

3.8. Corollary. *In the situation of 3.7, suppose that Y is locally defined by one equation; then there is a commutative diagram*

$$\begin{array}{ccc} H^p\Gamma(X, \mathscr{E}_X^{\cdot}(*Y)) & \xrightarrow{I} & H^p(U;\mathbb{C}) \\ \downarrow{\scriptstyle \cap_s} & & \downarrow{\scriptstyle \cap} \\ [H^{2n-p}\Gamma_c(U, \mathscr{E}_X^{\cdot})]^* & \xrightarrow[e^*]{} & [H_c^{2n-p}(U;\mathbb{C})]^* . \end{array} \tag{13}$$

Proof. Consider the restriction $H^p\Gamma(X, \mathscr{E}_X^{\cdot}(*Y)) \to H^p\Gamma(U, \mathscr{E}_U^{\cdot})$, and apply (11) to the space of U, in the place of X.

4. De Rham Homology of a Semianalytic Space

4.1. We suppose in this section that X and Y are closed semianalytic sets in a paracompact real analytic space, that $Y \subset X$, $U = X - Y$, and that $i: Y \to X$ and $j: U \to X$ are the inclusions. X will be considered together with its complex $\mathscr{E}_X^{\cdot}$ of smooth $\mathbb{C}$-valued differential forms ([2], 3.3) and Y together with the complex $\mathscr{E}_{Y\infty}^{\cdot} = \mathscr{E}_X^{\cdot} | Y$. There are epimorphisms

$$\mathscr{E}_X^{\cdot} \to i\,\mathscr{E}_{Y\infty}^{\cdot}, \quad \mathscr{E}_{Y\infty}^{\cdot} \to \mathscr{E}_Y^{\cdot}, \tag{1}$$

and the kernel of the first one is denoted by $\mathscr{E}_{X,Y}^{\cdot}$; clearly

$$\Gamma_c(X, \mathscr{E}_{X,Y}^{\cdot}) \simeq \Gamma_c(U, \mathscr{E}_X^{\cdot}).$$

The semianalytic chains in Y are mapped inyectively by i into the chains in X. This defines an epimorphism $\mathscr{S}_X^{\cdot} \to i\mathscr{S}_Y^{\cdot}$ between the complexes of semianalytic cochains, and its kernel is denoted by $\mathscr{S}_{X,Y}^{\cdot}$.

We use the standard notations $\mathbb{C}_Y = i(\mathbb{C}|Y)$ and $\mathbb{C}_U = j(\mathbb{C}|U)$, where $\mathbb{C}$ is the constant sheaf over X. Define complexes $\mathbb{C}^{\cdot}$, $\mathbb{C}_Y^{\cdot}$ and $\mathbb{C}_U^{\cdot}$ by $\mathbb{C}^0 = \mathbb{C}$ and $\mathbb{C}^p = 0$ if $p \neq 0$, and similarly for $\mathbb{C}_Y^{\cdot}$ and $\mathbb{C}_U^{\cdot}$.

There is a commutative diagram of complexes (2), where the

$$\begin{array}{ccccccccc} 0 \to & \mathbb{C}_U^{\cdot} & \to & \mathbb{C}^{\cdot} & \to & \mathbb{C}_Y^{\cdot} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathscr{E}_{X,Y}^{\cdot} & \to & \mathscr{E}_X^{\cdot} & \to & i\mathscr{E}_{Y\infty}^{\cdot} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathscr{S}_{X,Y}^{\cdot} & \to & \mathscr{S}_X^{\cdot} & \to & i(\mathscr{S}_Y^{\cdot}) & \to 0 \end{array} \tag{2}$$

horizontal sequences are exact, the bottom vertical arrows are obtained by integration and the top vertical arrows are augmentations (in degree 0). Composition of the vertical arrows, in degree zero, gives the augmentations $\mathbb{C} \to \mathscr{S}_X^{\cdot}$, $\mathbb{C}_Y \to i\mathscr{S}_Y^{\cdot}$ and $\mathbb{C}_U \to \mathscr{S}_{X,Y}^{\cdot}$; the first two of these complexes are resolutions ([2], 2.11), so that the last one also is.

Consider now the diagram of hypercohomology with compact supports associated to (2). After convenient identifications, taking into account that $\mathscr{E}_{X,Y}^{\cdot}$, $\mathscr{E}_X^{\cdot}$ and $\mathscr{E}_Y^{\cdot}$ are c-acyclic, one obtains the commutative diagram (3) of exact sequences, where the compositions of the vertical arrows are canonical isomorphisms.

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H_c^p(U;\mathbb{C}) & \longrightarrow & H_c^p(X;\mathbb{C}) & \longrightarrow & H_c^p(Y;\mathbb{C}) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^p\Gamma_c(U,\mathscr{E}_X^{\cdot}) & \longrightarrow & H^p\Gamma_c(X,\mathscr{E}_X^{\cdot}) & \longrightarrow & H^p\Gamma_c(Y,\mathscr{E}_X^{\cdot}) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^p\Gamma_c(U,\mathscr{S}_X^{\cdot}) & \longrightarrow & H^p\Gamma_c(X,\mathscr{S}_X^{\cdot}) & \longrightarrow & H^p\Gamma_c(Y,\mathscr{S}_Y^{\cdot}) & \longrightarrow \cdots.
\end{array} \tag{3}$$

4.2. *Currents on semianalytic sets.* Let M be a closed semianalytic subset of the open set V in $\mathbb{R}^n$. By definition of the complex $\mathscr{E}_M^{\cdot}$, there is an exact sequence ([2], 3.3)

$$0 \to \Gamma_c(V;\mathscr{E}_{V,M}^{\cdot}) \to \Gamma_c(V;\mathscr{E}_V^{\cdot}) \to \Gamma_c(V;\mathscr{E}_M^{\cdot}) \to 0\,,$$

where $\mathscr{E}_{V,M}^{\cdot} = \ker(\mathscr{E}_V^{\cdot} \to \mathscr{E}_M^{\cdot})$.

Consider the space $\Gamma_c(V,\mathscr{E}_V^{\cdot})$ with its usual locally convex inductive topology. This is the topology of convergence of the coefficients of forms in $\Gamma_c(V,\mathscr{E}_V^{\cdot})$, together with their derivatives, on the compact subsets of V. The projection $\Gamma_c(V,\mathscr{E}_V^{\cdot}) \to \Gamma_c(M,\mathscr{E}_M^{\cdot})$ has closed kernel, which allow us to define on $\Gamma_c(M,\mathscr{E}_M^{\cdot})$ the quotient Hausdorff topology.

In the case of our semianalytic space X, we define on $\Gamma_c(X,\mathscr{E}_X^{\cdot})$ the unique locally convex inductive topology that induces on the spaces $\Gamma_c(M,\mathscr{E}_M^{\cdot})$, associated to the local models (M, V) of X, the topology just described.

Define the space $'\mathscr{D}_q(X)$ of q-currents on X as the topological dual of $\Gamma_c(X,\mathscr{E}_X^q)$, and the border homomorphism

$$b_q : {}'\mathscr{D}_q(X) \to {}'\mathscr{D}_{q-1}(X) \quad (q \in \mathbf{Z})$$

by $b_q\mathrm{T}(\alpha) = (-)^{q+1}T(d\alpha)$, for $T \in {}'\mathscr{D}_q(X)$ and $\alpha \in \Gamma_c(X,\mathscr{E}_X^{q-1})$. The complex of sheaves $'\mathscr{D}_{\cdot X}$ of germs of currents on X is constructed accordingly; $'\mathscr{D}_q = 0$ if $q < 0$ or $q > \dim X$.

Denote by $'\mathscr{D}_{\cdot Y\infty}$ the subcomplex of $'\mathscr{D}_{\cdot X}$ of the currents on X with support on Y, and by $'\mathscr{D}_{\cdot X/Y\infty} = {}'\mathscr{D}_{\cdot X}/{}'\mathscr{D}_{\cdot Y\infty}$ the quotient complex. We have then an exact sequence

$$0 \to {}'\mathscr{D}_{\cdot Y\infty} \to {}'\mathscr{D}_{\cdot X} \to {}'\mathscr{D}_{\cdot X/Y\infty} \to 0 \tag{4}$$

of c-fine sheaves.

Denote also by $'\mathscr{S}_{\cdot X} = \Sigma(\mathscr{S}_{qX} : q \in \mathbf{Z})$ the complex of sheaves of germs of semianalytic chains in X, with the modified boundary $\partial_q = (-)^{q+1}\partial : \mathscr{S}_{qX} \to \mathscr{S}_{q-1,X}$. The complex $'\mathscr{S}_{\cdot Y}$ of semianalytic chains in Y is defined in the same way; it is clear that $i'\mathscr{S}_{\cdot Y}$ is a subcomplex of $'\mathscr{S}_{\cdot X}$. Define $'\mathscr{S}_{\cdot X/Y}$ as the quotient complex, so that there is an exact sequence

$$0 \to i'\mathscr{S}_{\cdot Y} \to {}'\mathscr{S}_{\cdot X} \to {}'\mathscr{S}_{\cdot X/Y} \to 0 \tag{5}$$

of soft sheaves.

Integration of forms in $\Gamma_c(X, \mathscr{E}_X^{\cdot})$ on the chains in $'\mathscr{S}_{\cdot X}$ defines a homomorphism $'\mathscr{S}_{\cdot X} \to {}'\mathscr{D}_{\cdot X}$ and, in fact, a homomorphism from (5) to (4). Taking cohomology of the global sections we deduce a commutative diagram of long exact sequences

$$\begin{array}{ccccccc} \to H_p(\Gamma(Y, {}'\mathscr{S}_{\cdot Y})) & \to & H_p(\Gamma(X, {}'\mathscr{S}_{\cdot X}) & \to & H_p(\Gamma(X, {}'\mathscr{S}_{\cdot X/Y})) & \to . \\ \downarrow & & \downarrow & & \downarrow & \\ \to H_p(\Gamma(Y, {}'\mathscr{D}_{\cdot Y\infty})) & \to & H_p(\Gamma(X, {}'\mathscr{D}_{\cdot X})) & \to & H_p(\Gamma(X, {}'\mathscr{D}_{\cdot X/Y\infty})) & \to . \end{array} \tag{6}$$

On this point we should note that $H_p\Gamma(X, {}'\mathscr{S}_{\cdot X}) \simeq H_p(X; \mathbb{C})$ and $H_p\Gamma(Y, {}'\mathscr{S}_{\cdot Y}) \simeq H_p(Y; \mathbb{C})$ (by [2], 2.8), and that $H_p\Gamma(X, {}'\mathscr{S}_{\cdot X/Y}) \simeq H_p(U; {}'\mathscr{S}_{\cdot U}) \simeq H_p(U; \mathbb{C})$. The last isomorphism is obtained here by the canonical (restriction) map $\mathscr{S}_{\cdot X/Y} \to {}'\mathscr{S}_{\cdot U}$, and the five lemma. In fact, this method realizes the exact sequence of Borel-Moore homology (complex coefficients)

$$\cdots \to H_p(X) \to H_p(U) \to H_{p-1}(Y) \to \cdots \tag{7}$$

by means of semianalytic chains.

We now want to define a canonical homomorphism from diagram (6) into the dual of the bottom diagram of (3):

$$\begin{array}{ccccccc} \to [H^p\Gamma_c(Y, \mathscr{S}_Y^{\cdot})]^* & \to & [H^p\Gamma_c(X, \mathscr{S}_X^{\cdot})]^* & \to & [H^p\Gamma_c(U, \mathscr{S}_X^{\cdot})]^* & \to \\ \downarrow I^* & & \downarrow I^* & & \downarrow I^* & \\ \to [H^p\Gamma_c(Y, \mathscr{E}_Y^{\cdot})]^* & \to & [H^p\Gamma_c(X, \mathscr{E}_X^{\cdot})]^* & \to & [H^p\Gamma_c(U, \mathscr{E}_X^{\cdot})]^* & \to , \end{array} \tag{8}$$

where $[\cdot]^* \approx \mathrm{Hom}(\cdot, \mathbb{C})$.

First, we observe that there is a commutative diagram

$$\begin{array}{ccc} H_p\Gamma(X, {}'\mathscr{S}_{\cdot X}) & \xrightarrow{v'} & [H^p\Gamma_c(X, \mathscr{S}_X^{\cdot})]^* \\ \downarrow & & \downarrow I^* \\ H_p\Gamma(X, {}'\mathscr{D}_{\cdot X}) & \xrightarrow[\tilde{v}]{} & [H^p\Gamma_c(X, \mathscr{E}_X^{\cdot})]^* , \end{array} \tag{9}$$

where the top arrow is the isomorphism 3(9) and the bottom one is deduced from the injection $\Gamma(X. {}'\mathscr{D}_{\cdot X}) \to \mathrm{Hom}(\Gamma_c(X, \mathscr{E}_X^{\cdot}), \mathbb{C})$.

If we follow $\tilde{v}$ by the transposed homomorphism $e^*: [H^p\Gamma_c(X, \mathscr{E}_X^{\cdot})]^* \to [H_c^p(X, \mathbb{C})]^*$ to the one in (3), we obtain a sequence of homomorphisms

$$H_p\Gamma(X, {}'\mathscr{S}_{\cdot X}) \to H_p\Gamma(X, {}'\mathscr{D}_{\cdot X}) \to [H_c^p(X; \mathbb{C})]^* \tag{10}$$

whose composition is an isomorphism, since it can be factored as $e^* \circ I^* \circ v'$, and both v' and $(e \circ I)^*$ are isomorphisms. Moreover, the end terms of (10) are canonically identified with $H_p(X, \mathbb{C})$, the first by [2], 2.8 and the second by 3(9).

We have so obtained a canonically splitting exact sequence

$$0 \to H_p(X;\mathbb{C}) \to H_p\Gamma(X,'\mathcal{D}_{\cdot X}) \to H_p(X;\mathbb{C}) \to 0\,. \tag{11}$$

The same method can be applied to the other terms in (6).

One constructs a right inverse τ to $H_p\Gamma(X;'\mathcal{S}_{\cdot X/Y}) \to H_p\Gamma(X,'\mathcal{D}_{\cdot X/Y\infty})$ by applying first $\tilde{v}: H_p\Gamma(X,'\mathcal{D}_{\cdot X/Y\infty}) \to [H^p\Gamma_c(U,\mathcal{E}_X^{\cdot})]^*$, which is deduced from the map

$$\Gamma(X,'\mathcal{D}_{pX/Y\infty}) \simeq \Gamma(X,'\mathcal{D}_{pX})/\Gamma(X,'\mathcal{D}_{pY\infty}) \to [\Gamma_c(U,\mathcal{E}_X^p)]^*\,,$$

and then following with the edge homomorphism's dual

$$e^*: [H^p\Gamma_c(U;\mathcal{E}_X^{\cdot})]^* \to [H_c^p(U;\mathbb{C})]^* \simeq H_p(U;\mathbb{C})\,. \tag{12}$$

Similarly, a right inverse τ to $H_p(Y;\mathbb{C}) \to H_p\Gamma(Y,'\mathcal{D}_{\cdot Y\infty})$ is constructed as shown in the following diagram,

$$\begin{array}{ccc} H_p\Gamma(Y,'\mathcal{D}_{\cdot Y\infty}) & \overset{\tau}{\dashrightarrow} & [H_c^p(Y;\mathbb{C})]^* \simeq H_p(Y;\mathbb{C}) \\ & {\scriptstyle\tilde{v}}\searrow \qquad \nearrow{\scriptstyle e^*} & \\ & [H^p\Gamma_c(Y,\mathcal{E}_X^{\cdot})]^*\,, & \end{array} \tag{13}$$

where e^* is dual to e (cf. (3)) and $\tilde{v}$ is deduced from the canonical homomorphism

$$\Gamma(Y,'\mathcal{D}_{\cdot Y\infty}) \to [\Gamma_c(Y,\mathcal{E}_X^{\cdot})]^*\,.$$

It is easy to check that the maps so constructed are compatible, as expressed in the following

4.3. Theorem. *Let $Y \subset X$ be closed semianalytic sets in a paracompact real analytic space. With the notations in 4.2, there is a commutative diagram* (14), *in which compositions of vertical homomorphisms are natural identifications.*

$$\begin{array}{ccccccc} \cdots \to & H_p(Y;\mathbb{C}) & \to & H_p(X;\mathbb{C}) & \to & H_p(U;\mathbb{C}) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_p\Gamma(Y;'\mathcal{D}_{\cdot Y\infty}) & \to & H_p\Gamma(X;'\mathcal{D}_{\cdot X}) & \to & H_p(X;'\mathcal{D}_{\cdot X/Y\infty}) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_p(Y;\mathbb{C}) & \to & H_p(X;\mathbb{C}) & \to & H_p(U;\mathbb{C}) & \to \cdots\,, \end{array} \tag{14}$$

4.4. *Remark.* In particular, we have obtained a natural splitting of the "De Rham homology" $H_p\Gamma(X;'\mathcal{D}_{\cdot X})$ of X, in which one factor is the Borel-Moore homology $H_p(X;\mathbb{C})$. Examples where the two homologies are different can be constructed, using examples where Poincaré lemma fails for $\mathcal{E}_X^{\cdot}$ (cf. [2], 3.7).

Also, Theorem 4.3 holds if one replaces the complex $'\mathcal{D}_{\cdot Y\infty}$ by the more natural one $'\mathcal{D}_{\cdot Y}$. However, from the point of view of residues, it seems that one has to work with $'\mathcal{D}_{\cdot Y\infty}$.

4.5. Corollary. *In the conditions of Theorem 4.3, suppose that X and Y are manifolds, and let $i: Y \to X$ denote the inclusion. Then all vertical homomorphisms in diagram* (14) *are isomorphisms.*

Proof. If X is a manifold of dimension m, the sheaf map $'\mathcal{S}._X \to '\mathcal{D}._X$ induces an isomorphism $\mathcal{H}('\mathcal{S}._X) \simeq \mathcal{H}('\mathcal{D}._X)$ in the homology level. In fact, $\mathcal{H}_p('\mathcal{D}._X) \simeq \mathbb{C}$ if $p=m$, and $\mathcal{H}_p('\mathcal{D}._X) \simeq 0$, otherwise (cf. [10], § 19); consequently, $\mathcal{H}.('\mathcal{D}._X)$ is isomorphic with the homology sheaf $\mathcal{H}.('\mathcal{S}._X)$ of X.

Moreover, the sheaf map $i'\mathcal{S}._Y \to '\mathcal{D}._{Y\infty}$ also induces isomorphism in the homology level. In fact, this map is composition of $i'\mathcal{S}._Y \xrightarrow{I_Y} i'\mathcal{D}._Y \xrightarrow{e_Y} '\mathcal{D}._{Y\infty}$, where I_Y is obtained by integration, and e_Y is a canonical inclusion. As shown above, I_Y induces an isomorphism in the homology level. To see that e_Y also does, it suffices to consider $\mathbb{R}^m = \mathbb{R}^s \oplus \mathbb{R}^{m-s}$ as a local model, $s = \dim Y$, and the homotopy formulas associated to the retraction $\mu_t(u,v) = (u, tv)$ $(t \in \mathbb{R})$ (cf. [10], § 14 and 19). With a standard notation, we have

$$\alpha - \mu_0^*(\alpha) = dM^*\alpha + M^*d\alpha \quad (\alpha \in \Gamma_c(\mathbb{R}^m, \mathcal{E}^{\cdot}_{\mathbb{R}^m}))\,, \tag{15}$$

where M^* transforms p-forms into $(p-1)$-forms. A current T in $\Gamma_{\mathbb{R}^s}(\mathbb{R}^m; '\mathcal{D}._{\mathbb{R}^m})$ can be applied to both members in (15), when α has compact support, because in this case $\mathbb{R}^s$ and the support of $M^*\alpha$ have compact intersection. Moreover, there is a well defined current $\tilde{T} \in \Gamma(\mathbb{R}^s, '\mathcal{D}._{\mathbb{R}^s})$ such that $i\tilde{T}(\alpha) = T(\mu_0^*(\alpha))$. This fact, together with (15), imply that the maps

$$e_Y: \mathcal{H}.(i'\mathcal{D}._Y) \to \mathcal{H}.('\mathcal{D}._{Y\infty})$$

are isomorphisms.

As a consequence, we have isomorphisms in hyperhomology $\boldsymbol{H}_p(X; '\mathcal{S}._X) \to \boldsymbol{H}_p(X; '\mathcal{D}._X)$ and $\boldsymbol{H}_p(X; i'\mathcal{S}._Y) \to \boldsymbol{H}_p(X; '\mathcal{D}._{Y\infty})$; the first one can be identified with the wanted isomorphism $H_p(X; \mathbb{C}) \simeq H_p\Gamma(X; '\mathcal{S}._X) \to H_p\Gamma(X; '\mathcal{D}._X)$ because $'\mathcal{S}._X$ and $'\mathcal{D}._X$ are acyclic ($'\mathcal{S}._X$ is soft and $'\mathcal{D}._X$ is fine); by the same reasons, the second isomorphism identifies with the mapping $H_p\Gamma(X; i'\mathcal{S}._Y) \to H_p\Gamma(X; '\mathcal{D}._{Y\infty})$, or, what is the same, with $H_p(Y; \mathbb{C}) \simeq H_p\Gamma(Y; '\mathcal{S}._Y) \to H_p\Gamma(Y; '\mathcal{D}._{Y\infty})$, considering that the sheaves involved have supports on Y. This in turn implies that the homomorphisms $H_p(U; \mathbb{C}) \to H_p\Gamma(X; '\mathcal{D}._{X/Y\infty})$ in (14) are also isomorphisms.

5. Residues on Analytic Spaces

5.1. Theorem. *Let X be a paracompact reduced complex analytic space of pure dimension n and Y a 1-codimensional closed analytic subspace locally defined by one equation.*

Consider the associated exact sequences of complexes of forms and currents on X (cf. no. 2 and 4):

$$0 \to \Omega^{\cdot}_X \to \Omega^{\cdot}_X(*Y) \to Q^{\cdot} \to 0 \tag{1}$$

and

$$0 \to '\mathcal{D}._{Y\infty} \to '\mathcal{D}._X \to '\mathcal{D}._{X/Y\infty} \to 0\,. \tag{2}$$

There are canonical homomorphisms

$$\mathrm{V}: \Omega_X^{\cdot} \to {}'\mathscr{D}_{2n-\cdot,X}, \quad \mathrm{PV}: \Omega_X^{\cdot}(*Y) \to {}'\mathscr{D}_{2n-\cdot,X/Y^\infty}$$

and

$$\mathrm{Res}: Q^{\cdot} \to {}'\mathscr{D}_{2n-1-\cdot,Y^\infty}$$

that induce a commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow H^p(X;\Omega_X^{\cdot}) & \longrightarrow & H^p(X;\Omega_X^{\cdot}(*Y)) & \longrightarrow & H^p(X;Q^{\cdot}) & \longrightarrow \cdots \\ \downarrow \mathrm{V} & & \downarrow \mathrm{PV} & & \downarrow \mathrm{Res} & \\ \cdots \to H_{2n-p}\Gamma(X,{}'\mathscr{D}_{\cdot X}) & \to & H_{2n-p}\Gamma(X;{}'\mathscr{D}_{\cdot X/Y}) & \to & H_{2n-p-1}\Gamma(X;{}'\mathscr{D}_{\cdot Y^\infty}) & \to \cdots \end{array} \tag{3}$$

which is compatible with the topological diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow H^p(X;\mathbb{C}) & \longrightarrow & H^p(X-Y;\mathbb{C}) & \longrightarrow & H_Y^{p+1}(X;\mathbb{C}) & \longrightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow & \\ \cdots \to H_{2n-p}(X;\mathbb{C}) & \to & H_{2n-p}(X-Y;\mathbb{C}) & \to & H_{2n-p-1}(Y) & \to \cdots \end{array} \tag{4}$$

obtained by cap product with the fundamental class of X (cf. 1(14)).

Proof. For each open set W in X, define $V: \Gamma(W,\Omega_X^p) \to \Gamma(W,{}'\mathscr{D}_{2n-p,X})$, $\omega \to V[\omega]$, by the formula

$$V[\omega](\alpha) = I[X](\omega \wedge \alpha) \quad (\alpha \in \Gamma_c(W,\mathscr{E}_X^{2n-p})), \tag{5}$$

where $I[X]$ is the integration current over X, oriented by its fundamental class $[X] \in H_{2n}(X;\mathbb{C})$ (cf. [2], 3.4). V is a sheaf map, and $V[\omega]$ *is a* (0-*continuous, cf.* [7], *II.A,2) current. If* $\alpha \in \Gamma_c(W;\mathscr{E}_X^{2n-p-1})$, Stokes' theorem ([7], II.B,2.9) implies that $I[X](d(\omega \wedge \alpha)) = I[X](d\omega \wedge \alpha) + (-1)^p I[X](\omega \wedge d\alpha) = 0$, from what we deduce

$$V[d\omega](\alpha) = (-1)^{p+1} V[\omega](d\alpha) = (-1)^{2n-p+1} V[\omega](d\alpha) = b.V[\omega](\alpha),$$

so that V is compatible with boundaries.

We now give local definitions of PV and Res.

5.2. Definition. *Consider an open subspace W of X, and a holomorphic equation $\varphi \in \Gamma(W,\mathcal{O}_X)$ of Y in W. For any form $\omega \in \Gamma(W,\Omega_X^p)$ and $q \in \mathbf{Z}_+$, the principal value* $\mathrm{PV}[\omega/\varphi^q]$ *and the residue* $\mathrm{Res}[\omega/\varphi^q]$ *of the meromorphic form $\omega/\varphi^q \in \Gamma(W,\Omega^p(*Y))$ are the currents*

$$\mathrm{PV}[\omega/\varphi^q](\alpha) = \lim_{\delta\to 0} I[W(>\delta)](\omega \wedge \alpha/\varphi^q), \quad (\alpha \in \Gamma_c(W;\mathscr{E}_X^{2n-p}))$$

$$\mathrm{Res}[\omega/\varphi^q](\beta) = \lim_{\delta\to 0} I[W(=\delta)](\omega \wedge \beta/\varphi^q), \quad (\beta \in \Gamma_c(W;\mathscr{E}_X^{2n-p-1})).$$

For any $\delta > 0$, $W(>\delta)$ and $W(=\delta)$ denote here the semianalytic chains in W (cf. [2], no. 2):

$$[W(>\delta)] = [W(>\delta),\ e(>\delta)] \in \mathscr{S}_{2n}(W;\mathbb{C}),$$
$$[W(=\delta)] = [W(=\delta),\ e(=\delta)] \in \mathscr{S}_{2n-1}(W;\mathbb{C}),$$

where $W(>\delta)=(x\in W:|\varphi^q(x)|>\delta)$, $W(=\delta)=(x\in W:|\varphi^q(x)|=\delta)$, $e(>\delta)\in H_{2n}(W(>\delta);\mathbb{C})$ *is the fundamental homology class of the space* $W(>\delta)$, *and* $e(=\delta)=-\partial e(>\delta)\in H_{2n-1}(W(=\delta);\mathbb{C})$ *is equal to minus the boundary of* $e(>\delta)$ *in the exact sequence of Borel-Moore homology*

$$\cdots\to H_{2n}(W(>\delta);\mathbb{C})\to H_{2n-1}(W(=\delta);\mathbb{C})\to H_{2n-1}(W(\geqq\delta);\mathbb{C})\to\cdots.$$

We have, in fact, that $\partial[W(>\delta)]=-[W(=\delta)]$, according to definitions in [2], n. 2.

The integrations in 5.2 are defined, since the forms $\omega\wedge\alpha/\varphi^q$ and $\omega\wedge\beta/\varphi^q$ are regular and their supports have compact intersection with $W(>\delta)$ and $W(=\delta)$, respectively. The existence of the limits in 5.2., and their continuous dependence of α and β, will be proved in sections 6 and 7 (cf. Theorem 7.1). Clearly, then, $\operatorname{Res}[\omega/\varphi^q]$ is a current with support in $Y\cap W$.

Let now $\tilde{W}$ be any open set in X, and $\tilde{\omega}\in(\tilde{W},\Omega_X^p(*Y))$. Each point $x\in\tilde{W}$ has a neighborhood W_x where $\tilde{\omega}$ can be represented by ω/φ^q, as in 5.2, so that we can define

$$\mathrm{PV}[\tilde{\omega}]=\mathrm{PV}[\omega/\varphi^q]\,,\quad \operatorname{Res}[\tilde{\omega}]=\operatorname{Res}[\omega/\varphi^q]\quad\text{on}\quad W_x\,.$$

These local definitions of PV and Res agree on overlapping neighborhoods $W_x(x\in\tilde{W})$, by 7.1 and 7.2, and define sheaf maps $\operatorname{Res}:\Omega_X^p(*Y)\to{}'\mathscr{D}_{2n-p-1,Y^\infty}$ and

$$\mathrm{PV}:\Omega_X^p(*Y)\to{}'\mathscr{D}_{2n-p,X}\,. \tag{9}$$

Moreover, $\operatorname{Res}(\tilde{\omega})=0$ when $\tilde{\omega}=\varphi\tilde{\omega}/\varphi$ is regular on Y, by 7.4. Hence, there is a map (also denoted by Res)

$$\operatorname{Res}:Q_X^p=\Omega_X^p(*Y)/\Omega_X^p\to{}'\mathscr{D}_{2n-p-1,Y^\infty}\,. \tag{10}$$

The construction of V, PV and Res is thus finished, and we study now the relation of PV and Res with boundaries.

In the conditions of 5.2, let $\alpha\in\Gamma_c(W,\mathscr{E}_X^{2n-p-1})$. By Stokes' theorem

$$I[W(>\delta)]\,(d(\omega\wedge\alpha/\varphi^q))=-I[W(=\delta)]\,(\omega\wedge\alpha/\varphi^q)\,,$$

since $\partial[W(>\delta)]=-[W(=\delta)]$. Letting $\delta\to 0$, we get

$$\mathrm{PV}[d(\omega/\varphi^q)]\,(\alpha)+(-1)^p\,\mathrm{PV}[\omega/\varphi^q]\,(d\alpha)=-\operatorname{Res}[\omega/\varphi^q]\,(\alpha)\,;$$

or

$$(-1)^{2n-p+1}b\,\mathrm{PV}[\omega/\varphi^q]\,(\alpha)-\mathrm{PV}[d(\omega/\varphi^q)]\,(\alpha)=\operatorname{Res}[\omega/\varphi^q]\,(\alpha)\,;$$

that is, the following formula holds

$$b.\mathrm{PV}-\mathrm{PV}\,d=\operatorname{Res}\,, \tag{11}$$

for representations $\omega/\varphi^q\in\Gamma(W,\Omega_X^p(*Y))$.

If we take now any section $\tilde{\omega} \in \Gamma(\tilde{W}, \Omega_X^{\cdot}(*Y))$ over an open set $\tilde{W}$, formula (11) applies locally, so that it is also true globally, and we can state the following proposition

5.3. Proposition. *The sheaf mappings* PV *and* Res, *considered as homomorphisms from* $\Omega_X^{\cdot}(*Y)$ *to* $'\mathscr{D}_{\cdot X}$, *satisfy the relation*

$$b.\,\mathrm{PV} - \mathrm{PV}\,d = \mathrm{Res}\,. \tag{11}$$

This formula, and the fact that $\mathrm{Res}[\omega/\varphi^q]$ has support on Y, imply the following

5.4. Corollary. $\mathrm{PV}: \Omega_X^{\cdot}(*Y) \to {}'\mathscr{D}_{2n-\cdot,X}/{}'\mathscr{D}_{2n-\cdot,Y^\infty} = {}'\mathscr{D}_{2n-\cdot,X/Y^\infty}$ *is compatible with boundaries.*

Consider again $\omega/\varphi^q \in \Gamma(W, \Omega^p(*Y))$ and $\beta \in \Gamma_c(W, \mathscr{E}_X^{2n-p-2})$. By Stokes' theorem

$$0 = I[W(=)]\,(d(\omega \wedge \beta/\varphi^q)) = I[W(=\delta)]\,(d(\omega/\varphi^q) \wedge \beta + (-1)^p(\omega/\varphi^q) \wedge d\beta)\,.$$

Letting $\delta \to 0$, we have

$$\mathrm{Res}[d(\omega/\varphi^q)]\,(\beta) = (-1)^{p+1}\,\mathrm{Res}[\omega/\varphi^q]\,(d\beta) = -(-1)^{2n-p} b\,\mathrm{Res}[\omega/\varphi^q]\,(\beta)\,.$$

As before, this local formula implies the following

5.5. Proposition. *The sheaf mapping* $\mathrm{Res}: \Omega_X^{\cdot}(*Y) \to {}'\mathscr{D}_{2n-1-\cdot,Y^\infty}$ *satisfies the relation*

$$\mathrm{Res} \circ d = b. \circ \mathrm{Res}\,.$$

The induced map $\Omega_X^{\cdot}(*Y)/\Omega_X^{\cdot} \to {}'\mathscr{D}_{2n-1-\cdot,Y^\infty}$ is therefore compatible with boundaries.

5.6. *Commutativity of diagram* (3). The diagram

$$\begin{array}{ccc} \Omega_X^{\cdot} & \longrightarrow & \Omega_X^{\cdot}(*Y) \\ \downarrow \mathrm{V} & & \downarrow \mathrm{PV} \\ {}'\mathscr{D}_{\cdot X} & \longrightarrow & {}'\mathscr{D}_{\cdot X/Y^\infty} \end{array} \tag{12}$$

commutes obviously, since $\mathrm{PV}[\omega] = \mathrm{V}[\omega]$ for all germs $\omega \in \Omega_X^{\cdot}$. In the deduced diagram of hypercohomology, one can identify the hypercohomology of the bottom complexes, which are fine, with the cohomology of their global sections. Hence, commutativity of one square in (3) is obtained.

To handle the rest of (3), we remark that homomorphisms

$$\mathrm{V}: \mathscr{E}_X^{\cdot} \to {}'\mathscr{D}_{2n-\cdot,X}, \quad \mathrm{PV}: \mathscr{E}_X^{\cdot}(*Y) \to {}'\mathscr{D}_{2n-\cdot,X/Y^\infty} \tag{13}$$

and

$$\mathrm{Res}: {}_s Q^{\cdot} = \mathscr{E}_X^{\cdot}(*Y)/\mathscr{E}_X^{\cdot} \to {}'\mathscr{D}_{2n-1-\cdot,Y^\infty}$$

can be defined, in the same way and with the same properties as the previous ones, and which are compatible with the canonical diagram

$$\begin{array}{ccccccccc} 0 \to & \Omega_X^{\cdot} & \to & \Omega_X^{\cdot}(*Y) & \to & Q^{\cdot} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathscr{E}_X^{\cdot} & \to & \mathscr{E}_X^{\cdot}(*Y) & \to & {}_sQ^{\cdot} & \to 0. \end{array} \tag{14}$$

So, (3) can be replaced by the hypercohomology diagram of (14), which commutes, together with a diagram similar to (3), but with complexes of smooth differential forms, which are acyclic. In this case we can replace hypercohomology by cohomology of global sections, which is easier to handle for diagram chasing. We are reduced, then, to prove commutativity in the diagram

$$\begin{array}{ccccc} H^p\Gamma(X;\mathscr{E}_X^{\cdot}(*Y)) & \longrightarrow & H^p\Gamma(X;{}_sQ^{\cdot}) & \xrightarrow{\delta} & H^{p+1}\Gamma(X;\mathscr{E}_X^{\cdot}) \\ \downarrow{\scriptstyle \mathrm{PV}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{V}} \\ H_{2n-p}\Gamma(X;'\mathscr{D}_{\cdot X/Y\infty}) & \xrightarrow{\partial} & H_{2n-p-1}\Gamma(X;'\mathscr{D}_{\cdot Y\infty}) & \xrightarrow{i} & H_{2n-p-1}\Gamma(X;'\mathscr{D}_{\cdot X}). \end{array} \tag{15}$$

Take a cycle $\tilde{\omega} \in Z^p\Gamma(X;\mathscr{E}_X^{\cdot}(*Y))$; by 5.3, $b.\mathrm{PV}[\tilde{\omega}] = \mathrm{Res}[\tilde{\omega}]$, and $b.\mathrm{PV}[\tilde{\omega}] \in Z\Gamma(X,'\mathscr{D}_{Y\infty})$ represents the image by ∂ of the class of $\mathrm{PV}[\tilde{\omega}]$ in $H_{2n-p}\Gamma(X, '\mathscr{D}_{\cdot X/Y\infty})$, so that the left square commutes.

Finally, a cohomology class $\omega' \in H^p\Gamma(X;{}^sQ^{\cdot})$ can be represented by a section $\tilde{\omega} \in \Gamma(X;\mathscr{E}_X^p(*Y))$ such that $d\tilde{\omega} \in \Gamma(X;\mathscr{E}_X^{p+1})$. Then $\mathrm{V}\circ\delta[\omega']$ is represented by $\mathrm{V}[d\tilde{\omega}]$, and $i\circ\mathrm{Res}[\omega']$ is represented by $\mathrm{Res}[\tilde{\omega}]$; by 5.3, $\mathrm{V}[d\tilde{\omega}] - \mathrm{Res}[\tilde{\omega}] = \mathrm{PV}[d\tilde{\omega}] - \mathrm{Res}[\tilde{\omega}] = b.\mathrm{PV}[\tilde{\omega}]$, a boundary, and the right square also commutes.

5.7. *Compatibility of diagram* (3) *and* (4). By this we mean that there is a canonical homomorphism from (3) to (4). In fact, the homomorphism between top sequences in (3) and (4) has been constructed in no. 2, and the homomorphism between bottom sequences in no. 4. Only commutativity of the "interior" squares is left to prove. As explained before, it suffices to consider the diagram associated to the complexes

$$0 \to \mathscr{E}_X^{\cdot} \to \mathscr{E}_X^{\cdot}(*Y) \to {}_sQ^{\cdot} \to 0. \tag{16}$$

(a) *Commutativity of*

$$\begin{array}{ccc} H^p\Gamma(X,\mathscr{E}_X^{\cdot}) & \xrightarrow{I} & H^p(X;\mathbb{C}) \\ \downarrow{\scriptstyle \mathrm{V}} & & \downarrow{\scriptstyle \cap} \\ H_{2n-p}\Gamma(X,'\mathscr{D}_{\cdot X}) & \xrightarrow[\tau]{} & H_{2n-p}(X;\mathbb{C}). \end{array} \tag{17}$$

We recall (cf. 4(11)) that τ is composition of the dual $e^*: H^{2n-p}\Gamma_c(X,\mathscr{E}_X^{\cdot}) \to [H_c^{2n-p}(X;\mathbb{C})]^* \simeq H_{2n-p}(X;\mathbb{C})$ of the edge homomorphism e, with the canonical map $\tilde{\nu}: H_{2n-p}\Gamma(X;'\mathscr{D}_{\cdot X}) \to [H^{2n-p}\Gamma_c(X,\mathscr{E}_X^{\cdot})]^*$ in 4(9). Clearly, $\tilde{\nu}\circ\mathrm{V} = {}_s\cap$, the cap homomorphism defined in diagram 3(11), whose commutativity assures that of (17).

(b) *Commutativity of*

$$\begin{array}{ccc} H^p\Gamma(X;\mathscr{E}_X^\cdot(*Y)) & \xrightarrow{\ I\ } & H^p(U;\mathbb{C}) \\ {\scriptstyle \mathrm{PV}}\downarrow & & \downarrow{\scriptstyle \cap} \\ H_{2n-p}\Gamma(X;'\mathscr{D}_{\cdot X/Y^\infty}) & \xrightarrow[\tau]{} & H_{2n-p}(U;\mathbb{C}). \end{array} \qquad (18)$$

Replace τ by $e^*\circ\tilde{\nu}$, as in 4(11), so that $\tau\circ\mathrm{PV}=e^*\circ\tilde{\nu}\circ\mathrm{PV}=e^*\circ{}_s\cap$, as defined in 3.8. Then (18) commutes, since the diagram 3.8(13) does.

(c) *Commutativity of*

$$\begin{array}{ccc} H^p\Gamma(X,Q_s^\cdot) & \xrightarrow{\ \mu\ } & H_Y^{p+1}(X;\mathbb{C}) \\ {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle \cap'} \\ H_{2n-p-1}\Gamma(X,'\mathscr{D}_Y) & \xrightarrow[\tau]{} & H_{2n-p-1}(Y;\mathbb{C}). \end{array} \qquad (19)$$

Considering the definition of τ and μ (cf. 4(13) and 2.2), (19) can be decomposed as follows:

$$\begin{array}{ccccc} H^p\Gamma(X,Q_s^\cdot) & \xrightarrow{\ \eta\ } & H_Y^{p+1}(X;\mathscr{E}_X^\cdot) & \xrightarrow{\ I\ } & H_Y^{p+1}(X;\mathbb{C}) \\ {\scriptstyle \mathrm{Res}}\downarrow & \searrow{\scriptstyle \tilde{R}} & \downarrow{\scriptstyle {}_s\cap'} & & \downarrow{\scriptstyle \cap'} \\ H_{2n-p-1}\Gamma(X,'\mathscr{D}_Y) & \xrightarrow[\tilde{\nu}]{} & [H^{2n-p-1}\Gamma_c(Y,\mathscr{E}_X^\cdot)]^* & \xrightarrow[e^*]{} & [H_c^{2n-p-1}(Y;\mathbb{C})]^*, \end{array} \qquad (20)$$

where $\tilde{R}=\tilde{\nu}\circ\mathrm{Res}$. The right square commutes here by 3.7, so that only the equality ${}_s\cap'\circ\eta=\tilde{R}$ is left to prove. We observe, to this purpose, that compatibility between cup product and conexion homomorphisms δ gives a commutative diagram

$$\begin{array}{ccc} H_Y^p(X;{}_sQ^\cdot)\otimes H_c^q(Y;\mathscr{E}_X^\cdot) & \xrightarrow{\ \delta\otimes 1\ } & H_Y^{p+1}(X,\mathscr{E}_X^\cdot)\otimes H_c^q(Y;\mathscr{E}_X^\cdot) \\ {\scriptstyle U'}\downarrow & & \downarrow{\scriptstyle U'} \\ H_{c\mid Y}^{p+q}(X;{}_sQ^\cdot\otimes\mathscr{E}_X^\cdot) & \xrightarrow[\delta]{} & H_{c\mid Y}^{p+q+1}(X;\mathscr{E}_X^\cdot\otimes\mathscr{E}_X^\cdot), \end{array} \qquad (21)$$

where the top map is deduced from the exact sequence (14), and the bottom one from the top sequence in the following diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{E}_X^\cdot\otimes\mathscr{E}_X^\cdot & \to & \mathscr{E}_X^\cdot(*Y)\otimes\mathscr{E}_X^\cdot & \to & {}_sQ^\cdot\otimes\mathscr{E}_X^\cdot & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{E}_X^\cdot & \longrightarrow & \mathscr{E}_X^\cdot(*Y) & \longrightarrow & {}_sQ^\cdot & \longrightarrow & 0. \end{array} \qquad (22)$$

The vertical maps of this diagram are exterior products, and induce a homomorphism from the bottom line in (21) to

$$H_{c\mid Y}^{p+q}(X;{}_sQ^\cdot)\xrightarrow{\ \delta\ }H_{c\mid Y}^{p+q+1}(X;\mathscr{E}_X^\cdot).$$

By inclusion of supports, we replace this homomorphism by the similar one

$$H^{p+q}\Gamma_c(X;{}_sQ^\cdot)\xrightarrow{\ \delta\ }H^{p+q+1}\Gamma_c(X,\mathscr{E}_X^\cdot), \qquad (23)$$

identifying hypercohomology and cohomology of global sections.

Suppose now that $p+q+1=2n$; the construction just described gives a commutative diagram

$$\begin{array}{ccc} H^p_Y(X;{}_sQ^\cdot)\otimes H^q_c(Y;\mathscr{E}^\cdot_X) & \xrightarrow{\delta\otimes 1} & H^{p+1}_Y(X;\mathscr{E}^\cdot_X)\otimes H^q_c(Y;\mathscr{E}^\cdot_X) \\ \downarrow & & \downarrow \\ H^{2n-1}\Gamma_c(X;{}_sQ^\cdot) & \xrightarrow[\delta]{} & H^{2n}\Gamma_c(X;\mathscr{E}^\cdot_X)\,. \end{array} \qquad (24)$$

Composition of the right arrow in this diagram with the integration homomorphism $I: H^{2n}\Gamma_c(X,\mathscr{E}^\cdot_X)\to\mathbb{C}$ (cf. 3(6)) induces the map ${}_s\frown'$ in (20).

We define now a homomorphism $\mathrm{R}:\Gamma_c(X;{}_sQ^{2n-1})\to\mathbb{C}$ as follows: if a form $\alpha\in\Gamma_c(X;\mathscr{E}^{2n-1}_X(*\,Y))$ is representable by ω/φ^q on the open set W, where φ is an equation of Y on W, then

$$\mathrm{R}(\alpha)=\lim_{\delta\to 0} I[W(=\delta)]\,(\omega/\varphi^q). \qquad (25)$$

By 7.1 and 7.3, this local definition can be patched into a global map $R:\Gamma_c(X,\mathscr{E}^\cdot_X(*\,Y))\to\mathbb{C}$, which is zero on forms bounded on Y (cf. 7.4) and on boundaries in $\Gamma_c(X;\mathscr{E}^\cdot(*\,Y))$; in fact, $I[W(=\delta)]\,(\omega/\varphi^q)=0$ if ω/φ^q is a boundary, by Stokes' theorem. It follows that homomorphisms:

$$\Gamma_c(X;{}_sQ^{2n-1})\simeq\Gamma_c(X;\mathscr{E}^{2n-1}_X(*\,Y))/\Gamma_c(X;\mathscr{E}^{2n-1}_X)\to\mathbb{C}$$

and

$$\mathrm{R}: H^{2n-1}\Gamma_c(X;{}_sQ^\cdot)\to\mathbb{C} \qquad (26)$$

can be constructed, and one checks easily that R, composed with the left arrow in (24), induces the homomorphism $\tilde{\mathrm{R}}=\tilde{\nu}\circ\mathrm{Res}$ in (20). Finally, we want to see that diagram

$$\begin{array}{ccc} H^{2n-1}\Gamma_c(X;{}_sQ^\cdot) & \xrightarrow{\delta} & H^{2n}\Gamma_c(X;\mathscr{E}^\cdot_X) \\ & {\scriptstyle \mathrm{R}}\searrow \quad \swarrow{\scriptstyle I} & \\ & \mathbb{C} & \end{array} \qquad (27)$$

anticommutes, which can be checked locally. In fact, suppose that the form $\omega/\varphi^q\in\Gamma_c(W;\mathscr{E}^{2n-1}_X(*\,Y))$ represents a cycle in $\Gamma_c(W;{}_sQ^{2n-1})$; this means that $d(\omega/\varphi^q)\in\Gamma_c(W;\mathscr{E}^{2n}_X)$. By Stokes' theorem and (6),

$$-I[W(=\delta)]\,(\omega/\varphi^q)=I[W(>\delta)]\,(d(\omega/\varphi^q))\,;$$

letting $\delta\to 0$ we get $-\mathrm{R}(\omega/\varphi^q)=I(d(\omega/\varphi^q))$, so that (27) anticommutes.

As a consequence, the following diagram anticommutes

$$\begin{array}{ccc} H^p_Y(X;{}_sQ^\cdot) & \xrightarrow{\delta} & H^{p+1}_Y(X;\mathscr{E}^\cdot_X) \\ & {\scriptstyle \tilde{\mathrm{R}}}\searrow \quad \swarrow{\scriptstyle {}_s\frown'} & \\ & [H^{2n-p-1}\Gamma_c(Y;\mathscr{E}^\cdot_X)]^*\,, & \end{array}$$

so that $\tilde{\mathrm{R}}=-{}_s\frown'\circ\delta={}_s\frown'\circ\eta$, and commutativity of (19) is proved.

5.8. Corollary. *In the conditions of Theorem 5.1, suppose that U has only simple points. Then* PV *splits canonically. If X has only simple points,* V *is an isomorphism and* Res *(and* PV*) splits canonically. If X and Y have only simple points, then* V, PV *and* Res *are isomorphisms, and diagram* (3) *is canonically isomorphic to diagram* (4).

Proof. Replace $\mathscr{E}_X^{\cdot}(*Y)$ by $\Omega_X^{\cdot}(*Y)$ in diagram (18); if U has only simple points, I is an isomorphism, by Grothendieck's theorem (cf. [5]), and $\cap: H^p(U;\mathbb{C})\to H_{2n-p}(U;\mathbb{C})$ is an isomorphism by Poincaré duality, so that $\tau\circ\mathrm{PV}$ is an isomorphism, which gives the splitting of PV.

If X is a manifold, then I, $\cap$ and τ in (17) are all isomorphisms, so that V also is (a direct proof is easy to give). Moreover, in diagram (19) μ and $\cap'$ are isomorphisms, the first by 2.4 and the second by Poincaré duality; we deduce that τ is a right inverse to Res, what gives the splitting.

If X and Y are manifolds, then Res (and V) is an isomorphism, since in (19) τ will also be an isomorphism (cf. no. 4). We deduce that PV is an isomorphism, as desired.

5.9. *Relation with the classical notion of residue.* Suppose that X is a complex Stein manifold of dimension n, and that Y is a 1-codimensional submanifold. If we compose the maps

$$H^p(X-Y);\mathbb{C})\to H_Y^{p+1}(X;\mathbb{C}) \quad \text{and} \quad H_Y^{p+1}(X;\mathbb{C})\to H_{2n-p-1}(Y;\mathbb{C})$$

of diagram (4) with the Poincaré duality isomorphism $H_{2n-p-1}(Y;\mathbb{C})\to H^{p-1}(Y;\mathbb{C})$, we obtain the homomorphism

$$\widetilde{\mathrm{Res}}: H^p(X-Y;\mathbb{C})\to H^{p-1}(Y;\mathbb{C}), \tag{28}$$

which has been realized by J. Leray (cf. [6] and [8]) as follows: Represent a class $[\tilde{\omega}]\in H^p(X-Y;\mathbb{C})$ by a closed form $\tilde{\omega}\in\Gamma(X,\mathscr{E}_X^p(*Y))$ that has only first order poles on Y ([6], Theorem 1). Locally, $\tilde{\omega}$ has the expression

$$\tilde{\omega}=\psi\wedge\frac{ds}{s}+\theta\,, \tag{29}$$

where s is a local equation for Y ([6], no. 2) and ψ and θ are holomorphic forms on X, locally defined.

The restrictions to Y of these local forms ψ define a closed global form $\mathrm{res}[\tilde{\omega}]\in\Gamma(Y;\mathscr{E}_Y^{p-1})$, whose cohomology class is $\widetilde{\mathrm{Res}}[\tilde{\omega}]$, as denoted in (28).

Using (29) one proves immediately, that

$$\mathrm{Res}[\tilde{\omega}]=2\pi i I[Y]\wedge\mathrm{res}[\tilde{\omega}]\,, \tag{30}$$

where the left current has been defined in 5.2, and the right one is the current $\alpha\to I[Y](\mathrm{res}[\tilde{\omega}]\wedge\alpha)$. $I[Y]$ denotes here integration on the canonically oriented manifold Y.

Equality (30) relates the residue current $\mathrm{Res}[\tilde{\omega}]$, as defined in this paper, and the residue form $\mathrm{res}[\tilde{\omega}]$ of Leray-Norguett theory. In fact, (30) is also true on spaces, for forms that have local expressions like (29), where s is any holomorphic function, not necessarily a coordinate. The proof of this result will be given in another place.

5.10. *Remark.* Diagram (19) answers a question posed by P. Dolbeault in [14], 3.7, namely that of the relationship between the local cohomology class and the homology class defined by a meromorphic form on X.

6. Residue and Principal Value in the Normal Crossing's Case

The purpose of this section is to prove the local existence of the residue and the principal value of meromorphic forms, defined on manifolds, whose polar set has only normal crossings. The general problem will be reduced to this case in no. 7.

The following notations and conventions will be used throughout this section, often without reference.

6.1. *Notations.* As before, $\boldsymbol{Z}$ and $\boldsymbol{Z}_+$ denote the sets of integers and natural numbers respectively, and $\boldsymbol{Z}^n$ and $\boldsymbol{Z}^n_+$ their n-Cartesian products. Let $\alpha=(\alpha_1, \ldots, \alpha_n)\in \boldsymbol{Z}^n$, $z=(z_1, \ldots, z_n)\in \mathbb{C}^n$, and choose an integer j, $1\leqq j\leqq n$. Then $|z_j|$ denotes the absolute value of $z_j=x_j+iy_j\in\mathbb{C}$,

$$z^\alpha=\Pi(z_s^{\alpha_s}:s=1,\ldots,n)\,,\quad |z^\alpha|=\Pi(|z_s^{\alpha_s}|:s=1,\ldots,n)$$

and

$$z(j)=(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_n)\in\mathbb{C}^{n-1},\ \alpha(j)=(\alpha_1,\ldots,\alpha_{j-1},\alpha_{j+1},\ldots,\alpha_n)\in Z^{n-1};$$

in this case, we abbreviate $Z(j)^{\alpha(j)}=\Pi(Z_s^{\alpha_s}:1\leqq s\leqq n, s\neq j)$ by $Z(j)^\alpha$, so that $z^\alpha=z_j^{\alpha_j}z(j)^\alpha$.

We also denote $\|z\|=\max(|z_j|:j=1,\ldots,n)$, and

$$B=(z\in\mathbb{C}^n:\|z\|<1)\,,\quad B(j)=(z(j)\in\mathbb{C}^{n-1}:\|z(j)\|<1)\,.$$

For a fixed $\delta>0$, $\alpha\in Z^n_+$ and $j=1,\ldots,n$, we use the notations

$$\begin{aligned}&B_\delta^\alpha=(z\in B:\delta<|z^\alpha|<1)\,,\quad S_\delta^\alpha=(z\in B:|z^\alpha|=\delta)\\ &{}^jB_\delta^\alpha=(z\in\mathbb{C}^n:\|z(j)\|<1,|z_j|=1,\delta<|z(j)^\alpha|<1)\,,\end{aligned}\tag{1}$$

and

$$B_\delta^\alpha(j)=(\|z(j)\|<1,\delta<|z(j)^\alpha|<1)\,,\quad S_\delta^\alpha(j)=(\|z(j)\|<1,|z(j)^\alpha|=\delta)\,.$$

The domain B_δ^α is always considered with its canonical complex orientation, and S_δ^α is given the opposite orientation to the one induced by B_δ^α. This is equivalent to define the semianalytic chains (cf. [2], 2) $B_\delta^\alpha=[B_\delta^\alpha, e(>\delta)]\in\mathscr{S}_{2n}(B,\mathbb{C})$ and $S_\delta^\alpha=[S_\delta^\alpha, e(=\delta)]\in\mathscr{S}_{2n-1}(B;\mathbb{C})$, where $e(>\delta)\in H_{2n}(B_\delta^\alpha;\mathbb{C})$ is the canonical fundamental class of B_δ^α, and $e(=\delta)=-\partial e(>\delta)\in H_{2n-1}(S_\delta^\alpha;\mathbb{C})$, where ∂ is the boundary in the exact sequence of Borel-Moore homology

$$H_{2n}(B_\delta^\alpha;\mathbb{C})\xrightarrow{\ \partial\ }H_{2n-1}(S_\delta^\alpha;\mathbb{C})\longrightarrow H_{2n-1}(\bar{B}_\delta^\alpha;\mathbb{C})\,.$$

With these conventions, we have the formula

$$\partial B_\delta^\alpha = -S_\delta^\alpha + \Sigma({}^sB_\delta^\alpha : s=1, \ldots, n), \tag{2}$$

in the space $\mathscr{S}.(\bar{B};\mathbb{C})$ of semianalytic chains in $\bar{B}=(\|z\| \leqq 1)$, where the chains ${}^sB_\delta^\alpha$ are given appropriate orientations.

We also use the notations $|\alpha| = \alpha_1 + \cdots + \alpha_n$, for $\alpha \in Z_+^n$, and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_1}}, \qquad \bar{D}^\alpha f = \frac{\partial^{|\alpha|} f}{\partial \bar{z}_1^{\alpha_1} \ldots \partial \bar{z}_n^{\alpha_n}}$$
$$D^{\alpha_j} f = \frac{\partial^{\alpha_j} f}{\partial z_j^{\alpha_j}}, \qquad D^{-1} f = 0, \tag{3}$$

for a smooth function $f \in \mathscr{E}^0(\mathbb{C}^n)$, $f = f(z, \bar{z})$. Ocasionally, we write $f = f(z(j))$ to emphasize that f does not depend on the variables z_j and $\bar{z}_j$.

Finally, we write

$$dz \wedge d\bar{z} = \Pi(dz_s \wedge d\bar{z}_s : s=1, \ldots, n),$$
$$dz(j) \wedge d\bar{z}(j) = \Pi(dz_s \wedge d\bar{z}_s : s=1, \ldots. n, s \neq j), \tag{4}$$
$$dz(i,j) \wedge d\bar{z}(i,j) = \Pi(dz_s \wedge d\bar{z}_s : s=1, \ldots, n, s \neq i, s \neq j).$$

6.2. Lemma. *Let $k = k(z, \bar{z})$ be a smooth complex valued function defined in $\mathbb{C}^n$, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in Z_+^n$. There exists a decomposition*

$$k(z, \bar{z}) = \sum_{j=1}^{n} \left(\Sigma z_j^r . \bar{z}_j^s g_{r,s}^j(z(j), \bar{z}(j)) : r+s < \alpha_j\right) + K(z, \bar{z}),$$
$$K(z, \bar{z}) = \Sigma(z^\beta . \bar{z}^\gamma . K_{\beta,\gamma}(z, \bar{z}) : \beta + \gamma = \alpha),$$

such that:

a) *$g_{r,s}^j$ and $K_{\beta,\gamma}$ are smooth functions,*

b) $g_{r,s}^1(z(j), \bar{z}(j)) = \frac{1}{r!s!} \cdot \left.\frac{\partial^{r+s} k}{\partial z_1^r \partial \bar{z}_1^s}\right|_{z_1=0}$, *$r+s < \alpha_1$, and*

c) *K depends continuously on k, with respect to the seminorms* $\max(|D^\nu \bar{D}^\mu f|$: $|\nu| + |\mu| \leqq l)$, *for all compacts C in $\mathbb{C}^n$ and $l \in Z_+$.*

Proof. The case $\alpha = 0$ is trivial, so that we can always suppose that $\alpha_j \neq 0$ for all $j \leqq k$, for some $k > 0$, and that $\alpha_j = 0$ if $j > k$. By Taylor's formula, we have that

$$k(z, \bar{z}) = \Sigma(z_1^r \bar{z}_1^s g_{r,s}^1(z(1), \bar{z}(1)) : r+s < \alpha_1)$$
$$+ \Sigma(z_1^r \bar{z}_1^s K_{r,s}^1(z, \bar{z}) : r+s = \alpha_1), \tag{5}$$

where $g_{r,s}^1$ and $K_{r,s}^1$ are smooth functions and $K_{r,s}^1(r+s=\alpha_1)$ depend continuously on k, in the sense described in (c); this follows from the integral expression of the rest in Taylor's formula.

We now apply Taylor's formula to each function $K_{r,s}^1(z, \bar{z})$, in a similar way, with respect of z_2 and $\bar{z}_2$, and up to the degree α_2. The second line in (5) can then be written as

$$\Sigma(z_2^r \bar{z}_2^s g_{r,s}^2(z(2), \bar{z}(2)) : r+s < \alpha_2)$$
$$+ \Sigma(z_1^r \bar{z}_1^s z_2^u \bar{z}_2^v K_{r,s,u,v}^2(z, \bar{z}) : r+s = \alpha_1, u+v = \alpha_2),$$

where the functions $K^2_{r,s,u,v}$ depend continuously on $K^1_{r,s}$, hence on k. It is clear that iteration of the method gives the wanted decomposition, and that a) and c) are satisfied. As for b), it is obvious from (5).

6.3. Lemma. *Let $g \in \mathscr{E}^0(\mathbb{C}^n)$ be independent of $z_j = x_j + iy_j$, and $\alpha \in \mathbf{Z}^n_+$. Then*

(a) $\int\limits_{B^\alpha_\delta} z^r_j \bar{z}^s_j z^{-\alpha} g\, dz \wedge d\bar{z} = 0$

(b) $\int\limits_{S^\alpha_\delta} z^r_j \bar{z}^s_j z^{-\alpha} g\, d\bar{z} \wedge dz(j) \wedge d\bar{z}(j) = 0$

for all $r, s \in \mathbf{Z}_+$ such that $r+s<\alpha_j$, and each $\delta>0$, and

(c) $\int\limits_{S^\alpha_\delta} z^r_j \bar{z}^s_j z^{-\alpha} g\, dz_j \wedge dz(j) \wedge d\bar{z}(j) = 0$

for all $r, s \in \mathbf{Z}_+$ such that $r+s<\alpha_j$, $s>0$, and each $\delta>0$.

Proof. For each $z(j) \in B^\alpha_\delta(j)$, define

$$B(z(j), \delta) = (z_j \in \mathbb{C};\ \delta/|z(j)^\alpha| < |z_j^{\alpha_j}| < 1).$$

The partial integrals

$$I(z(j), \delta) = \int\limits_{B(z(j),\delta)} z_j^{r-\alpha_j} \bar{z}^s_j dz_j \wedge d\bar{z}_j = \int\limits_{\delta/|z(j)^\alpha|}^{1} \varrho^{r-\alpha_j+s+1} d\varrho \int\limits_0^{2\pi} e^{i\theta(r-\alpha_j-s)} d\theta$$

are zero for all $z(j) \in B^\alpha_\delta(j)$, because $r-s \leqq r+s < \alpha_j$. Then the integral in (a) is zero, since it is equal to

$$\int\limits_{B^\alpha_\delta(j)} I(z(j), \delta) . z(j)^{-\alpha} g\, dz(j) \wedge d\bar{z}(j).$$

To compute (c), one integrates over $B^\alpha_\delta(j)$ the differential form $z(j)^{-\alpha}.g\, dz(j) \wedge d\bar{z}(j)$ times the integrals

$$\int (z_j^{r-\alpha_j} \bar{z}^s_j dz_j : |z_j^{\alpha_j}| = \delta/|z(j)^\alpha|), \quad z(j) \in B^\alpha_\delta(j),$$

which are all zero if $s>0$, since in such case $r-\alpha_j-s+1 < r-\alpha_j+s+1 \leqq 0$. We deduce that the integral in (c) is zero, and a similar argument shows that the integral in (b) is also zero, for all $r+s<\alpha_j$.

6.4. Lemma.

$$\lim_{\delta\to 0} \int\limits_{S^\alpha_\delta} b\, dz_j \wedge dz(j) \wedge d\bar{z}(j) = 0$$

and (6)

$$\lim_{\delta\to 0} \int\limits_{S^\alpha_\delta} b\, d\bar{z}_j \wedge dz(j) \wedge d\bar{z}(j) = 0$$

for any integrable bounded function b defined in $\mathbb{C}^n$, and any $\alpha \in \mathbf{Z}^n_+$.

Proof. If $\alpha_j = 0$, the integrals themselves are zero, considering the definition of S^α_δ. Suppose then $\alpha_j \neq 0$, and rearrange α so that $j=1$ and $\alpha_s = 0$ if $s>k$, for some $k \geqq 1$, and $\alpha_s \neq 0$ if $s \leqq k$. By means of the following parametrization of S^α_δ:

$$(\theta_1, z(1)) \to \left(\left(\delta / \prod_{s=2}^{k} \varrho_s^{\alpha_s}\right)^{1/\alpha_1} e^{i\theta_1}, z(1)\right)$$

where $z_s = \varrho_s e^{i\theta_s}$ $(s=1, \ldots, n)$, one verifies that the integrals in (6) are bounded, up to a constant, by

$$\delta^{1/\alpha_1} \cdot \int_{E_\delta} \prod_{s=2}^{k} \varrho_s^{1-\alpha_s/\alpha_1} d\varrho_s, \quad E_\delta = \left(\delta < \prod_{s=2}^{k} \varrho_s < 1\right).$$

Induction on k proves that the limit of this expression, as $\delta \to 0$, is zero.

6.5. Proposition. *For any $\alpha \in \mathbf{Z}_+^n$ and $k \in \mathscr{E}^0(\mathbb{C}^n)$, the limits*

$$\lim_{\delta \to 0} \int_{S_\delta^\alpha} z^{-\alpha} k \, dz_j \wedge dz(j) \wedge d\bar{z}(j) \quad (j=1, \ldots, n) \tag{7}$$

and

$$\lim_{\delta \to 0} \int_{B_\delta^\alpha} z^{-\alpha} . k \, dz \wedge d\bar{z} \tag{8}$$

exist, and are continuous on $\mathscr{E}^0(\mathbb{C}^n)$, if this space is considered with the seminorms $\max_{\bar{B}}(|D^\nu k| : |\nu| \leqq l)$, $l \in \mathbf{Z}_+$.

Moreover,

$$\lim_{\delta \to 0} \int_{S_\delta^\alpha} z^{-\alpha} k \, d\bar{z}_j \wedge dz(j) \wedge d\bar{z}(j) = 0 \tag{9}$$

and

$$\begin{aligned} &\lim_{\delta \to 0} \int_{S_\delta^\alpha} z^{-\alpha} . k \, dz_j \wedge dz(j) \wedge d\bar{z}(j) \\ &= \lim_{\delta \to 0} \frac{2\pi i}{(\alpha_j - 1)!} \int_{B_\delta^\alpha(j)} z(j)^{-\alpha} (D^{\alpha_j - 1} k)_{z_j = 0} \, dz(j) \wedge d\bar{z}(j) . \end{aligned} \tag{10}$$

Proof. We can always suppose that $\alpha_j = 0$ for all $j > k \geqq 0$, and that $\alpha_j \neq 0$ if $j \leqq k$, since the case $\alpha = 0$ is trivial. Consider the decomposition of k given in Lemma 6.2; by 6.3(a), terms like $z_j^r \bar{z}_j^s g(z(j))$ times $z^{-\alpha}$ give no contribution when integrated over B_δ^α, so that the limit in (8) reduces to

$$\lim_{\delta \to 0} \int_{B_\delta^\alpha} z^{-\alpha} K \, dz \wedge d\bar{z} = \int_B z^{-\alpha} K \, dz \wedge d\bar{z} , \tag{11}$$

since $z^{-\alpha} . \mathrm{K}$ is bounded on B.

As for the limit in (7), only the case $j \leqq k$ needs to be analyzed, since the integral itself is zero when $j > k$. Then, if one considers the decomposition of k in 6.2, terms like $z_i^r \bar{z}_i^s g(z(i), \bar{z}(i))$ with $i \neq j$ do not contribute to the integral over S_δ^α. In fact,

$$\int_{S_\delta^\alpha} z^{-\alpha} z_i^r \bar{z}_i^s g \, dz_j \wedge dz(j) \wedge d\bar{z}(j) = \int_{\delta < |z_i^{\alpha_i}| < 1} z_i^{r-\alpha_i} \bar{z}_i^s J_\delta(|z_i|) \, dz_i \wedge d\bar{z}_i ,$$

where $J_\delta(|z_i|)$ is the integral of $z(i)^{-\alpha} g(z(i), \bar{z}(i)) . dz_j \wedge dz(i,j) \wedge d\bar{z}(i,j)$ over the domain $(\|z(i)\| < 1, |z(i)^\alpha| = \delta / |z_i^{\alpha_i}|)$ and depends only on the absolute value $\varrho = |z_i|$ of $z_i = \varrho e^{i\theta}$. Consequently, the last integral is equal to

$$\int_{\delta^{1/\alpha_i}}^{1} J_\delta(\varrho) \, \varrho^{r - \alpha_i + s + 1} d\varrho \int_0^{2\pi} e^{i\theta(r - \alpha_i - s)} d\theta = 0 ,$$

since $r - s \leqq r + s < \alpha_i$.

In the case $i=j$, the contribution to the integral over S^α_δ of terms like $z_j^r \bar{z}_j^s g_{r,s}(z(j), \bar{z}(j))$, with $s>0$, is zero (6.3(c)); in the case $s=0$, it is equal to

$$\int_{B^\alpha_\delta(j)} z(j)^{-\alpha} g_{r,0}\, dz(j) \wedge d\bar{z}(j) \int_{|z_j^{\alpha_j}|=\delta/|z(j)^\alpha|} z_j^{r-\alpha_j} dz_j\,,$$

which is zero if $r-\alpha_j \neq -1$, and is equal to

$$2\pi i \int_{B^\alpha_\delta(j)} z(j)^{-\alpha} g_{\alpha_j-1,0}\, dz(j) \wedge d\bar{z}(j)\,, \tag{12}$$

if $r-\alpha_j=-1$.

Then, the integral in (7) reduces to (12), plus the integral over S^α_δ due to the term K in the decomposition of k. The limit of this last integral is zero by 6.4, so that the limit in (7) is equal to

$$\lim_{\delta\to 0} 2\pi i \int_{B^\alpha_\delta(j)} z(j)^{-\alpha} g_{\alpha_j-1,0}(z(j), \bar{z}(j))\, dz(j) \wedge d\bar{z}(j)\,, \tag{13}$$

which exists by the first part of this proof.

Equality (10) is trivial if $\alpha_j=0$, in which case both integrals are zero, for fixed values of δ. If $\alpha_j \neq 0$, permuting coordinates we reduce the problem to the case $j=1$, $\alpha_1 \neq 0$, and $g_{\alpha_1-1,0}(z(1), \bar{z}(1)) = (\alpha_1-1)!^{-1}(D^{\alpha_1-1}k)_{z_1=0}$, by 6.2(b).

The proof of (9) is similar to that of the existence of (7). Finally, the continuity of the limit in (8) with respect to the given seminorms is clear by (11), since K depends continuously of k (cf. lemma 6.2), and the continuity of the limit in (7) follows from (10).

6.6. Proposition. *Let f be a never vanishing holomorphic function on $\bar{B} \subset \mathbb{C}^n$, and choose $\alpha \in Z^n_+$. Denote*

$$S^\alpha_\delta(f) = (z \in B : |z^\alpha f(z)| = \delta)\,,$$
$$B^\alpha_\delta(f) = (z \in B : |z^\alpha f(z)| > \delta)\,.$$

Consider $B^\alpha_\delta(f)$ with its canonical complex orientation and $S^\alpha_\delta(f)$ with the orientation such that $\partial B^\alpha_\delta(f) = -S^\alpha_\delta(f)$, as in 6.1(2).

There exists an open neighborhood $\tilde{B} \subset B$ of the origin $0 \in \mathbb{C}^n$ such that, if $\tilde{S}^\alpha_\delta = S^\alpha_\delta(f) \cap \tilde{B}$ and $\tilde{B}^\alpha_\delta = B^\alpha_\delta(f) \cap \tilde{B}$, then

$$\lim_{\delta\to 0} \int_{\tilde{S}^\alpha_\delta} z^{-\alpha} k\, dz_1 \wedge dz(1) \wedge d\bar{z}(1) = \lim_{\delta\to 0} \int_{S^\alpha_\delta} z^{-\alpha} k\, dz_1 \wedge dz(1) \wedge d\bar{z}(1)$$

and

$$\lim_{\delta\to 0} \int_{\tilde{S}^\alpha_\delta} z^{-\alpha} k\, d\bar{z}_1 \wedge dz(1) \wedge d\bar{z}(1) = 0 \tag{15}$$

for all $k \in \mathscr{E}^0(\bar{B})$, and

$$\lim_{\delta\to 0} \int_{\tilde{B}^\alpha_\delta} z^{-\alpha} b\, dz \wedge d\bar{z} = \lim_{\delta\to 0} \int_{B^\alpha_\delta} z^{-\alpha} b\, dz \wedge d\bar{z} \tag{16}$$

for all $b \in \Gamma_c(\tilde{B}; \mathscr{E}^0)$.

Proof of (14). If $\alpha_1=0$, we can choose $\tilde{B}=B$ and both integrals in (14) will be zero, for values of δ. We suppose, then, that $\alpha_1 \neq 0$, and can always assume that $|f|>1$ on $\bar{B}$.

Choose an holomorphic function h such that $h^{\alpha_1}=f$ on $\bar{B}$, and consider the mapping $\mathbb{C}^n(z)\to\mathbb{C}^n(\zeta)$ defined by the functions $\zeta_1: z_1 h(z)$ and $\zeta_j=z_j$ $(j=2, \ldots, n)$. This mapping defines an isomorphism on some neighborhood W of $0\in\mathbb{C}^n$, and after a suitable change of z-coordinates we can suppose that $B\subset W$; denote

$$\tilde{B}=(z\in B: |z_1 h(z)|<1), \tag{17}$$

so that our mapping defines an isomorphism λ from $\tilde{B}$ to the unit ball $B(\zeta)$ in $\mathbb{C}^n(\zeta)$. According to notations in 6.1 and above,

$$\lambda(\tilde{B}^\alpha_\delta)=B^\alpha_\delta(\zeta)=(\zeta\in\mathbb{C}^n: \|\zeta\|<1 \quad\text{and}\quad |\zeta^\alpha|>\delta),$$
$$\lambda(\tilde{S}^\alpha_\delta)=S^\alpha_\delta(\zeta)=(\zeta\in\mathbb{C}^n: \|\zeta\|<1 \quad\text{and}\quad |\zeta^\alpha|=\delta).$$

The isomorphism λ has an inverse $\mu: B(\zeta)\to\tilde{B}$ given by functions $z_1=\zeta_1 g(\zeta)$, $z_j=\zeta_j$ $(j=2, \ldots, n)$, where g is holomorphic never vanishes on $B(\zeta)$.

Consider now any function $k\in\mathscr{E}^0(\bar{B})$. By Taylor's formula

$$k(z,\bar{z})=\sum_{r+s<\alpha_1} z_1^r\bar{z}_1^s g_{r,s}(z(1),\bar{z}(1))+\sum_{r+s=\alpha_1} z_1^r\bar{z}_1^s G_{r,s}(z,\bar{z}),$$

where all functions $g_{r,s}$ and $G_{r,s}$ are smooth. Applying the change of coordinates $z=\mu(\zeta)$, we deduce

$$\int_{\tilde{S}^\alpha_\delta} z^{-\alpha} z_1^r\bar{z}_1^s G_{r,s}(z,\bar{z})\,dz_1\wedge dz(1)\wedge d\bar{z}(1)$$
$$=\int_{S^\alpha_\delta(\zeta)} \zeta^{-\alpha} H_{r,s}(\zeta,\bar{\zeta})\,d\zeta_1\wedge d\zeta(1)\wedge d\bar{\zeta}(1), (r+s=\alpha_1)$$

where $H_{r,s}(\zeta,\bar{\zeta})=\zeta_1^r\bar{\zeta}_1^s g^{r-\alpha_1}\bar{g}^s G_{r,s}(\mu,\bar{\mu})\cdot\left(g+\zeta_1\dfrac{\partial g}{\partial\zeta_1}\right)$. Now $r+s=\alpha_1$ implies that $(D^{\alpha_1-1}H_{r,s}(\zeta,\bar{\zeta}))_{\zeta_1=0}=0$, so that according to 6.5(10) the limit of the last expression, as $\delta\to 0$, is zero.

By the same argument one proves that the contribution to the left limit in (14) of terms like $z_1^r\bar{z}_1^s g_{r,s}(z(1),\bar{z}(1))$, with $s>0$, is zero. Consequently, the left limit in (14) reduces to

$$\lim_{\delta\to 0}\int_{\tilde{S}^\alpha_\delta}\sum_{r<\alpha_1} z^{-\alpha} z_1^r g_{r,0}(z(1),\bar{z}(1))\,dz_1\wedge dz(1)\wedge d\bar{z}(1) \tag{18}$$

Moreover, it is clear from the definition of $\tilde{B}$ (17) that, with the notation of 6.1,

$$\tilde{S}^\alpha_\delta=(z(1)\in B^\alpha_\delta(1): |z_1^{\alpha_1}f(z)|=\delta/|z(1)^\alpha|),$$

so that the integral in (18) is equal to

$$\sum_{r<\alpha_1}\int_{B^\alpha_\delta(1)} z(1)^{-\alpha} g_{r,0}(z(1),\bar{z}(1))\,dz(1)\wedge d\bar{z}(1)\int_{C(\delta,z(1))} z_1^{r-\alpha_1}dz_1, \tag{19}$$

where $C(\delta,z(1))=(z_1\in\mathbb{C}: |z_1^{\alpha_1}f(z_1,z(1))|=\delta\cdot|z(1)^{-\alpha}|)$. For any $z(1)\in B^\alpha_\delta(1)$, the second integral is equal to $2\pi i$ if $r-\alpha_1=-1$ and is zero otherwise. The limit in (18) reduces then to

$$\lim_{\delta\to 0} 2\pi i \int_{B_\delta^\alpha(1)} z(1)^{-\alpha} g_{\alpha_1-1,0}(z(1),\bar z(1))\, dz(1)\wedge d\bar z(1),$$

which is equal to the right limit in (14), as shown by 6.5(13).

The proof of (15) is simpler, since we can apply directly 6.4, after changing to ζ-coordinates by $z=\mu(\zeta)$.

Proof of (16). Choose $b\in\Gamma_c(\tilde B,\mathscr{E}^0)$, and a smooth function b^* such that $\dfrac{\partial b^*}{\partial\bar z_1}=-b$ on $\bar B$; if $\gamma=z^{-\alpha}b^*dz_1\wedge dz(1)\wedge d\bar z(1)$, we have

$$d\gamma=z^{-\alpha}b\,dz\wedge d\bar z.$$

By Stokes' theorem

$$\int_{\tilde B_\delta^\alpha} z^{-\alpha}b\,dz\wedge d\bar z=\int_{\partial\tilde B_\delta^\alpha}\gamma=\left(\int_{{}^1\tilde B_\delta^\alpha}-\int_{\tilde S_\delta^\alpha}\right)\gamma, \tag{20}$$

considering that

$$\tilde B_\delta^\alpha=(|z_1h_1|<1,\ \|z(1)\|<1,\ |z^\alpha f|>\delta)$$

and

$$\partial\tilde B_\delta^\alpha={}^1\tilde B_\delta^\alpha-\tilde S_\delta^\alpha+\sum_{l=2}^{n}{}^l\tilde B_\delta^\alpha, \tag{21}$$

where equality is understood in the sense of semianalytic chains in $\bar B$, and where

$${}^1\tilde B_\delta^\alpha=(|z_1h|=1,\ \|z(1)\|<1,\ |z^\alpha f|>\delta)=(|z_1h|=1,\ \|z(1)\|<1,\ |z(1)^\alpha|>\delta)$$

and

$${}^l\tilde B_\delta^\alpha=(|z_1h|<1,\ |z_l|=1,\ \|z(l)\|<1,\ |z^\alpha f|>\delta),\quad (l=2,\dots,n)$$

are given appropriate orientation; it is clear that the integral of γ on ${}^l\tilde B_\delta^\alpha$ $(l\neq 1)$ is zero.

Define now

$$U_\delta=(z\in B:|z_1h|>1,\ |z(1)^\alpha|>\delta),$$

and its boundary

$$\partial U_\delta=-{}^1\tilde B_\delta+U'_\delta+\sum_{j=1}^{n}{}^jU_\delta \tag{22}$$

in the sense of semianalytic chains, where ${}^1\tilde B_\delta^\alpha$ is oriented as in (21), and the chains

$${}^jU_\delta=(\|z(j)\|<1,\ |z_j|=1,\ |z(j)^\alpha|>\delta,\ |z_1h|>1),\quad (j=1,\dots,n)$$
$$U'_\delta=(z\in B:|z(1)^\alpha|=\delta,\ |z_1h|>1)$$

are given convenient orientations.

The integral of γ over U'_δ and ${}^jU_\delta$, with $j=2,\dots,n$, is zero, and it is clear that ${}^1U_\delta={}^1B_\delta^\alpha=(\|z(1)\|<1,\ |z_1|=1,\ |z(1)^\alpha|>\delta)$, since $|h|>1$ on $\bar B$, and that the orientations of this chain in (21) and (22) are the same. The integral of $z^{-\alpha}bdz$

$\wedge\, d\bar{z}$ over U_δ is also zero, since b has support in $\tilde{B} \subset \bar{B} - U_\delta$. By Stokes' theorem,

$$0 = \int_{\partial U_\delta} \gamma = \left(\int_{{}^1\tilde{B}^\alpha_\delta} - \int_{{}^1B^\alpha_\delta} \right) \gamma$$

and we deduce that

$$\begin{aligned} \lim_{\delta\to 0} \int_{{}^1\tilde{B}^\alpha_\delta} \gamma &= \lim_{\delta\to 0} \int_{{}^1B^\alpha_\delta} \gamma \\ &= \lim_{\delta\to 0} \int_{B^\alpha_\delta(1)} z(1)^{-\alpha} dz(1) \wedge d\bar{z}(1) \int_{|z_1|=1} z_1^{-\alpha_1} b^*(z,\bar{z}) dz_1, \end{aligned} \tag{23}$$

where the last limit exists by 6.5(8).

Moreover,

$$\lim_{\delta\to 0} \int_{\tilde{S}^\alpha_\delta} \gamma = \lim_{\delta\to 0} \int_{S^\alpha_\delta} \gamma \tag{24}$$

by the first part of this proposition, so that taking limit in (20) we have, by (23) and (24), that

$$\lim_{\delta\to 0} \int_{\tilde{B}^\alpha_\delta} z^{-\alpha} b\, dz \wedge d\bar{z} = \lim_{\delta\to 0} \left(\int_{{}^1B^\alpha_\delta} - \int_{S^\alpha_\delta} \right) \gamma.$$

Finally, Stokes' theorem applied to the chain in 6.1(2) implies that the second limit is equal to

$$\lim_{\delta\to 0} \int_{B^\alpha_\delta} z^{-\alpha} b\, dz \wedge d\bar{z},$$

which proves (16).

6.7. Proposition. *Let* α *and* $\beta \in \mathbf{Z}^n_+$. *Then*

$$\lim_{\delta\to 0} \int_{S^{\alpha+\beta}_\delta} z^{-\alpha} k\, dz_1 \wedge dz(1) \wedge d\bar{z}(1) = \lim_{\delta\to 0} \int_{S^\alpha_\delta} z^{-\alpha} k\, dz_1 \wedge dz(1) \wedge d\bar{z}(1) \tag{25}$$

and

$$\lim_{\delta\to 0} \int_{B^{\alpha+\beta}_\delta} z^{-\alpha} k\, dz \wedge d\bar{z} = \lim_{\delta\to 0} \int_{B^\alpha_\delta} z^{-\alpha} k\, dz \wedge d\bar{z} \tag{26}$$

for all $k \in \mathscr{E}^0(\mathbb{C}^n)$.

Proof. We proceed by induction on n. The assertions are clear for $n=1$, since in this case $S^{\alpha+\beta}_\delta = S^\alpha_{\delta'}$ and $B^{\alpha+\beta}_\delta = B^\alpha_{\delta'}$ for some δ'. In the general case, the left limit in (25) is equal to

$$\begin{aligned} &\lim_{\delta\to 0} \int_{S^{\alpha+\beta}_\delta} z^{-\alpha-\beta} (z^\beta k)\, dz_1 \wedge dz(1) \wedge d\bar{z}(1) \\ &= \lim_{\delta=0} \frac{2\pi i}{(\alpha_1+\beta_1-1)!} \int_{B^{\alpha+\beta}_\delta(1)} z(1)^{-\alpha-\beta} (D^{\alpha_1+\beta_1-1} z^\beta k)_{z_1=0}\, dz(1) \wedge d\bar{z}(1), \end{aligned}$$

by 6.5(10), and this limit reduces to

$$\lim_{\delta\to 0} \frac{2\pi i}{(\alpha_1-1)!} \int_{B^{\alpha+\beta}_\delta(1)} z(1)^{-\alpha} (D^{\alpha_1-1} k)_{z_1=0}\, dz(1) \wedge d\bar{z}(1), \tag{27}$$

since $(D^{\alpha_1+\beta_1-1} z^\beta)_{z_1=0} = \beta_1! \binom{\alpha_1+\beta_1-1}{\beta_1} z(1)^\beta\, (D^{\alpha_1-1} k)_{z_1=0}$. By the inductive hypothesis, this last limit is equal to

$$\lim_{\delta\to 0}\frac{2\pi i}{(\alpha_1-1)!}\int_{B_\delta^\alpha(1)} z(1)^{-\alpha}(D^{\alpha_1-1}k)_{z_1=0}dz(1)\wedge d\bar{z}(1),$$

which is equal to the right limit in (25), according to 6.5(10).

To prove (26), choose a form $\gamma=z^{-\alpha}k^*dz_1\wedge d\bar{z}(1)\wedge dz(1)$ such that $d\gamma=z^{-\alpha}kdz\wedge d\bar{z}$. By Stokes' theorem (cf. 6.1(2)),

$$\int_{B_\delta^{\alpha+\beta}} z^{-\alpha}kd\ \wedge d\bar{z}=\left(\int_{{}^1B_\delta^{\alpha+\beta}}-\int_{S_\delta^{\alpha+\beta}}\right)\gamma, \tag{28}$$

and

$$\int_{{}^1B_\delta^{\alpha+\beta}}\gamma=\int_{B_\delta^{\alpha+\beta}(1)} z(1)^{-\alpha}dz(1)\wedge d\bar{z}(1)\int_{|z_1|=1} z_1^{-\alpha_1}k^*dz_1;$$

by inductive hypothesis, the limit of this integral, as $\delta\to 0$, is equal to

$$\lim_{\delta\to 0}\int_{B_\delta^\alpha} z(1)^{-\alpha}dz(1)\wedge d\bar{z}(1)\int_{|z_1|=1} z_1^{-\alpha_1}k^*dz_1=\lim_{\delta\to 0}\int_{{}^1B_\delta^\alpha}\gamma.$$

This fact, together with (25) and Stokes' theorem, applied to 6.1(2), imply that

$$\lim_{\delta\to 0}\int_{B_\delta^{\alpha+\beta}} z^{-\alpha}adz\wedge d\bar{z}=\lim_{\delta\to 0}\left(\int_{{}^1B_\delta^\alpha}-\int_{S_\delta^\alpha}\right)\gamma$$
$$=\lim_{\delta\to 0}\int_{B_\delta^\alpha} z^{-\alpha}adz\wedge d\bar{z},$$

as wanted.

7. Existence of Residue and Principal Value in the General Case

Let W be a paracompact reduced complex space of (pure) dimension n and structural sheaf $\mathcal{O}_W$, and let Y be a 1-codimensional subspace defined by one global holomorphic equation. We endow the space of smooth forms with compact support $\Gamma_c(W;\mathcal{E}^{\cdot}_W)$ with the topology defined in 4.2. The semianalytic chains $[W(>\delta)]=[|\varphi^q|>\delta]$ and $[W(=\delta)]=[|\varphi^q|=\delta]$ of the following theorem are defined as in 5.2, and the intersection of these chains with an open subset G of W will be denoted by $G(>\delta)$ and $G(=\delta)$.

7.1. Theorem. *Let $\xi\in\Gamma_c(W;\mathcal{E}_W^{2n})$, $\theta\in\Gamma_c(W;\mathcal{E}_W^{2n-1})$, and consider the semi-meromorphic forms $\tilde{\xi}=\xi/\varphi^q$ and $\tilde{\theta}=\theta/\varphi^q$ in $\Gamma_c(W;\mathcal{E}^{\cdot}_W(*Y))$, for some function $\varphi\in\Gamma(W,\mathcal{O}_W)$ which is an equation of Y, and some $q\in\mathbf{Z}_+$.*

The limits

$$P(\xi/\varphi^q)=\lim_{\delta\to 0}I[|\varphi^q|>\delta]\,(\xi/\varphi^q) \tag{1}$$

$$R(\theta/\varphi^q)=\lim_{\delta\to 0}I[|\varphi^q|=\delta]\,(\theta/\varphi^q) \tag{2}$$

exist, are independent of the particular representations ξ/φ^q and θ/φ^q of $\tilde{\xi}$ and $\tilde{\theta}$ in terms of the chosen equation of Y, and define continuous functionals on $\Gamma_c(W,\mathcal{E}_W^j)$, $j=2n,\ 2n-1$.

Proof. Take a point $x\in Y$ and a function $\varrho\in\Gamma(W_x,\mathcal{O}_W)$ on some neighborhood $W_x\subset W$ of x such that W_x-Y_0 has only simple points and is dense in W_x, where $Y_0 = (\omega \in W_x : \varphi^q . \varrho = 0)$. By Hironaka's resolution of singularities [9], W_x can be chosen so small that a proper holomorphic map $\pi : W' \to W_x$ exists with the following properties: W' is a manifold, $Y'_0 = \pi^{-1}(Y_0)$ is a subspace of W' with only normal crossings and π induces an isomorphism of $U' = W' - Y'_0$ onto $U = W_x - Y_0$.

To prove the theorem, it will be enough to consider forms ξ and θ with support contained in W_x. In this case, the integrals in (1) and (2) are equal to $I[W_x(>\delta) - Y_0](\xi/\varphi^q)$ and $I[W_x(=\delta) - Y_0](\theta/\varphi^q)$ respectively, since the real codimension of Y_0 is one in $W_x(=\delta)$, and is two in $W(>\delta)$. Moreover, these last two integrals are also equal to

$$I[W'(>\delta) - Y'_0](\pi^* \tilde{\xi}) \quad \text{and} \quad I[W'(=\delta) - Y'_0]\, \pi^* \tilde{\theta}),$$

because π is an isomorphism outside Y'_0; we denote here $W'(>\delta) = W'(|\varphi'| > \delta)$, $W'(=\delta) = (|\varphi'| = \delta)$ and $\varphi' = \varphi^q \circ \pi$. The chain $[W'(>\delta)]$ is oriented by the fundamental class of W', and $[W'(=\delta)] = -\partial[W'(>\delta)]$. The same codimension argument, applied to these integrals, gives

$$I[W_x(>\delta)](\tilde{\xi}) = I[W'(>\delta)](\pi^* \tilde{\xi}) \tag{3}$$

and

$$I[W_x(=\delta)](\tilde{\theta}) = I[W'(=\delta)](\pi^* \tilde{\theta}). \tag{4}$$

To prove the existence of the limits of the right integrals, we can proceed locally, since $\pi^* \tilde{\xi} = \pi^* \xi/\varphi'$ and $\pi^* \tilde{\theta} = \pi^* \theta/\varphi'$ have compact support. By Lemma 7.3, there is a neighborhood Q of a given point in Y'_0, and a coordinate system $\omega = (\omega_1, \ldots, \omega_n)$ on Q, such that $\varphi' \varrho'(\omega) = \omega^\alpha$ on Q, for $\varrho' = \varrho \circ \pi$ and some $\alpha \in \mathbf{Z}^n_+$. Then $\varphi' = \omega^\beta f$ and $\varrho' = \omega^\gamma f^{-1}$ on Q, for β and γ in $\mathbf{Z}^n_+$ such that $\beta + \gamma = \alpha$, and for some holomorphic function $f \neq 0$ on Q.

By 6.6(14), (15) and (16), and 6.5(7) and (8), there is a neighborhood $\tilde{Q} \subset Q$ such that the limits

$$\lim_{\delta \to 0} I[\tilde{Q}(|\omega^\beta f| > \delta)](\xi'/\omega^\beta f) \tag{5}$$

and

$$\lim_{\delta \to 0} I[\tilde{Q}(|\omega^\beta f| = \delta)](\theta'/\omega^\beta f) \tag{6}$$

exist, for all forms ξ' and θ' with support contained in $\tilde{Q}$; this fact implies the existence of $P(\xi/\varphi^q)$ and $R(\theta/\varphi^q)$ in (1) and (2).

Moreover, the limits in (5) and (6) are respectively equal, according to 6.6, to

$$\lim_{\delta \to 0} I[Q(|\omega^\beta| > \delta)](\xi'/\omega^\beta f)$$

and

$$\lim_{\delta \to 0} I[Q(|\omega^\beta| = \delta)](\theta'/\omega^\beta f)$$

which depend continuously of ξ' and θ', by 6.5. We deduce that the limits in (3) and (4) are continuous functions of $\pi^* \xi$ and $\pi^* \theta$, hence of ξ and θ.

Finally, the following proposition implies the asserted independence of $P(\xi/\varphi^q)$ and $R(\theta/\varphi^q)$ with respect of the equation φ of Y.

7.2. Proposition. *In the conditions of Theorem 7.1, let ψ be a holomorphic function on W which is not identically zero on any irreducible component of W. Then*

$$P(\psi\xi/\psi\varphi^q) = P(\xi/\varphi^q) \tag{7}$$

and

$$R(\psi\theta/\psi\varphi^q) = R(\theta/\varphi^q)\,. \tag{8}$$

Proof. The left members of these equalities are limits of the integrals

$$I[W(|\psi\varphi^q| > \delta)]\,(\xi/\varphi^q) \quad \text{and} \quad I[W(|\psi\varphi^q| = \delta)\,(\theta/\varphi^q)\,. \tag{9}$$

As before, given a point $x \in (\psi\varphi^q = 0)$, we can choose a neighborhood W_x, a function $\varrho \in \Gamma(W_x, \mathcal{O}_W)$ such that $W_x(\varrho \neq 0)$ has only simple points and is dense in W_x, and a resolution $\pi : W' \to W_x$ of $T = W_x(\varrho\psi\varphi^q = 0)$.

We restrict ourselves to consider forms with support in W_x. In such case, reasoning as above we deduce that the integrals in (9) are equal to

$$I[W'(|\psi'\varphi'| > \delta)]\,(\pi^*\xi/\varphi') \quad \text{and} \quad I[W'(|\psi'\varphi'| = \delta)]\,(\pi^*\theta/\varphi')\,, \tag{10}$$

where $\psi' = \psi_0\pi$ and $\varphi' = \varphi^q \circ \pi$. By 7.3, there is a coordinate system $\omega = (\omega_1, \ldots, \omega_n)$ on some neighborhood Q of a point in $T' = \pi^{-1}(T)$, such that $\varrho'\psi'\varphi' = \omega^\alpha$ on Q, where $\varrho' = \varrho \circ \pi$ and $\alpha \in \mathbf{Z}^n_+$. It follows that $\varphi' = \omega^\beta f$, $\psi' = \omega^\gamma g$ and $\varrho' = \omega^\nu h$, where β, γ and ν belong to $\mathbf{Z}^n_+$ and $\beta + \gamma + \nu = \alpha$, and where f, g and h are never vanishing functions in $\Gamma(Q, \mathcal{O}_W)$ such that $f.\,g.h = 1$.

We have now to study the limits of the integrals in (10) when $\delta \to 0$. By 6.6(16), there is a neighborhood $\tilde{Q} \subset Q$ such that

$$\lim_{\delta \to 0} I[W'(|\psi'\varphi'| > \delta)]\,(\xi'/\varphi') = \lim_{\delta \to 0} I[Q(|\omega^{\beta+\gamma}| > \delta)]\,(\xi'/\varphi')$$

for all forms ξ' with support in $\tilde{Q}$. The last limit is equal to $\lim_{\delta \to 0} I[Q(|\omega^\beta| > \delta)]\,(\xi'/\varphi')$ by 6.7(26), which is in turn equal to $\lim_{\delta \to 0} I[Q(|\varphi'| > \delta)]\,(\xi'/\varphi')$ for forms with support in a possibly smaller $\tilde{Q}$, by 6.6(16). This local result implies that, globally,

$$\begin{aligned} P(\psi\xi/\psi\varphi^q) &= \lim_{\delta \to 0} I[W'(|\psi'\varphi'| > \delta)]\,(\pi^*\xi/\varphi') \\ &= \lim_{\delta \to 0} I[W'(|\varphi'| > \delta)]\,(\pi^*\xi/\varphi')\,, \end{aligned}$$

which is equal to $P(\xi/\varphi^q)$, as explained before.

In a similar way, using 6.6(14), (15) and 6.7(25), one proves that

$$\begin{aligned} R(\psi\theta/\psi\varphi^q) &= \lim_{\delta \to 0} I[W'(|\psi'\varphi'| = \delta)]\,(\pi^*\theta/\varphi') \\ &= \lim_{\delta \to 0} I[W'(|\varphi'| = \delta)]\,(\pi^*\theta/\varphi') = R(\theta/\varphi^q)\,, \end{aligned}$$

as wanted.

7.3. Lemma. *Let ψ be a holomorphic function on some neighborhood of the origin $0 \in \mathbb{C}^n$ such that $(\psi = 0)$ has only normal crossings at 0. Then there exists a coordinate system $\omega = (\omega_1, \ldots, \omega_n)$ on a neighborhood Q of 0 such that $\psi(\omega) = \omega^\alpha$ on Q for some $\alpha \in Z^n_+$.*

Proof. We can always write $\psi(z) = z^\alpha \cdot g(z)$, for some $\alpha \in Z^n_+$ and some holomorphic g with $g(0) \neq 0$. Suppose that $\alpha_1 \neq 0$, and find a function h such that $h^{\alpha_1} = g$ around 0. Then $\omega_1 = z_1 h(z)$, $\omega_j = z_j\,(j > 1)$ define a coordinate system on some neighborhood of 0, and is clear that $\psi(\omega) = \omega^\alpha$.

7.4. *Remark.* The limit $R(\xi) = R(\varphi\xi/\varphi)$ of a form ξ regular on Y is zero, as follows from the proof of 7.1 and from 6.4.

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