Jost Functions for the Harmonic Oscillator.

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Summary. — The Jost functions for the harmonic oscillator (in one and three dimensions) are computed explicitly. They are entire analytic functions in the complex E plane. Its zeros give the well-known bound states of the system. An integral representation is given for the Jost functions of the perturbed harmonic oscillator.

1. – Introduction.

It is a well-known fact that the functions introduced by JOST (1) in the nonrelativistic theory of the scattering of a particle by a spherically symmetric potential, when calculated, solve completely the problem. The phase of such functions $f_l(k)$, are the phase shifts $\delta_l(k)$ of the l-partial wave, and it can be shown that the zeros associated with the bound states are located on the negative imaginary axis of the k-complex plane. However, these results are valid only for short-range potentials that have finite first and second moments with respect to the origin; *i.e.*,

$$\int_0^\infty r\,|V(r)|\,\mathrm{d}r<\infty\,, \tag{1a}$$

$$\int_0^\infty r^2\,|V(r)|\,\mathrm{d}r<\infty\,. \tag{1b}$$

(1) R. JOST: *Helv. Phys. Acta*, **20**, 256 (1947).

Given the importance of these functions for this kind of problems, it is of interest to extend the method for potentials that do not verify conditions (1.a) and (1.b), as, for example, the case of the harmonic oscillator. In Section **2** a brief review is given of the most relevant formulas of the theory of Jost functions, in order to compare them with the following results; in Section **3** they are generalized to the case of the three-dimensional isotropic oscillator, and an integral representation of the Jost functions is given for the perturbed oscillator. Finally, in Section **4** the linear oscillator is discussed.

2. – The Jost functions.

For a spherically symmetric potential, the radial Schrödinger equation is

$$\frac{\mathrm{d}^2\varphi_l}{\mathrm{d}r^2}+\left[k^2-\frac{l(l+1)}{r^2}-v(r)\right]\varphi_l(k,r)=0\,;\qquad v(r)=\frac{2mV}{\hbar^2}\,. \tag{2.1}$$

If the potential fulfills condition (1.a), it can be shown [2] that a regular solution $\varphi_l(k,r)$ exists around the origin, defined by the boundary condition:

$$\lim_{r\to\infty}\varphi_l(k,r)=r^{l+1}+0(r^{l+3})\,. \tag{2.2}$$

As this condition does not depend on k, $\varphi_l(k,r)$ will be an even function of such variable. Condition (2.b) implies the existence of two irregular solutions at infinity, $f_l(\pm k,r)$, defined by the boundary condition:

$$\lim_{r\to\infty}f_l(\pm k,r)=\exp[\mp ikr] \tag{2.3}$$

which are linearly independent.

The Jost function is defined as the Wronskian:

$$f_l(k)=W[f_l(k,r),\,\varphi_l(k,r)]=f_l\frac{\mathrm{d}\varphi_l}{\mathrm{d}r}-\frac{\mathrm{d}f_l}{\mathrm{d}r}\varphi_l\,, \tag{2.4}$$

and it is independent of r as the Wronskian of two solutions of a differential equation is a constant different from zero if they are linearly independent. As the $f_l(\pm k,r)$ are linearly independent, $\varphi_l(k,r)$ can be written as a linear combination of them:

$$\varphi_l(k,r)=\alpha_l(k)f_l(k,r)+\beta_l(k)f_l(-k,r) \tag{2.5}$$

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[2] R. NEWTON: *Journ. Math. Phys.*, **1**, 319 (1960).

and it can be easily verified that

$$\beta_l(k) = \frac{W[f_l(k,r),\varphi_l(k,r)]}{W[f_l(k,r), f_l(-k,r)]} = \frac{f_l(k)}{W[f_l(k,r), f_l(-k,r)]}, \tag{2.6}$$

$$\alpha_l(k) = -\frac{f_l(-k)}{W[f_l(k,r), f_l(-k,r)]}. \tag{2.7}$$

The Wronskian of the denominator can be calculated in $r\to\infty$. Using (2.3), we obtain

$$W[f_l(k,r), f_l(-k,r)] = 2ik$$

and the (2.5) goes into

$$\varphi_l(k,r) = \frac{1}{2ik}[f_l(k)f_l(-k,r) - f_l(-k)f_l(k,r)]. \tag{2.8}$$

Using the asymptotic behavior of this expression, it can be shown that the scattering matrix $S_l(k)$ can be written

$$S_l(k) = \frac{f_l(k)}{f_l(-k)}, \tag{2.9}$$

and the unitarity condition implies that

$$f_l^*(-k^*) = f_l(k). \tag{2.10}$$

Moreover, $f_l(k) = |f_l(k)|\exp[i\,\delta_l(k)]$, where $\delta_l(k)$ is the phase shift of the l-partial wave; so

$$\operatorname{tg}\delta_l(k) = \frac{\operatorname{Im} f_l(k)}{\operatorname{Re} f_l(k)}. \tag{2.11}$$

If $f_l(k_0) = W[f_l(k_0,r),\varphi_l(k_0,r)] = 0$ in $k_0 = -iK$; $K>0$; this means that the two solutions are not linearly independent, *i.e.*

$$f_l(k_0, r) = C\varphi_l(k_0,r). \tag{2.12}$$

We know that $\varphi_l(k_0,r)$ is zero at the origin, and that $f_l(-iK,r)$ tends to zero exponentially as $r\to\infty$; then both sides of (2.12) are square integrable, and k_0^2 is a discrete eigenvalue of the Schrödinger equation. It is a bound state of the system.

Evaluating (2.4) at $r\to\infty$, and making use of (2.2), one gets

$$f_l(k) = \lim_{r\to\infty}(2l+1)\,r^l f_l(k,r). \tag{2.13}$$

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3. – The three-dimensional harmonic oscillator.

3·1. *The unperturbed oscillator.* – In this case, we consider the potential $V(r) = \frac{1}{2} m\omega^2 r^2$; so

$$v(r) = \frac{2m}{\hbar^2} V(r) = \beta^2 r^2 , \qquad \beta = \frac{m\omega}{\hbar} , \tag{3.1}$$

and writing $\varepsilon = 2mE/\hbar^2$ the radial Schrödinger eq. (2.1) is

$$\frac{\mathrm{d}^2\varphi_l(\varepsilon, r)}{\mathrm{d}r^2} + \left[\varepsilon - \frac{l(l+1)}{r^2} - \beta^2 r^2\right] \varphi_l(\varepsilon, r) = 0 . \tag{3.2}$$

As there are no free particle states, we cannot write $\varepsilon = k^2$.

Making the substitution $\varphi_l(\varepsilon, r) = r^{l+1} \exp[-\frac{1}{2}\beta r^2] \Phi_l(\varepsilon, r)$, it can be shown that $\Phi_l(\varepsilon, r)$ satisfies

$$\frac{\mathrm{d}^2\Phi_l}{\mathrm{d}r^2} + \left[\frac{2(l+1)}{r} - 2\beta r\right] \frac{\mathrm{d}\Phi_l}{\mathrm{d}r} + [\varepsilon - \beta(2l+3)]\Phi_l = 0 , \tag{3.3}$$

or, with $\xi = \beta r^2$,

$$\xi \frac{\mathrm{d}^2\Phi_l}{\mathrm{d}\xi^2} + \left[\left(l + \frac{3}{2}\right) - \xi\right] \frac{\mathrm{d}\Phi_l}{\mathrm{d}\xi} + \left[\frac{\varepsilon}{4\beta} - \frac{2l+3}{4}\right] \Phi_l = 0 , \tag{3.4}$$

which is a confluent hypergeometric equation

$$z \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + (c - z) \frac{\mathrm{d}y}{\mathrm{d}z} - ay = 0 \tag{3.5}$$

of indices $a = \big((2l+3)/4\big) - \varepsilon/4\beta$ and $c = l + \frac{3}{2}$; $\Phi_l(\varepsilon, \beta r^2)$ will then be the corresponding confluent hypergeometric functions.

The two linearly independent solutions of the confluent hypergeometric eq. (3.5) about the origin are (see MORSE and FESHBACH, page 604 and following)

$$y_1^0 = F(a|c|z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n) \cdot z^n}{\Gamma(c+n) \cdot n!} = 1 + \frac{a}{c} z + \frac{a(a+1)}{c(c+1)} \frac{z^2}{z!} + \dots . \tag{3.6}$$

The series being convergent for any finite $|z|$,

$$y_2^0 = z^{1-c} F(a - c + 1 | 2 - c | z)$$

and the linearly independent solutions of (3.5) about the irregular point at infinity are

$$(3.7)\qquad y_1^\infty = U_1(a|c|z) = \frac{e^z z^{a-c}}{\Gamma(c-a)}\int_0^\infty \exp[-u]u^{c-a-1}\left(1-\frac{u}{z}\right)^{a-1} du\,,$$

$$(3.8)\qquad y_2^\infty = U_2(a|c|z) = \frac{\exp[i\pi a]z^{-a}}{\Gamma(a)}\int_0^\infty \exp[-u]u^{a-1}\left(1+\frac{u}{z}\right)^{c-a-1} du\,,$$

which are called confluent hypergeometric functions of third kind.

The Jost function is here defined as before, as

$$(3.9)\qquad f_l(E) = W[f_l(\varepsilon, r), \varphi_l(\varepsilon, r)]\,, \qquad \varepsilon = \frac{2mE}{\hbar^2}\,,$$

where $\varphi_l(\varepsilon, r)$ is the regular solution at the origin, and $f_l(\varepsilon, r)$ is the irregular solution at infinity, which goes to zero as $r\to\infty$.

First we analyse the irregular solutions at $r\to\infty$. The solutions defined by the boundary conditions at infinity are

$$(3.10)\qquad f_l(\varepsilon, r) = r^{l+1}\exp[-\tfrac{1}{2}\beta r^2]\, U_2\left(\frac{2l+3}{4}-\frac{\varepsilon}{4\beta}\,\middle|\, l+\frac{3}{2}\,\middle|\,\beta r^2\right),$$

$$(3.11)\qquad g_l(\varepsilon, r) = r^{l+1}\exp[-\tfrac{1}{2}\beta r^2]\, U_1\left(\frac{2l+3}{4}-\frac{\varepsilon}{4\beta}\,\middle|\, l+\frac{3}{2}\,\middle|\,\beta r^2\right).$$

Using the integral representations of the U functions given in (3.7) and (3.8) it can be shown that

$$(3.12)\qquad g_l(\varepsilon, r) = r^{l+1}\exp[+\tfrac{1}{2}\beta r^2]\, U_2\left(\frac{2l+3}{4}+\frac{\varepsilon}{4\beta}\,\middle|\, l+\frac{3}{2}\,\middle|\,-\beta r^2\right) = + i^{l+1} f_l(-\varepsilon, ir)\,.$$

So our linearly independent solutions about $r\to\infty$ are $f_l(\varepsilon, r)$ and $f_l(-\varepsilon, ir)$

It is necessary to know the asymptotic behavior of these functions. This can be obtained by recalling that the asymptotic behavior of $U_2(a|c|z)$ is given by

$$(3.13)\qquad U_2(a|c|z) \underset{z\to\infty}{\to} \exp[i\pi a]z^{-a}\,, \qquad \text{if } |\arg z|<\pi,$$

so

$$(3.14)\qquad f_l(\varepsilon, r) \underset{r\to\infty}{\to} (\exp[-i\pi]\beta)^{((\varepsilon/4\beta)-(2l+3)/4)}\, r^{(\varepsilon/2\beta)-\frac{1}{2}}\exp[-\tfrac{1}{2}\beta r^2] \to 0\,,$$

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and also

$$f_l(-\varepsilon, ir) \underset{r\to\infty}{\Rightarrow} \left(\exp[-i\pi]\beta\right)^{-((\varepsilon/4\beta)+(2l+3)/4)} \frac{\exp[+\frac{1}{2}\beta r^2]}{(ir)^{(\varepsilon/2\beta)+\frac{1}{2}}}\,. \tag{3.15}$$

Next we study the regular solution about the origin. It is given by

$$\varphi_l(\varepsilon, r) = r^{l+1}\exp[-\tfrac{1}{2}\beta r^2] F\left(\frac{2l+4}{4} - \frac{\varepsilon}{4\beta}\,\middle|\, l+\frac{3}{2}\,\middle|\,\beta r^2\right). \tag{3.16}$$

If $r\to 0$; $\exp[-\frac{1}{2}\beta r^2] \simeq 1 - \frac{1}{2}\beta r^2$ and, recalling (3.6), it can be easily verified that $\varphi_l(\varepsilon, r)$ satisfies the boundary condition

$$\lim_{r\to 0} \varphi_l(\varepsilon, r) \simeq r^{l+1} + 0(r^{l+3})\,. \tag{3.17}$$

Its asymptotic behavior as $r\to\infty$ is obtained recalling that (MORSE and FESHBACH, page 607)

$$F(a\,|\,c\,|\,z) \underset{z\to\infty}{\Rightarrow} \frac{\Gamma(c)}{\Gamma(a)} z^{a-c} e^z, \qquad \text{if } \arg z = 0, \tag{3.18}$$

and we obtain

$$\varphi_l(\varepsilon, r) \underset{r\to\infty}{\Rightarrow} \beta^{((\varepsilon/4\beta)+(2l+3)/4)} \frac{\Gamma(l+\frac{3}{2})}{\Gamma\big((2l+3)/4 - (\varepsilon/4\beta)\big)} r^{((\varepsilon/2\beta)+\frac{1}{2})} \exp[+\tfrac{1}{2}\beta r^2]\,. \tag{3.19}$$

We can now use (3.9) to find the Jost function. Using (3.14) and (3.19) we compute the Wronskian in $r\to\infty$.

We obtain

$$f_l(E) = 2\beta^{-(l+\frac{1}{2})} \frac{\Gamma(l+\frac{3}{2})}{\Gamma\big((2l+3)/4 - (\varepsilon/4\beta)\big)} \exp\left[i\pi\left(l+\frac{3}{2}-\frac{\varepsilon}{4\beta}\right)\right]. \tag{3.20}$$

As before, the zeros of $f_l(E)$ are the bound states of the system. Then only zeros of $f_l(E)$ are given by the poles of the $\Gamma\big((2l+3)/4 - (\varepsilon/4\beta)\big)$ of the denominator; and recalling that all the singularities of $\Gamma(z)$ are poles when $Z=0$, $-n$; n integer, the bound states will be located at

$$\left\{\begin{array}{ll} \dfrac{2l+3}{4} - \dfrac{\varepsilon}{4\beta} = -k\,, & k = 0, 1, 2, \ldots, \\ \text{or be:} & \\ \dfrac{E}{h\nu} = \dfrac{\varepsilon}{2\beta} = (2k + l + \frac{3}{2})\,, & E = (2k+l+\frac{3}{2})\cdot h\nu, \end{array}\right. \tag{3.21}$$

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which are the correct energy levels for the three-dimensional harmonic oscillator.

Next we can verify that the usual relation

$$f_l(E) = \lim_{r\to 0}\,(2l+1)r^l f_l(\varepsilon, r) \tag{3.22}$$

is also valid in our problem. This can be done directly, taking power series expansions of $f_l(\varepsilon, r)$ around the origin. Using the relation

$$U_2(a|c|z) = \frac{\Gamma(1-c)}{\Gamma(z-c+1)}\exp[i\pi a]F(a|c|z) + \\ + \frac{\Gamma(c-1)}{\Gamma(a)}\exp[i\pi a]z^{1-c}F(a-c+1|z-c|z) \tag{3.23}$$

and recalling the power series expansion (3.6) of $F(a|c|z)$, we obtain

$$U_2\left(\frac{2l+3}{4}-\frac{\varepsilon}{4\beta}\bigg| l+\frac{3}{2}\bigg|\beta r^2\right) \simeq \\ \simeq \frac{\Gamma(-l-\frac{1}{2})}{\Gamma(-(2l+3)/4-(\varepsilon/4\beta)+1)}\exp\left[i\pi\left(\frac{2l+3}{4}-\frac{\varepsilon}{4\beta}\right)\right]\{1+\alpha\beta r^2+\ldots\}+ \\ + \frac{\Gamma(l+\frac{1}{2})}{\Gamma((2l+3)/4-(\varepsilon/4\beta))}\exp\left[i\pi\left(\frac{2l+3}{4}-\frac{\varepsilon}{4\beta}\right)\right]\beta^{-(l+\frac{1}{2})}r^{-(2l+1)}\{1+\alpha'\beta r^2+\ldots\}, \tag{3.24}$$

where we have written

$$\alpha = \frac{a}{c} = \frac{(2l+3)/4-(\varepsilon/4\beta)}{l+\frac{3}{2}}; \qquad \alpha' = \frac{a-c+1}{2-c}. \tag{3.25}$$

Replacing (3.24) and (3.25) in (3.10) and omitting the powers of r of higher order, we obtain

$$f_l(\varepsilon, r) \simeq \frac{\Gamma(l+\frac{1}{2})}{\Gamma((2l+3)/4-(\varepsilon/4\beta))}\beta^{-(l+\frac{1}{2})}\exp\left[i\pi\left(\frac{2l+3}{4}-\frac{\varepsilon}{4\beta}\right)\right]r^{-l}. \tag{3.26}$$

The limit (3.22) is then

$$f_l(E) = \lim_{r\to 0}(2l+1)\,r^l\,f_l(\varepsilon, r) = \\ = \frac{(2l+1)\,\Gamma(l+\frac{1}{2})}{((2l+3)/4-(\varepsilon/4\beta))}\beta^{-(l+\frac{1}{2})}\exp\left[i\pi\left(l+\frac{3}{2}-\frac{\varepsilon}{4\beta}\right)\right]. \tag{3.27}$$

Using $\Gamma(z+1) = z\Gamma(z)$; $(2l+1)\Gamma(l+\frac{1}{2}) = 2\Gamma(l+\frac{3}{2})$, it is evident that (3.27) agrees with (3.20).

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Let us write again

$$\varphi_l(\varepsilon, r) = \alpha_l(E) f_l(\varepsilon, r) + \beta_l(E) f_l(-\varepsilon, ir)\,. \tag{3.28}$$

We get as before

$$\beta_l(E) = \frac{W[f_l(\varepsilon, r), \varphi_l(\varepsilon, r)]}{W[f_l(\varepsilon, r), \varphi_l(-\varepsilon, ir)]} = \frac{f_l(E)}{W[f_l(\varepsilon, r), f_l(-\varepsilon, ir)]}\,, \tag{3.29}$$

$$\alpha_l(E) = -\frac{W[f_l(-\varepsilon, ir), \varphi_l(\varepsilon, r)]}{W[f_l(\varepsilon, r), f_l(-\varepsilon, ir)]}\,. \tag{3.30}$$

The Wronskian of the denominator can be computed in $r \to \infty$; using the asymptotic formulas (3.14) and (3.15) we obtain

$$W[f_l(\varepsilon, r), f_l(-\varepsilon, ir)] = 2i^{-(\varepsilon/2\beta+\frac{1}{2})}\beta^{-(l+\frac{1}{2})}\exp[i\pi(l+\tfrac{3}{2})]\,. \tag{3.31}$$

Next we compute $W[f_l(-\varepsilon, ir), \varphi_l(\varepsilon, r)]$ at the origin. Using $\varphi_l(\varepsilon, r) \simeq r^{l+1}$ and

$$f_l(-\varepsilon, ir) \simeq \frac{\Gamma(l+\frac{1}{2})}{\Gamma\big((2l+3)/4+(\varepsilon/4\beta)\big)}\exp\left[i\pi\left(l+\frac{3}{2}\right)\right]\exp\left[+i\pi\frac{\varepsilon}{4\beta}\right]\beta^{-(l+\frac{1}{2})}(ir)^{-l}\,.$$

We find

$$W[f_l(-\varepsilon, ir), \varphi_l(\varepsilon, r)] = \\ = \frac{(2l+1)\Gamma(l+\frac{1}{2})\exp\big[i\pi\big((2l+3)/4+(\varepsilon/4\beta)\big)\big]}{\Gamma\big((2l+3)/4+(\varepsilon/4\beta)\big)}\beta^{-(l+\frac{1}{2})}(-i)^l f_l(-E)\,,$$

so $\varphi_l(\varepsilon, r)$ can be written

$$\varphi_l(\varepsilon, r) = \tfrac{1}{2}i^{(\varepsilon/2\beta+\frac{1}{2})}\exp[-i\pi(l+\tfrac{3}{2})]\,[f_l(E)f_l(-\varepsilon, ir) - (-i)^l f_l(-E) f_l(\varepsilon, r)]. \tag{3.32}$$

3'2. *The perturbed oscillator.* – We now perturb the harmonic oscillator with a potential $v(r)$, such that

$$\lim_{r\to\infty}\frac{v(r)}{r^2} \to 0\,.$$

In this way, the asymptotic behavior of the solutions of the new Schrödinger equation

$$\frac{d^2\psi_l(\varepsilon, r)}{dr^2} + \left[\varepsilon - \beta^2 r^2 - \frac{l(l+1)}{r^2}\right]\psi_l(\varepsilon, r) = v(r)\psi_l(\varepsilon, r) \tag{3.33}$$

is just the same as the asymptotic behavior of the already known solutions of the unperturbed equation, *i.e.*, that corresponding to the harmonic oscillator.

We call then $\Phi_l(\varepsilon, r)$ the solution of (3.33) which satisfies the boundary condition

$$\lim_{r\to 0} \Phi(\varepsilon, r) \simeq r^{l+1} \tag{3.34}$$

and $F_l(\varepsilon, r)$ the solution which tends to zero as $r \to \infty$. Next we define as usual, the Jost functions as

$$F_l(E) = W[F_l(\varepsilon, r), \Phi_l(\varepsilon, r)] \tag{3.35}$$

and here again it is evident that the roots of $F_l(E) = 0$ will be the energy levels of the system.

Integral equations can be obtained for the functions $\Phi_l(\varepsilon, r)$, $F_l(\varepsilon, r)$. In the Appendix I of BOTTINO, LONGONI and REGGE ([3]) it is shown how to get the integral equations corresponding to equations of the kind of (3.33) with the inclusion of boundary conditions. We recall that the linearly independent solutions of the homogeneous equations are:

about $r = 0$

$$\begin{cases} \varphi_1(l, \varepsilon, r) = \varphi_l(\varepsilon, r)\,, \quad \text{given by (3.16)} \\ \varphi_2(l, \varepsilon, r) = \chi_l(\varepsilon, r) = \\ \qquad = r^{-l} \exp[-\tfrac{1}{2}\beta r^2]\beta^{-(l+\frac{1}{2})} F\left(-\dfrac{2l+3}{4} - \dfrac{\varepsilon}{4\beta} + 1 \,\middle|\, -\left(l+\dfrac{1}{2}\right) \,\middle|\, \beta r^2\right), \end{cases}$$

and about $r \to \infty$

$$\begin{cases} f_1(l, \varepsilon, r) = f_l(\varepsilon, r)\,, \\ f_2(l, \varepsilon, r) = f_l(-\varepsilon, ir)\,. \end{cases}$$

The integral equation for $\Phi_l(\varepsilon, r)$ is

$$\Phi_l(\varepsilon, r) = \varphi_l(\varepsilon, r) + \frac{1}{W[\varphi_l, \chi_l]}\int_r^\infty [\varphi_l(\varepsilon, r')\,\chi_l(\varepsilon, r) - \varphi_l(\varepsilon, r)\,\chi_l(\varepsilon, r')]v(r')\,\Phi_l(\varepsilon, r')\,\mathrm{d}r'$$

or, computing the Wronskian,

$$\Phi_l(\varepsilon, r) = \varphi_l(\varepsilon, r) + \\ + \frac{\beta^{(l+\frac{1}{2})}}{(2l+1)}\int_0^r [\varphi_l(\varepsilon, r)\,\chi_l(\varepsilon, r') - \varphi_l(\varepsilon, r')\,\chi_l(\varepsilon, r)]v(tr')\,\Phi_l(\varepsilon, r')\,\mathrm{d}r', \tag{3.36}$$

([3]) A. BOTTINO, A. M. LONGONI and T. REGGE: *Nuovo Cimento*, **23**, 954 (1962).

and for $F_l(\varepsilon, r)$

$$F_l(\varepsilon, r) = f_l(\varepsilon, r) + \tag{3.37}$$

$$+ \frac{1}{W[f_1, f_2]} \int_r^\infty [f_l(\varepsilon, r') f_l(-\varepsilon, ir) - f_l(\varepsilon, r) f_l(-\varepsilon, ir')] v(r') \, F_l(\varepsilon, r') \, \mathrm{d}r' .$$

It is evident that in this perturbed case the Jost functions will also be given by

$$F_l(E) = \lim_{r \to 0} (2l+1) r^l F_l(\varepsilon, r) . \tag{3.38}$$

Replacing (3.37) in (3.38) we find

$$F_l(E) = \lim_{r \to 0} (2l+1) r^l f_l(\varepsilon, r) + \frac{1}{W[f_1, f_2]} \lim_{r \to 0} (2l+1) r^l \cdot$$

$$\int_r^\infty [f_l(\varepsilon, r') f_l(-\varepsilon, ir) - f_l(\varepsilon, r) f_l(-\varepsilon, ir')] v(r') F_l(\varepsilon, r') \, \mathrm{d}r' ,$$

and recalling (3.22) for the Jost function $f_l(E)$ of the unperturbed oscillator, this expression becomes

$$F_l(E) = f_l(E) +$$

$$+ \frac{1}{W[f_1, f_2]} \int_0^\infty [(-i)^l f_l(-E) f_l(\varepsilon, r') - f_l(E) f_l(-\varepsilon, ir')] v(r') F_l(\varepsilon, r') \, \mathrm{d}r' ,$$

and finally, using (3.32)

$$F_l(E) = f_l(E) - \int_0^\infty \varphi_l(\varepsilon, r') v(r') F_l(\varepsilon, r') \, \mathrm{d}r' . \tag{3.39}$$

This integral representation of the Jost function can be used for practical evaluations, combined with the integral eq. (3.37).

4. – The linear oscillator.

In this case, the potential is $V(x) = \frac{1}{2} m\omega^2 x^2$; and the one-dimensional Schrödinger equation is

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (\varepsilon - \beta^2 x^2)\psi = 0 , \tag{4.1}$$

where, as usual, $\varepsilon = 2mE/\hbar^2$, $\beta = m\omega/\hbar$ and $\beta^2 x^2 = 2mV/\hbar^2$.

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With $\xi = x\sqrt{2\beta}$, eq. (4.1) goes into

$$\frac{d^2\psi}{d\xi^2} + \left(\frac{\varepsilon}{2\beta} - \frac{1}{4}\xi^2\right)\psi = 0 . \tag{4.2}$$

This is the Weber equation, whose solutions are $\psi = D_n(x\sqrt{2\beta})$, where $D_n(\xi)$ are the Weber functions, and

$$n = \frac{\varepsilon}{2\beta} - \frac{1}{2} = \frac{E}{n\nu} - \frac{1}{2} .$$

The Weber functions are (see MORSE and FESHBACH, page 1565)

$$D_m(z) = 2^{\frac{1}{2}m} \exp\left[-\frac{1}{4}z^2 + \frac{i\pi m}{2}\right] U_2\left(-\frac{m}{2}\middle|\frac{1}{2}\middle|\frac{z^2}{2}\right) . \tag{4.3}$$

They are not defined for $\operatorname{Re} z \leqslant 0$ because U_2 has a branch line along the real negative axis; and z^2 would cross that singularity. The Weber equation satisfied by $D_m(z)$

$$\frac{d^2 D_m(z)}{dz^2} + \left[\left(m + \frac{1}{2}\right) - \frac{1}{4}z^2\right] D_m(z) = 0 , \tag{4.4}$$

is also satisfied by $D_m(-z)$ and $D_{-m-1}(iz)$; and among the three solutions it exists the relation

$$D_m(z) = \exp[-i\pi m] D_m(-z) + \frac{\sqrt{2\pi}}{\Gamma(-m)} \exp[-i\pi(m+1)] D_{-m-1}(iz) . \tag{4.5}$$

It can be shown (KEMBLE: *Quantum Mechanics*, Appendix C), that for a given value of the energy it exists one and only one solution that goes to zero as $x \to -\infty$, and diverges exponentially as $x \to +\infty$; and one and only one solution that goes to zero as $x \to +\infty$ and diverges as $x \to -\infty$. We call $\psi_L(E, x)$ the solution that satisfies the first boundary condition and $\psi_R(E, x)$ that which satisfies the second.

We define in this case a «Jost function» as

$$F(E) = W[\psi_R(E, x), \psi_L(E, x)] . \tag{4.6}$$

As in general ψ_R, ψ_L are linearly independent, this expression will be a constant different from zero; $F(E)$ will be zero only for the eigenvalues of the system, because then ψ_R $(= c\psi_L)$ will be integrable square.

We must study the asymptotic behavior of $D_m(\xi)$ in order to identify

$\psi_R(E, x)$ and $\psi_L(E, x)$. Using (3.13) it follows that

$$U_2\left(-\frac{m}{2}\left|\frac{1}{2}\right|\frac{1}{2}z^2\right)_{\mathrm{Re}\,z\to+\infty} \exp\left[-\frac{i\pi m}{2}\right]\left(\frac{z^2}{z}\right)^{\frac{1}{2}m} = \frac{\exp[-i\pi m/2]z^m}{2^{\frac{1}{2}m}}, \tag{4.7}$$

and the asymptotic behavior of $D_m(z)$, as $\mathrm{Re}\, a \to +\infty$, is

$$D_m(z)_{\mathrm{Re}\,z\to+\infty}\, z^m \exp[-z^2] \to 0 . \tag{4.8}$$

In order to study the behavior of $D_m(z)$ as $\mathrm{Re}\, z \to -\infty$ we must use the relation (4.5) (see WITTAKER and WATSON: *Modern Analysis*, page 348) to find

$$D_m(z)_{\mathrm{Re}\,z\to-\infty} \frac{\sqrt{2\pi}}{\Gamma(-m)} \frac{\exp[+\frac{1}{4}z^2]}{(-z)^{m+1}} . \tag{4.9}$$

Finally, we obtain, for ξ real

$$D_m(\xi) \to \begin{cases} \xi^m \exp[-\frac{1}{4}\xi^2] \to 0 , & \text{as } \xi \to +\infty, \\ \dfrac{\sqrt{2\pi}}{\Gamma(-m)} \dfrac{\exp[+\frac{1}{4}\xi^2]}{(-\xi)^{m+1}} \to \infty , & \text{as } \xi \to -\infty. \end{cases} \tag{4.10}$$

It is evident that $D_m(\xi)$ verifies the boundary conditions imposed on $\psi_R(E, x)$, then

$$\psi_R(E, x) = D_m(x\sqrt{2\beta}) , \qquad m = \frac{E}{h\nu} - \frac{1}{2} . \tag{4.11}$$

As $D_m(-\xi)$ is also a solution of (4.1), it follows that

$$\psi_L(E, x) = D_m(-x\sqrt{\beta 2}) , \tag{4.12}$$

with the asymptotic behavior

$$\psi_L(E, x) = D_m(-\xi) \to \begin{cases} (-\xi)^m \exp[-\frac{1}{4}\xi^2] \to 0 , & \text{as } \xi \to -\infty, \\ \dfrac{\sqrt{2\pi}}{\Gamma(-m)} \dfrac{\exp[+\frac{1}{4}\xi^2]}{\xi^{m+1}} \to \infty , & \text{as } \xi \to +\infty. \end{cases} \tag{4.13}$$

We can now calculate the « Jost function » $F(E)$, by evaluating the Wronskian (4.6) at $x \to +\infty$. Using (4.10) and (13.*ab*), the result is

$$F(E) = \frac{2\sqrt{2\pi}}{\Gamma(\frac{1}{2} - E/h\nu)} , \tag{4.14}$$

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the zeros being located at the poles of the $\Gamma(\frac{1}{2}-(E/h\nu))$

$$\frac{1}{2}-\frac{E}{h\nu}=-n\,; \qquad n=0,1,2,\ldots \text{ so } E=(n+\tfrac{1}{2})h\nu,$$

which is the well-known result.

5. – Discussion.

We see that the Jost functions for the harmonic oscillator

$$f_l(E)=2\beta^{-(l+\frac{1}{2})}\frac{\Gamma(l+\frac{1}{2})}{\Gamma\big((2l+3)/4-(E/2h\nu)\big)}\exp\left[i\pi\left(l+\frac{3}{2}\right)\right]\exp\left[-\frac{i\pi E}{2h\nu}\right] \tag{5.1}$$

are analytic entire functions in the complex E-plane.

The unitarity condition (2.10), valid for potentials satisfying (1.*a*) and (1.*b*), when expressed in terms of the energy complex variable by means of the transformation $E=k^2$ is

$$f_l^*(E)=f_l(E^*)\,. \tag{5.2}$$

The Jost functions that we obtained (5.1) clearly do not satisfyes (5.2). This is consistent with the fact that a unitary S-matrix can not be defined for this kind of problems because of the absence of dispersion phenomena.

The Regge trajectories will be given by the zeros of $f_l(E)$. Looking at this expression we notice that these trajectories are given by

$$\frac{2l+3}{4}-\frac{E}{2h\nu}=-k\,, \quad k=0,1,2,\ldots \quad \text{or} \quad l=\frac{E}{2h\nu}-2k-\frac{3}{2}, \tag{5.3}$$

so there are an infinite number of Regge poles, and, when the energy E goes from $-\infty$ to $+\infty$ this poles move along the real axis of the l-complex plane from $-\infty$ to $+\infty$.

Writing

$$z=1-\frac{2l+3}{4}+\frac{E}{2h\nu},$$

the Jost function (5.1) can be written

$$f_l(z)=A_l\frac{\exp[-i\pi z]}{\Gamma(1-z)}\,;\; A_l=2\beta^{-(l+\frac{1}{2})}\Gamma(l+\tfrac{3}{2})\exp[-i\pi(l+\tfrac{3}{2})]\exp\left[-i\pi\left(\frac{2l-1}{4}\right)\right],$$

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and recalling the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we obtain

$$f_l(z) = \frac{A_l}{\pi} \exp[-i\pi z]\,\Gamma(z) \sin \pi z . \tag{5.4}$$

The asymptotic behavior of $\Gamma(z)$ for $|z| \to \infty$, $|\arg z| < \pi$ is given by

$$\Gamma(z) \to \sqrt{2\pi}\, z^{z-\frac{1}{2}} \exp[z] .$$

Using this property, we can analyse the asymptotic behavior of $f_l(z)$ as $|z| \to \infty$, $|\arg z| < \pi$ and we find that $f_l(z)$ tends to zero in the left half-plane of the z complex plane, except on the negative real axis; but becomes strongly divergent on the right half-plane, as $\Gamma(z)$ is.

We conclude that our Jost function does not satisfy dispersion relations of the usual type.

* * *

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RIASSUNTO (*)

Si computano esplicitamente le funzioni di Jost per l'oscillatore armonico (in una e tre dimensioni). Esse sono funzioni analitiche intere nel piano E complesso. I loro zeri dànno i ben noti stati legati del sistema. Si dà una rappresentazione integrale delle funzioni di Jost dell'oscillatore armonico perturbato.

(*) *Traduzione a cura della Redazione.*

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