Numer. Math. (2004) 98: 389–425
Digital Object Identifier (DOI) 10.1007/s00211-004-0523-z

Numerische
Mathematik

Accurate pressure post-process of a finite element method for elastoacoustics

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Received January 11, 2002 /
Published online July 14, 2004 –

Summary. This paper deals with a post-process to obtain a more accurate approximation of the fluid pressure from a finite element computation of the vibration modes of a fluid-structure coupled system. The underlying finite element method, based on a displacement formulation for both media, consists of using *Raviart-Thomas* elements for the fluid combined with standard continuous elements for the solid.

An easy to compute post-process of the pressure is derived. The relation between this post-process and an alternative finite element approximation of the problem based on discretizing the fluid pressure by enriched *Crouzeix-Raviart* elements is studied. Higher order estimates for the L^2 norm of the post-processed pressure are proved by exploiting this relation. As a by-product, higher order L^2 estimates for the solid displacements obtained with the original method are also proved.

⋆ Member of CIC, Provincia de Buenos Aires, Argentina
⋆⋆ Member of CONICET, Argentina. Partially supported by FONDECYT 7.990.075 and FONDAP in Applied Mathematics, Chile
⋆⋆⋆ Partially supported by FONDECYT 1.990.346 and FONDAP in Applied Mathematics, Chile

1 Introduction

The need of computing fluid-solid interactions arises in many important engineering problems. A large amount of work has been devoted to this subject during the last years. A general overview can be found in the monographs [10, 19], where numerical methods and further references are also given.

This paper deals with one of these interactions: the *elastoacoustic* vibration problem. It concerns with the determination of harmonic vibrations of an elastic structure interacting with a compressible fluid. In this case, the displacements are small and then we can suppose a linear response of the structure. We neglect gravity effects and consider a homogeneous fluid, for which its reference density is constant. Other usual simplifications for this kind of problems are that viscous effects are not relevant and that velocities are small enough for convective effects to be neglected (see, for instance, [19]).

The problem of determining the vibrations of a fluid is usually treated by choosing the pressure as primary variable. However, for coupled systems, such choice leads to non-symmetric eigenvalue problems, whose computational solution involves considerable complications. To avoid this drawback the fluid has been described using different variables (see, for instance, [13, 18, 19]).

Since the solid is generally described in terms of displacements, to choose the same variable for the fluid presents several advantages. In particular, this approach could be in principle applied to the solution of a broad range of problems [4] and it leads to sparse symmetric matrices. Nevertheless, it is well known that the displacement formulation for the fluid suffers from the presence of zero-frequency spurious circulation modes with no physical meaning [18]. After discretization by standard finite elements, these modes are approximated by others with non-zero frequencies interspersed among the physical ones.

Several approaches have been proposed to circumvent this drawback [17, 4, 14]. One particularly successful has been introduced and analyzed in [5, 9]. It consists of using lowest-order Raviart-Thomas elements for the fluid displacements and standard Courant elements for those of the solid, both coupled in a weak way across the fluid-solid interface. This method has been extended to deal with, for instance, three-dimensional problems [8], incompressible fluids [6], dissipative acoustics [7], etc.

The fluid pressure is typically the most relevant variable in acoustic applications. However, when Raviart-Thomas elements are used on a displacement formulation, the computed pressure is only piecewise constant and, thus, the order of this approximation is necessarily low.

In this paper we introduce a post-process to compute a more accurate approximation of the pressure by means of piecewise linear Crouzeix-Raviart elements. This post-process requires only explicit local computations involving the computed approximate solution and element fluid mass matrices from the Raviart-Thomas discretization.

We prove higher order error estimates for the L^2 norm of the post-processed pressure. To do this, we show the relation between this post-process an other finite element approximation of the same problem based on discretizing the pressure by Crouzeix-Raviart elements enriched with element cubic bubbles. Let us remark that the solution of this alternative finite element method is only needed to prove the estimates, but not for the actual computation of the post-process. As a by-product, we also prove higher order L^2 estimates for the solid displacements computed with the original method.

The outline of the paper is as follows. We recall in Sect. 2 the fluid-structure vibration problem and the main convergence results when the fluid displacements are discretized by Raviart-Thomas elements. In Sect. 3 we introduce the post-process and prove its relation with the enriched Crouzeix-Raviart elements. We prove in Sect. 4 that this discretization is of higher order in L^2 norm and use this to conclude the same order of approximation for the post-processed pressure. This section requires of several technical lemmas whose statements and proofs are postponed to the last section.

2 The fluid-structure vibration problem

We consider the problem of determining the free vibration modes of a linear elastic structure containing an ideal acoustic (barotropic, inviscid, and compressible) fluid. Our model problem consists of a two-dimensional vessel completely filled with fluid as that in Fig. 1.

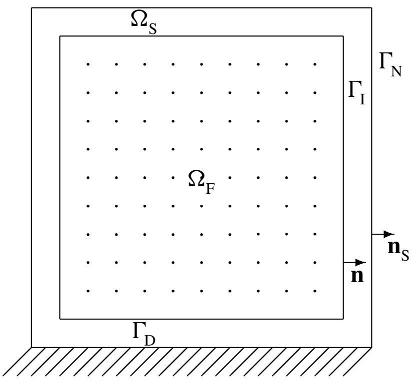

Fig. 1. Fluid and solid domains

Let Ω_F and Ω_S denote polygonal domains occupied by fluid and solid, respectively. Let Γ_I be the interface between both media, and $\mathbf{n}$ its unit normal vector pointing outwards Ω_F. We denote Γ_j, $j = 1, \dots, J$, the edges of the polygonal Γ_I (namely, $\Gamma_I = \cup_{j=1}^{J} \Gamma_j$). The exterior boundary of the solid is the union of Γ_D and Γ_N, the structure being fixed along Γ_D and free of stress along Γ_N; we assume $|\Gamma_D| > 0$. Finally, $\mathbf{n}_S$ denotes the unit outward normal vector along Γ_N.

Throughout this paper we use standard notation for Sobolev spaces, norms, and seminorms. We also denote $H^1_{\Gamma_D}(\Omega_S)$ the closed subspace of functions in $H^1(\Omega_S)$ with a vanishing trace on Γ_D, and $H(\mathrm{div}, \Omega_F) := \{\mathbf{u} \in L^2(\Omega_F)^2 : \mathrm{div}\, \mathbf{u} \in L^2(\Omega_F)\}$, with its corresponding norm defined by $\|\mathbf{u}\|^2_{\mathrm{div},\Omega_F} := \|\mathbf{u}\|^2_{0,\Omega_F} + \|\mathrm{div}\, \mathbf{u}\|^2_{0,\Omega_F}$.

We use the following notation for the physical magnitudes; in the fluid:

$\mathbf{u}$: the displacement field,
p: the pressure,
ρ_F: the density,
c: the acoustic speed,

and in the solid:

$\mathbf{w}$: the displacement field,
ρ_S: the density,
λ_S and μ_S: the Lamé coefficients,
$\boldsymbol{\varepsilon}(\mathbf{w})$: the strain tensor defined by $\varepsilon_{ij}(\mathbf{w}) := \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)$, $i, j = 1, 2$,
$\boldsymbol{\sigma}(\mathbf{w})$: the stress tensor which we assume related to the strain tensor by Hooke's law:

$$\sigma_{ij}(\mathbf{w}) = \lambda_S \sum_{k=1}^{2} \varepsilon_{kk}(\mathbf{w})\delta_{ij} + 2\mu_S \varepsilon_{ij}(\mathbf{w}), \quad i, j = 1, 2.$$

The classical elastoacoustics approximation for small amplitude motions yields the following eigenvalue problem for the free vibration modes of the coupled system and their corresponding frequencies ω (see for instance [19]):

Find $\omega > 0$, $\mathbf{u} \in H(\mathrm{div}, \Omega_F)$, $\mathbf{w} \in H^1(\Omega_S)^2$ *and* $p \in H^1(\Omega_F)$, $(\mathbf{u}, \mathbf{w}, p) \neq (\mathbf{0}, \mathbf{0}, 0)$, *such that:*

$$\nabla p - \omega^2 \rho_F \mathbf{u} = \mathbf{0} \ \text{ in } \Omega_F, \tag{2.1}$$

$$p + \rho_F c^2 \,\mathrm{div}\, \mathbf{u} = 0 \ \text{ in } \Omega_F, \tag{2.2}$$

$$\mathbf{div}\, [\boldsymbol{\sigma}(\mathbf{w})] + \omega^2 \rho_S \mathbf{w} = \mathbf{0} \ \text{ in } \Omega_S, \tag{2.3}$$

$$\boldsymbol{\sigma}(\mathbf{w})\mathbf{n} + p\mathbf{n} = \mathbf{0} \ \text{ on } \Gamma_I, \tag{2.4}$$

$$\mathbf{u} \cdot \mathbf{n} - \mathbf{w} \cdot \mathbf{n} = 0 \ \text{ on } \Gamma_I, \tag{2.5}$$

$$\boldsymbol{\sigma}(\mathbf{w})\mathbf{n}_S = \mathbf{0} \ \text{ on } \Gamma_N, \tag{2.6}$$

$$\mathbf{w} = \mathbf{0} \ \text{ on } \Gamma_D. \tag{2.7}$$

Two different variables are used in the equations above to describe the fluid: pressure and displacements. Each of them can be eliminated in terms of the other to obtain two alternative variational formulations of the problem above.

By multiplying equations (2.1) and (2.3) by adequate test functions, integrating by parts, and using (2.2) to eliminate the pressure p in terms of the fluid displacement $\mathbf{u}$, we obtain the following symmetric pure displacement formulation:

Find $\lambda \in \mathbb{R}$ *and* $(\mathbf{u}, \mathbf{w}) \in \mathcal{V}$, $(\mathbf{u}, \mathbf{w}) \neq (\mathbf{0}, \mathbf{0})$, *such that:*

$$\begin{aligned}(2.8)\qquad &\int_{\Omega_F} \rho_F c^2 \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} + \int_{\Omega_S} \boldsymbol{\sigma}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{z}) \\ &= \lambda \left(\int_{\Omega_F} \rho_F \mathbf{u} \cdot \mathbf{v} + \int_{\Omega_S} \rho_S \mathbf{w} \cdot \mathbf{z} \right) \quad \forall (\mathbf{v}, \mathbf{z}) \in \mathcal{V},\end{aligned}$$

In the problem above, $\lambda = \omega^2$ and $\mathcal{V}$ is the space of coupled displacements satisfying the so-called *kinematic constraint* (2.5), namely:

$$\mathcal{V} := \left\{ (\mathbf{u}, \mathbf{w}) \in \mathcal{X} : \ \mathbf{u} \cdot \mathbf{n} = \mathbf{w} \cdot \mathbf{n} \ \text{ on } \Gamma_I \right\},$$

with

$$\mathcal{X} := \mathrm{H}(\operatorname{div}, \Omega_F) \times \mathrm{H}^1_{\Gamma_D}(\Omega_S)^2.$$

On the other hand, by multiplying equations (2.2) and (2.3) by adequate test functions, integrating by parts, and using (2.1) to eliminate the displacement $\mathbf{u}$ in terms of the pressure p, we obtain the following non-symmetric pressure/displacement formulation:

Find $\lambda \in \mathbb{R}$ *and* $(p, \mathbf{w}) \in \mathrm{H}^1(\Omega_F) \times \mathrm{H}^1_{\Gamma_D}(\Omega_S)^2$, $(p, \mathbf{w}) \neq (0, \mathbf{0})$, *such that:*

$$\begin{aligned}(2.9)\qquad &\int_{\Omega_F} \frac{1}{\rho_F} \nabla p \cdot \nabla q + \int_{\Omega_S} \boldsymbol{\sigma}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{z}) - \int_{\Gamma_I} p \, \mathbf{z} \cdot \mathbf{n} \\ &= \lambda \left(\int_{\Omega_F} \frac{1}{\rho_F c^2} pq + \int_{\Omega_S} \rho_S \mathbf{w} \cdot \mathbf{z} + \int_{\Gamma_I} \mathbf{w} \cdot \mathbf{n} \, q \right) \\ &\qquad\qquad \forall (q, \mathbf{z}) \in \mathrm{H}^1(\Omega_F) \times \mathrm{H}^1_{\Gamma_D}(\Omega_S)^2.\end{aligned}$$

Both variational problems attain the same non-zero eigenvalues with corresponding eigenfunctions with the same solid displacements, and fluid variables related by equations (2.1) and (2.2). More precisely, the following lemma holds:

Lemma 2.1 *Let* $(\lambda, (\mathbf{u}, \mathbf{w}))$ *be an eigenpair of Problem (2.8) with* $\lambda \neq 0$ *and let* $p = -\rho_{\mathrm{F}} c^2 \operatorname{div} \mathbf{u}$. *Then,* $(\lambda, (p, \mathbf{w}))$ *is an eigenpair of Problem (2.9).*

Conversely, let $(\lambda, (p, \mathbf{w}))$ *be an eigenpair of Problem (2.9) with* $\lambda \neq 0$ *and let* $\mathbf{u} = \frac{1}{\lambda \rho_{\mathrm{F}}} \nabla p$. *Then,* $(\lambda, (\mathbf{u}, \mathbf{w}))$ *is an eigenpair of Problem (2.8).*

Proof. For $\lambda = \omega^2 \neq 0$, both problems are equivalent to Problem (2.1)-(2.7). In fact, as it was mentioned above, the solutions of (2.1)-(2.7) satisfy equations (2.8) and (2.9). Conversely, by testing each of these two equations with adequate smooth functions, it is easy to show that, any solution of each of them, with $\lambda \neq 0$, also satisfy (2.1)-(2.7). □

The following spectral characterization was proved in [5]:

Theorem 2.1 *Problem (2.8) has two kinds of solutions:*

1. $\lambda_0 = 0$, *with corresponding eigenspace*

$$\mathcal{K} := \{(\mathbf{u}, \mathbf{0}) \in \mathcal{V} : \ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega_{\mathrm{F}} \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{I}}\};$$

2. *a sequence of finite-multiplicity strictly positive eigenvalues* λ_n, $n \in \mathbb{N}$, *converging to* $+\infty$, *with corresponding eigenfunctions* $(\mathbf{u}_n, \mathbf{w}_n) \in \mathcal{V}$, *satisfying* $\mathbf{u}_n = \nabla \varphi_n$ *for some* $\varphi_n \in \mathrm{H}^1(\Omega_{\mathrm{F}})$.

By virtue of Lemma 2.1, the theorem above shows that the spectrum of the non-symmetric problem (2.9) consists of the same non-negative real eigenvalues converging to $+\infty$. The following result concerning further regularity for the eigenfunctions of Problem (2.8) (and hence of those of (2.9) too) was proved in [5,21]:

Theorem 2.2 *Let* $(\mathbf{u}, \mathbf{w})$ *be an eigenfunction of problem (2.8) associated with an eigenvalue* $\lambda > 0$. *Let* $p = -\rho_{\mathrm{F}} c^2 \operatorname{div} \mathbf{u}$ *be the corresponding fluid pressure. Then, there exist constants* $s \in (\frac{1}{2}, 1]$, $t \in (0, 1]$, *and* $C > 0$, *such that* $\mathbf{u} \in \mathrm{H}^s(\Omega_{\mathrm{F}})^2$, $\mathbf{w} \in \mathrm{H}^{1+t}(\Omega_{\mathrm{S}})^2$, $p \in \mathrm{H}^{1+s}(\Omega_{\mathrm{F}})$, *and*

$$\|\mathbf{u}\|_{s,\Omega_{\mathrm{F}}} + \|\mathbf{w}\|_{1+t,\Omega_{\mathrm{S}}} + \|p\|_{1+s,\Omega_{\mathrm{F}}} \leq C \left(\|\mathbf{u}\|_{0,\Omega_{\mathrm{F}}} + \|\mathbf{w}\|_{0,\Omega_{\mathrm{S}}} \right).$$

In this theorem, s is either 1, if Ω_{F} is convex, or any $s < \frac{\pi}{\theta}$, with θ being the largest reentrant corner, otherwise. On the other hand, t depends on the reentrant angles of Ω_{S}, the angles between Γ_{D} and Γ_{N}, and the Lamé coefficients (see [16]).

The pressure/displacement formulation (2.9) was the first one considered in the literature (see for instance [22]). However, it has not been so widely used because of the non-symmetric character of the problem. In fact, after discretization, it leads to a non-symmetric generalized eigenvalue problem, which hinders the use of most standard eigensolvers.

Instead, the pure displacement formulation (2.8) leads to a sparse symmetric generalized eigenvalue problem. However, a severe drawback of this approach was early noticed (see [18] and [17]): non-zero frequency spurious modes pollute the spectrum when standard conforming finite elements are used to discretize the fluid displacements. Indeed, in such case, divergence free flows are approximated by nearly divergence free elements only. Therefore, the splitting described by Theorem 2.1 is spoiled by the discretization.

This is different for Raviart-Thomas elements. Because of this, they have been used to discretize the fluid displacements in [5,9], where it was shown that these elements, conveniently coupled with standard conforming elements for the solid, lead to a spurious-free method. We end this section by recalling this method and some of its approximation properties which will be used in the sequel.

Let $\{\mathcal{T}_h\}$ be a family of triangulations of $\bar{\Omega}_F \cup \bar{\Omega}_S$, regular in the sense of a minimum angle condition. The index h denotes, as usual, the mesh size of $\mathcal{T}_h$: $h := \max_{T\in\mathcal{T}_h} h_T$, with h_T being the diameter of T. We assume that all the triangles of each mesh are completely contained either in $\bar{\Omega}_F$ or in $\bar{\Omega}_S$, and that the end points of Γ_D coincide with nodes of the triangulation. We denote by $\mathcal{T}_h^F$ and $\mathcal{T}_h^S$ the triangulations induced by $\mathcal{T}_h$ in $\bar{\Omega}_F$ and $\bar{\Omega}_S$, respectively.

Let $\mathcal{E}_h$ denote the set of all the edges of triangles $T \in \mathcal{T}_h^F$. We split this set as follows: $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^o$, with $\mathcal{E}_h^I := \{\ell \in \mathcal{E}_h : \ \ell \subset \Gamma_I\}$ and $\mathcal{E}_h^o := \{\ell \in \mathcal{E}_h : \ \ell \not\subset \Gamma_I\}$ being the sets of boundary and inner edges, respectively. For each inner edge $\ell \in \mathcal{E}_h^o$, we choose a unit vector normal to ℓ which we denote $\mathbf{n}_\ell$. We also denote $[\![\cdot]\!]_\ell$ the jump across ℓ along $\mathbf{n}_\ell$.

Let $\mathcal{RT}_h(\Omega_F)$ be the lowest-order Raviart-Thomas space on $\mathcal{T}_h^F$ (see [20]):

$$\mathcal{RT}_h(\Omega_F) := \left\{\mathbf{u}_h \in \mathrm{H}(\mathrm{div}, \Omega_F) : \ \mathbf{u}_h|_T \in \mathcal{RT}_0(T) \ \ \forall T \in \mathcal{T}_h^F\right\},$$

where

$$\mathcal{RT}_0(T) := \left\{\mathbf{u}_h \in \mathcal{P}_1(T)^2 : \ \mathbf{u}_h(x, y) = (a + bx, c + by), \ a, b, c \in \mathbb{R}\right\}.$$

Let

$$\mathcal{L}_h(\Omega_S) := \left\{w_h \in \mathrm{H}^1(\Omega_S) : \ w_h|_T \in \mathcal{P}_1(T) \ \ \forall T \in \mathcal{T}_h^S\right\}$$

be the standard finite element space of piecewise linear continuous functions, and $\mathcal{L}_{h\Gamma_D}(\Omega_S) := \{w_h \in \mathcal{L}_h(\Omega_S) : \ w_h = 0 \text{ on } \Gamma_D\}$. Let

$$\mathcal{X}_h := \mathcal{RT}_h(\Omega_F) \times \mathcal{L}_{h\Gamma_D}(\Omega_S)^2$$

and

$$\mathcal{V}_h := \left\{(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{X}_h : \ \int_\ell (\mathbf{u}_h \cdot \mathbf{n} - \mathbf{w}_h \cdot \mathbf{n}) = 0 \ \ \forall \ell \in \mathcal{E}_h^I\right\}.$$

Thus, the following discrete analogue of the spectral problem (2.8) is obtained:

Find $\lambda_h \in \mathbb{R}$ *and* $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{V}_h$, $(\mathbf{u}_h, \mathbf{w}_h) \neq (\mathbf{0}, \mathbf{0})$, *such that:*

$$\text{(2.10)}\quad \int_{\Omega_F} \rho_F c^2 \operatorname{div} \mathbf{u}_h \operatorname{div} \mathbf{v}_h + \int_{\Omega_S} \boldsymbol{\sigma}(\mathbf{w}_h) : \boldsymbol{\varepsilon}(\mathbf{z}_h) = \lambda_h \left(\int_{\Omega_F} \rho_F \mathbf{u}_h \cdot \mathbf{v}_h + \int_{\Omega_S} \rho_S \mathbf{w}_h \cdot \mathbf{z}_h \right) \quad \forall (\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{V}_h.$$

The following spectral characterization of the discrete problem above has been proved in [5]:

Theorem 2.3 *Problem (2.10) has two kinds of solutions:*

1. $\lambda_0 = 0$, *with corresponding eigenspace*

$$\mathcal{K}_h := \{(\mathbf{u}_h, \mathbf{0}) \in \mathcal{V}_h : \ \operatorname{div} \mathbf{u}_h = 0 \text{ in } \Omega_F \ \text{ and } \ \mathbf{u}_h \cdot \mathbf{n} = 0 \text{ on } \Gamma_I\};$$

2. *a set of positive eigenvalues* λ_h, *with corresponding eigenfunctions* $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{V}_h$, *such that* $\mathbf{u}_h \in \mathcal{K}_h^{\perp}$ *(where* $\mathcal{K}_h^{\perp}$ *denotes the orthogonal complement of* $\mathcal{K}_h$ *in* $\mathcal{V}_h$*).*

Non-existence of spurious modes and spectral convergence of the solutions of (2.10) to those of (2.8), with optimal order error estimates, have been proved in [5,21], including a double order for the eigenvalues. In particular, the following error estimates hold:

Theorem 2.4 *Let* $(\lambda, (\mathbf{u}, \mathbf{w}))$ *be an eigenpair of Problem (2.8) with* $\lambda > 0$. *Then, there exist strictly positive constants* C *and* h_0 *such that, if* $h \leq h_0$, *Problem (2.10) attains an eigenpair* $(\lambda_h, (\mathbf{u}_h, \mathbf{w}_h))$, *with* $\lambda_h > 0$, *satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\operatorname{div}, \Omega_F} + \|\mathbf{w} - \mathbf{w}_h\|_{1, \Omega_S} \leq C h^r \left(\|\mathbf{u}\|_{0, \Omega_F} + \|\mathbf{w}\|_{0, \Omega_S} \right)$$

and

$$|\lambda - \lambda_h| \leq C h^{2r},$$

where $r := \min\{s, t\}$, *with* s *and* t *being the regularity constants in Theorem 2.2.*

Here and thereafter C denotes a positive constant independent of the mesh size h.

Notice that, as a consequence of this theorem, the discrete pressure $-\rho_F c^2 \operatorname{div} \mathbf{u}_h$ approximate the pressure $p = -\rho_F c^2 \operatorname{div} \mathbf{u}$ also with order $\mathcal{O}(h^r)$. Since $\operatorname{div} \mathbf{u}_h$ is only piecewise constant for Raviart-Thomas elements, this order is optimal. In the following section we introduce a post-process to compute a more accurate approximation of the pressure by means of piecewise linear Crouzeix-Raviart elements.

3 A post-processed pressure

We consider the family of meshes $\{\mathcal{T}_h\}$ and the notation introduced above. Let

$$\mathcal{CR}_h(\Omega_{\mathrm{F}}) := \{q_h \in \mathrm{L}^2(\Omega_{\mathrm{F}}) : \; q_h|_T \in \mathcal{P}_1(T) \; \forall T \in \mathcal{T}_h^{\mathrm{F}}, \\ q_h \text{ continuous at midpoints of all } \ell \in \mathcal{E}_h^{\mathrm{o}}\},$$

be the space of lowest-order Crouzeix-Raviart elements. Let $\{\psi_\ell : \; \ell \in \mathcal{E}_h\}$ be the natural basis of this space consisting of piecewise linear functions attaining the value 1 at the midpoint of ℓ and vanishing at the midpoints of all the other edges $\ell' \neq \ell$.

For $(\lambda_h, (\mathbf{u}_h, \mathbf{w}_h))$ a solution of Problem (2.10), we define the post-processed pressure p_h^{L} by

$$p_h^{\mathrm{L}} := \sum_{\ell \in \mathcal{E}_h} \alpha_\ell \psi_\ell, \tag{3.1}$$

with

$$\alpha_\ell := -\rho_{\mathrm{F}} c^2 \operatorname{div} (\mathbf{u}_h|_T) + \frac{\lambda_h}{|\ell|} \int_T \rho_{\mathrm{F}} \mathbf{u}_h \cdot \boldsymbol{\phi}_\ell, \tag{3.2}$$

where $T \in \mathcal{T}_h^{\mathrm{F}}$ is a triangle such that $\ell \subset \partial T$ and $\boldsymbol{\phi}_\ell$ is the basis function of the Raviart-Thomas space on T, $\mathcal{RT}_0(T)$, associated with ℓ (i.e., the constant outer normal component of $\boldsymbol{\phi}_\ell$ is equal to 1 on ℓ and 0 on the other edges of T). We show below that this definition does not depend on the chosen triangle T with $\ell \subset \partial T$. On the other hand, let us remark that the integral in the definition of α_ℓ can be easily computed by using the same element fluid mass matrix used to solve Problem (2.10).

The aim of this paper is to prove that p_h^{L} is a higher order approximation of the pressure p. To do this, we will show first that p_h^{L} can also be seen as the linear part of the computed pressure obtained from a particular discretization of Problem (2.9), based on using Crouzeix-Raviart piecewise linear elements enriched with local cubic bubbles for the fluid pressure.

It is well known that Crouzeix-Raviart elements are strongly related with Raviart-Thomas ones when applied to the Laplace equation (see Section 2 of [2]; in particular Lemma 2.4 therein). In what follows, we show that similar results hold for the elastoacoustic vibration problem. Let

$$\begin{aligned} P &: \mathrm{L}^2(\Omega_{\mathrm{F}}) \longrightarrow \mathcal{Q}_h, \\ P_\ell &: \mathrm{L}^2(\ell) \longrightarrow \mathcal{P}_0(\ell) \qquad \forall \ell \in \mathcal{E}_h, \\ P_{\Gamma_{\mathrm{I}}} &: \mathrm{L}^2(\Gamma_{\mathrm{I}}) \longrightarrow \mathcal{C}_h, \\ \Pi_T &: \mathrm{L}^2(T)^2 \longrightarrow \mathcal{RT}_0(T) \qquad \forall T \in \mathcal{T}_h^{\mathrm{F}} \end{aligned}$$

be the corresponding L^2 orthogonal projections, with

$$\mathcal{Q}_h := \left\{q_h \in \mathrm{L}^2(\Omega_{\mathrm{F}}) : \; q_h|_T \in \mathcal{P}_0(T) \; \forall T \in \mathcal{T}_h^{\mathrm{F}}\right\}$$

and

$$\mathcal{C}_h := \{\delta_h \in \mathrm{L}^2(\Gamma_{\mathrm{I}}) : \ \delta_h|_\ell \in \mathcal{P}_0(\ell) \ \forall \ell \in \mathcal{E}_h^{\mathrm{I}}\}.$$

Let $\mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}})$ be the space of piecewise linear Crouzeix-Raviart elements enriched with local cubic bubbles; namely,

$$\mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}}) := \mathcal{CR}_h(\Omega_{\mathrm{F}}) \oplus \mathcal{B}_h(\Omega_{\mathrm{F}}),$$

with

$$\mathcal{B}_h(\Omega_{\mathrm{F}}) := \left\{\beta_h \in \mathrm{H}^1(\Omega_{\mathrm{F}}) : \ \beta_h|_T \in \mathrm{H}_0^1(T) \cap \mathcal{P}_3(T)\right\}.$$

Consider the following discretization of Problem (2.9):

Find $\lambda_h \in \mathbb{R}$ *and* $(p_h, \mathbf{w}_h) \in \mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}}) \times \mathcal{L}_{h\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})^2$, $(p_h, \mathbf{w}_h) \neq (0, \mathbf{0})$, *such that:*

(3.3)
$$\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \int_T \frac{1}{\rho_{\mathrm{F}}} \Pi_T \nabla p_h \cdot \Pi_T \nabla q_h + \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{w}_h) : \boldsymbol{\varepsilon}(\mathbf{z}_h) - \int_{\Gamma_{\mathrm{I}}} P_{\Gamma_{\mathrm{I}}} p_h \, \mathbf{z}_h \cdot \mathbf{n}$$
$$= \lambda_h \left(\int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}} c^2} P p_h \, P q_h + \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} \mathbf{w}_h \cdot \mathbf{z}_h + \int_{\Gamma_{\mathrm{I}}} \mathbf{w}_h \cdot \mathbf{n} \, P_{\Gamma_{\mathrm{I}}} q_h \right)$$
$$\forall (q_h, \mathbf{z}_h) \in \mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}}) \times \mathcal{L}_{h\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})^2.$$

For $\lambda_h \neq 0$, the eigenvalue problem above is equivalent to Problem (2.10) in a sense made precise in the following lemma:

Lemma 3.1 *Let* $(\lambda_h, (\mathbf{u}_h, \mathbf{w}_h))$ *be an eigenpair of Problem (2.10) with* $\lambda_h \neq 0$. *Then, there exists* $p_h \in \mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}})$ *such that* $(\lambda_h, (p_h, \mathbf{w}_h))$ *is an eigenpair of Problem (3.3). Moreover,* p_h *and* $\mathbf{u}_h$ *are related by*

$$P p_h = -\rho_{\mathrm{F}} c^2 \operatorname{div} \mathbf{u}_h, \tag{3.4}$$

and

$$\Pi_T \nabla p_h = \lambda_h \rho_{\mathrm{F}} \mathbf{u}_h|_T \qquad \forall T \in \mathcal{T}_h^{\mathrm{F}}. \tag{3.5}$$

Conversely, let $(\lambda_h, (p_h, \mathbf{w}_h))$ *be an eigenpair of Problem (3.3) with* $\lambda_h \neq 0$. *Then, there exists* $\mathbf{u}_h \in \mathcal{RT}_h(\Omega_{\mathrm{F}})$ *such that* $(\lambda_h, (\mathbf{u}_h, \mathbf{w}_h))$ *is a solution of Problem (2.10), and (3.4)–(3.5) hold true.*

Proof. Let $\mathcal{RT}_h^{\mathrm{d}}(\Omega_{\mathrm{F}})$ be the space of discontinuous Raviart-Thomas fields on Ω_{F}

$$\mathcal{RT}_h^{\mathrm{d}}(\Omega_{\mathrm{F}}) := \left\{\mathbf{v}_h \in \mathrm{L}^2(\Omega_{\mathrm{F}})^2 : \ \mathbf{v}_h|_T \in \mathcal{RT}_0(T) \ \forall T \in \mathcal{T}_h^{\mathrm{F}}\right\}.$$

First, we show that $(\lambda_h, (\mathbf{u}_h, \mathbf{w}_h))$ is an eigenpair of Problem (2.10) if and only if there exists unique

$$\gamma_h \in \mathcal{C}_h \qquad \text{and} \qquad \gamma_h' \in \mathcal{C}_h' := \left\{ \delta_h \in \prod\nolimits_{\ell \in \mathcal{E}_h^{\mathrm{o}}} \mathrm{L}^2(\ell) : \ \delta_h|_\ell \in \mathcal{P}_0(\ell) \right\},$$

such that

(3.6)
$$\int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 \operatorname{div} \mathbf{u}_h \operatorname{div} \mathbf{v}_h + \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{w}_h) : \boldsymbol{\varepsilon}(\mathbf{z}_h) + \int_{\Gamma_{\mathrm{I}}} \gamma_h \, (\mathbf{v}_h \cdot \mathbf{n} - \mathbf{z}_h \cdot \mathbf{n})$$
$$+ \sum_{\ell \in \mathcal{E}_h^{\mathrm{o}}} \int_\ell \gamma_h' [\![\mathbf{v}_h \cdot \mathbf{n}_\ell]\!]_\ell = \lambda_h \left(\int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \mathbf{u}_h \cdot \mathbf{v}_h + \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} \mathbf{w}_h \cdot \mathbf{z}_h \right)$$
$$\forall (\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{RT}_h^{\mathrm{d}}(\Omega_{\mathrm{F}}) \times \mathcal{L}_{h\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})^2,$$

(3.7) $$\int_{\Gamma_{\mathrm{I}}} (\mathbf{u}_h \cdot \mathbf{n} - \mathbf{w}_h \cdot \mathbf{n}) \delta_h = 0 \qquad \forall \delta_h \in \mathcal{C}_h,$$

(3.8) $$\sum_{\ell \in \mathcal{E}_h^{\mathrm{o}}} \int_\ell [\![\mathbf{u}_h \cdot \mathbf{n}_\ell]\!]_\ell \delta_h' = 0 \qquad \forall \delta_h' \in \mathcal{C}_h'.$$

Clearly, any solution of this problem provides a solution of Problem (2.10). Indeed, $\mathbf{u}_h \in \mathcal{RT}_h(\Omega_{\mathrm{F}})$ because of (3.8), whereas $\int_\ell (\mathbf{u}_h \cdot \mathbf{n} - \mathbf{w}_h \cdot \mathbf{n}) = 0$ $\forall \ell \in \mathcal{E}_h^{\mathrm{I}}$ because of (3.7). Hence $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{V}_h$, and (3.6) implies (2.10).

The converse is also true. In fact, let λ_h and $(\mathbf{u}_h, \mathbf{w}_h)$ be an eigenpair of Problem (2.10). Equations (3.7) and (3.8) are satisfied since $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{V}_h$, whereas equation (3.6) is true for $(\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{V}_h$, independently of the particular values of $\gamma_h \in \mathcal{C}_h$ and $\gamma_h' \in \mathcal{C}_h$. Now, for each edge $\ell \in \mathcal{E}_h$, let $T \in \mathcal{T}_h^{\mathrm{F}}$ be one triangle such that $\ell \subset \partial T$ (if $\ell \in \mathcal{E}_h^{\mathrm{I}}$, there is only one such triangle; if $\ell \in \mathcal{E}_h^{\mathrm{o}}$, we choose T such that $\mathbf{n}_\ell$ is the outer normal to ∂T). Let $\boldsymbol{\phi}_\ell$ denote the nodal basis functions of $\mathcal{RT}_0(T)$ associated with ℓ as in (3.2), extended by zero to the rest of Ω_{F}. Then $\mathcal{RT}_h^{\mathrm{d}}(\Omega_{\mathrm{F}}) \times \mathcal{L}_{h\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})^2 = \mathcal{V}_h \oplus \langle \{ (\boldsymbol{\phi}_\ell, \mathbf{0}) : \ell \in \mathcal{E}_h \} \rangle$. Thus, it is enough to prove that there exist unique $\gamma_h \in \mathcal{C}_h$ and $\gamma_h' \in \mathcal{C}_h'$ such that (3.6) holds for $(\mathbf{v}_h, \mathbf{z}_h) = (\boldsymbol{\phi}_\ell, \mathbf{0})$ $\forall \ell \in \mathcal{E}_h$. To prove this, let $\{\chi_\ell : \ \ell \in \mathcal{E}_h\}$ denote the canonical basis of $\mathcal{C}_h \oplus \mathcal{C}_h'$ (i.e., $\chi_\ell|_\ell \equiv 1$ and $\chi_\ell|_{\ell'} \equiv 0$ $\forall \ell' \in \mathcal{E}_h, \ell' \neq \ell$). Then, by writing in this basis $\gamma_h = \sum_{\ell \in \mathcal{E}_h^{\mathrm{I}}} c_\ell \chi_\ell$ and $\gamma_h' = \sum_{\ell \in \mathcal{E}_h^{\mathrm{o}}} c_\ell' \chi_\ell$, it is clear that it is enough to verify that there are unique coefficients c_ℓ and c_ℓ' such that (3.6) holds true for $(\mathbf{v}_h, \mathbf{z}_h) = (\boldsymbol{\phi}_\ell, \mathbf{0})$ $\forall \ell \in \mathcal{E}_h$; that is

$$|\ell| c_\ell = \sum_{\ell' \in \mathcal{E}_h^{\mathrm{I}}} \left(\int_\ell \chi_{\ell'} \boldsymbol{\phi}_\ell \cdot \mathbf{n}_\ell \right) c_{\ell'}$$
$$= \lambda_h \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \mathbf{u}_h \cdot \boldsymbol{\phi}_\ell - \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 \operatorname{div} \mathbf{u}_h \operatorname{div} \boldsymbol{\phi}_\ell \qquad \forall \ell \in \mathcal{E}_h^{\mathrm{I}}.$$

$$|\ell|c'_\ell = \sum_{\ell'\in\mathcal{E}_h^{\mathrm{o}}} \left(\int_\ell \chi_{\ell'}\boldsymbol{\phi}_\ell \cdot \mathbf{n}_\ell\right) c'_{\ell'}$$
$$= \lambda_h \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \mathbf{u}_h \cdot \boldsymbol{\phi}_\ell - \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 \operatorname{div} \mathbf{u}_h \operatorname{div} \boldsymbol{\phi}_\ell \qquad \forall \ell \in \mathcal{E}_h^{\mathrm{o}}.$$

Since these equations are clearly uniquely solvable, we conclude that there exists unique $\gamma_h \in \mathcal{C}_h$ and $\gamma'_h \in \mathcal{C}'_h$ such that (3.6)-(3.8) hold true.

Then, the arguments in [2] can be readily adapted to prove the lemma, with $p_h \in \mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}})$ being defined by

$$P_\ell p_h = \begin{cases} \gamma_h|_\ell & \text{if } \ell \in \mathcal{E}_h^{\mathrm{I}}, \\ \gamma'_h|_\ell & \text{if } \ell \in \mathcal{E}_h^{\mathrm{o}}, \end{cases} \tag{3.9}$$

$$P p_h = -\rho_{\mathrm{F}} c^2 \operatorname{div} \mathbf{u}_h. \tag{3.10}$$

□

Remark 3.1 According to (3.9) in the proof of the previous lemma, $P_{\Gamma_{\mathrm{I}}} p_h = \gamma_h$. This function $\gamma_h \in \mathcal{C}_h$ coincides with the Lagrange multiplier introduced in [6, 8] to impose the kinematic constraint on fluid and solid displacements, which is an approximation of the interface pressure $p|_{\Gamma_{\mathrm{I}}}$.

Remark 3.2 As a by-product of this lemma and Theorem 2.3, we have proved that the eigenvalues of the non-symmetric generalized eigenvalue problem (3.3) are real and non-negative.

The following lemma shows that the pressure p_h computed from Problem (3.3) equals the post-processed pressure p_h^{L} plus bubble functions, and an estimate is given for the latter:

Lemma 3.2 *For* $(\lambda_h, (\mathbf{u}_h, \mathbf{w}_h))$ *and* $(\lambda_h, (p_h, \mathbf{w}_h))$ *as in the previous lemma, let* p_h^{L} *be defined by (3.1)-(3.2). Then*

$$p_h - p_h^{\mathrm{L}} \in \mathcal{B}_h(\Omega_{\mathrm{F}})$$

and

$$\left\| p_h - p_h^{\mathrm{L}} \right\|_{0,\Omega_{\mathrm{F}}} \le C h^2 \| p_h \|_{0,\Omega_{\mathrm{F}}}.$$

Proof. Since $p_h \in \mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}})$, we write

$$p_h = \sum_{\ell\in\mathcal{E}_h} \alpha'_\ell \psi_\ell + \beta_h,$$

with $\beta_h \in \mathcal{B}_h(\Omega_{\mathrm{F}})$. Because of (3.9), for each edge $\ell \in \mathcal{E}_h^{\mathrm{o}}$,

$$\gamma'_h|_\ell = P_\ell p_h = \frac{1}{|\ell|} \int_\ell \left(\sum_{\ell'\in\mathcal{E}_h} \alpha'_{\ell'} \psi_{\ell'} + \beta_h \right) = \alpha'_\ell.$$

On the other hand, by testing (3.6) with $(\boldsymbol{\phi}_\ell, \mathbf{0})$, we obtain

$$\gamma_h'|_\ell = -\rho_{\mathrm{F}} c^2 \operatorname{div}(\mathbf{u}_h|_T) + \frac{\lambda_h}{|\ell|} \int_T \rho_{\mathrm{F}} \mathbf{u}_h \cdot \boldsymbol{\phi}_\ell = \alpha_\ell.$$

Since the same two equalities also hold true for $\gamma_h|_\ell$, $\ell \in \mathcal{E}_h^{\mathrm{I}}$, then

$$p_h - p_h^{\mathrm{L}} = \beta_h \in \mathcal{B}_h(\Omega_{\mathrm{F}}).$$

Finally, by testing (3.3) with $(\beta_{h,T}, \mathbf{0})$, where $\beta_{h,T} \in \mathcal{B}_h(\Omega_{\mathrm{F}})$ is a bubble function supported in $T \in \mathcal{T}_h^{\mathrm{F}}$, we have

$$\int_T \frac{1}{\rho_{\mathrm{F}}} \Pi_T \nabla p_h \cdot \Pi_T \nabla \beta_{h,T} = \lambda_h \int_T \frac{1}{\rho_{\mathrm{F}} c^2} P p_h \, P \beta_{h,T}.$$

Then, the arguments in the proof of Lemma 2.1 in [12] can be easily adapted to our case to show that

$$\|\beta_h\|_{0,\Omega_{\mathrm{F}}} \leq C h^2 \|p_h\|_{0,\Omega_{\mathrm{F}}}. \qquad \square$$

Remark 3.3 As a by-product of this lemma, since p_h^{L} coincides with the Crouzeix-Raviart part of $p_h \in \mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}})$, we have that p_h^{L} is uniquely defined, independently of the triangle T chosen in (3.2) to compute α_ℓ.

Remark 3.4 The pressure approximation p_h in Problem (3.3) can also be computed directly from the displacement approximation $\mathbf{u}_h$ in Problem (2.10). Indeed, because of the previous lemma we can write

$$p_h = p_h^{\mathrm{L}} + \sum_{T \in \mathcal{T}_h^{\mathrm{F}}} \alpha_T b_T,$$

with b_T being the bubble function attaining the value 1 at the barycenter of T. Then, as a consequence of (3.10), α_T can be explicitly computed from

$$\alpha_T \frac{9}{20} = P\beta_h|_T = P p_h|_T - P p_h^{\mathrm{L}}|_T = -\rho_{\mathrm{F}} c^2 \operatorname{div}(\mathbf{u}_h|_T) + \frac{1}{3} \sum_{\ell \subset \partial T} \alpha_\ell.$$

Thus, p_h can be seen as an alternative post-processed pressure. However, this lemma shows that p_h does not approximate the pressure with a higher order than p_h^{L} (see Theorem 4.3 below).

In the following section we will study the non-conforming method (3.3). In particular, we will prove higher order error estimates in L^2 norm for p_h, which, combined with the previous lemma, will allow us to show that the post-processed pressure p_h^{L} provides an accurate approximation of the pressure.

4 Analysis of the enriched Crouzeix-Raviart approximation

First, as a direct consequence of Lemma 3.1, we show that the discrete eigenvalue problem (3.3) provides approximations of the solutions of Problem (2.1)–(2.7) with the same order as those of Problem (2.10) in H^1-like norms:

Theorem 4.1 *Let* $(\lambda, (p, \mathbf{w}))$ *be an eigenpair of Problem (2.9). Then, there exist strictly positive constants* C *and* h_0 *such that, if* $h \leq h_0$*, Problem (3.3) attains an eigenpair* $(\lambda_h, (p_h, \mathbf{w}_h))$*, with* $\lambda_h > 0$*, satisfying*

$$\left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \|p - p_h\|_{1,T}^2\right)^{\frac{1}{2}} + \|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega_{\mathrm{S}}} \leq Ch^r \left(\|\boldsymbol{\nabla} p\|_{0,\Omega_{\mathrm{F}}} + \|\mathbf{w}\|_{0,\Omega_{\mathrm{S}}}\right)$$

and

$$|\lambda - \lambda_h| \leq Ch^{2r},$$

where $r := \min\{s, t\}$*, as in Theorem 2.4.*

Proof. For $(\lambda, (p, \mathbf{w}))$ being an eigenpair of Problem (2.9), let $(\lambda, (\mathbf{u}, \mathbf{w}))$ be an eigenpair of Problem (2.8) as in Lemma 2.1, $(\lambda_h, (\mathbf{u}_h, \mathbf{w}_h))$ an eigenpair of Problem (2.10) as in Theorem 2.4, and $(\lambda_h, (p_h, \mathbf{w}_h))$ an eigenpair of Problem (3.3) as in Lemma 3.1. By virtue of these lemmas and theorem, it only remains to prove that $\left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \|p - p_h\|_{1,T}^2\right)^{1/2} \leq Ch^r$. Now, because of (2.2), (3.4), (2.1), and (3.5), we have

$$\begin{aligned}
\left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \|p - p_h\|_{1,T}^2\right)^{\frac{1}{2}} &\leq \|p - p_h\|_{0,\Omega_{\mathrm{F}}} + \left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \|\boldsymbol{\nabla} p - \boldsymbol{\nabla} p_h\|_{0,T}^2\right)^{\frac{1}{2}} \\
&\leq \left\|\rho_{\mathrm{F}} c^2 \operatorname{div}(\mathbf{u} - \mathbf{u}_h)\right\|_{0,\Omega_{\mathrm{F}}} \\
&\quad + \|Pp_h - p_h\|_{0,\Omega_{\mathrm{F}}} + |\lambda - \lambda_h| \left\|\rho_{\mathrm{F}} \mathbf{u}\right\|_{0,\Omega_{\mathrm{F}}} \\
&\quad + \left\|\lambda_h \rho_{\mathrm{F}} (\mathbf{u} - \mathbf{u}_h)\right\|_{0,\Omega_{\mathrm{F}}} \\
&\quad + \left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \left\|\Pi_T \boldsymbol{\nabla} p_h - \boldsymbol{\nabla} p_h\right\|_{0,T}^2\right)^{\frac{1}{2}}.
\end{aligned}$$

The first, third, and fourth terms in the right hand side are appropriately bounded by means of Theorem 2.4. Regarding the second one we have

$$
\begin{aligned}
\|Pp_h - p_h\|_{0,\Omega_{\mathrm{F}}} &\le Ch \left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \|\nabla p_h\|_{0,T}^2 \right)^{\frac{1}{2}} \\
&\le Ch \left[\left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \left\|\Pi_T \nabla p_h\right\|_{0,T}^2 \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \left\|\nabla p_h - \Pi_T \nabla p_h\right\|_{0,T}^2 \right)^{\frac{1}{2}} \right] \\
&= Ch\lambda_h \left\|\rho_{\mathrm{F}} \mathbf{u}_h\right\|_{0,\Omega_{\mathrm{F}}} \\
&\quad + Ch \left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \left\|\Pi_T \nabla p_h - \nabla p_h\right\|_{0,T}^2 \right)^{\frac{1}{2}},
\end{aligned}
$$

where we have used (3.5) for the last equality. The last term in the inequality above is Ch times the last one in the previous inequality. Finally, we estimate this as follows:

$$
\begin{aligned}
\left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \left\|\Pi_T \nabla p_h - \nabla p_h\right\|_{0,T}^2 \right)^{\frac{1}{2}} &\le Ch \left(\sum_{T\in\mathcal{T}_h^{\mathrm{F}}} |\nabla p_h|_{1,T}^2 \right)^{\frac{1}{2}} \\
&\le Ch\lambda_h \rho_{\mathrm{F}} \| \operatorname{div} \mathbf{u}_h \|_{0,\Omega_{\mathrm{F}}},
\end{aligned}
$$

where the last inequality is obtained by repeating the arguments in the proof of Theorem 3.1 in [1] (see in particular the estimate previous to (3.21) in this reference). Thus, since $\mathbf{u} = \frac{1}{\lambda\rho_{\mathrm{F}}} \nabla p$ because of Lemma 3.1, we conclude the proof from Theorems 2.2 and 2.4. □

Our next goal is to obtain higher order error estimates in L^2 norms. We do this by using the abstract spectral approximation theory (see, for instance, [3]) in four steps. First, in Section 4.1, we put our problem in this framework by defining adequate continuous and discrete operators, $\mathbf{T}$ and $\mathbf{T}_h$, respectively, on a same space, with spectra and eigenfunctions coinciding with those of Problems (2.9) and (3.3), respectively. Then, in Section 4.2, we prove that the discrete operators $\mathbf{T}_h$ converge in norm to $\mathbf{T}$. In Section 4.3, by means of a duality argument, we prove a higher order estimate for the norm of the restriction of $(\mathbf{T}_h - \mathbf{T})$ to an eigenspace. Finally, in Section 4.4, we conclude the spectral approximation result, and we use it combined with Lemma 3.2

to prove the main result of the paper: a higher order error estimate for the post-processed pressure p_h^{L}.

4.1 Functional framework

We introduce the following spaces (recall that $\Gamma_{\mathrm{I}} = \cup_{j=1}^{J}\Gamma_j$):

$$\mathcal{Y} := \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})^2 \times \prod_{j=1}^{J} \mathrm{H}^{\frac{1}{2}}(\Gamma_j) \times \mathrm{H}^1(\Omega_{\mathrm{F}}),$$
$$\mathcal{W} := \left\{(\mathbf{w}, \xi, p) \in \mathcal{Y} : \ \xi = \mathbf{w}\cdot\mathbf{n} \text{ on } \Gamma_{\mathrm{I}}\right\},$$

endowed with their corresponding norms defined by

$$\|(\mathbf{w}, \xi, p)\|^2_{\mathcal{Y}} := \|\mathbf{w}\|^2_{1,\Omega_{\mathrm{S}}} + \sum_{j=1}^{J} \|\xi\|^2_{\frac{1}{2},\Gamma_j} + \|p\|^2_{1,\Omega_{\mathrm{F}}},$$
$$\|(\mathbf{w}, \xi, p)\|^2_{\mathcal{W}} := \|\mathbf{w}\|^2_{1,\Omega_{\mathrm{S}}} + \|p\|^2_{1,\Omega_{\mathrm{F}}}.$$

In view of these definitions, it is clear that $\mathcal{W}$ is a closed subspace of $\mathcal{Y}$ and that the $\mathcal{W}$-norm is equivalent to the $\mathcal{Y}$-norm on $\mathcal{W}$. We also introduce, $\forall\epsilon \geq 0$, the space

$$\mathcal{H}_\epsilon := \mathrm{L}^2(\Omega_{\mathrm{S}})^2 \times \prod_{j=1}^{J} \mathrm{H}^{\epsilon}(\Gamma_j) \times \mathrm{L}^2(\Omega_{\mathrm{F}}),$$

endowed with its corresponding product norm defined by

$$\|(\mathbf{w}, \xi, p)\|^2_{\mathcal{H}_\epsilon} := \|\mathbf{w}\|^2_{0,\Omega_{\mathrm{S}}} + \sum_{j=1}^{J} \|\xi\|^2_{\epsilon,\Gamma_j} + \|p\|^2_{0,\Omega_{\mathrm{F}}}.$$

Since, for $\epsilon \in [0, \frac{1}{2})$, $\prod_{j=1}^{J} \mathrm{H}^{\epsilon}(\Gamma_j) \equiv \mathrm{H}^{\epsilon}(\Gamma_{\mathrm{I}})$, we will not distinguish in this case between these spaces.

Let a be the bilinear and continuous form defined on $\mathcal{W}\times\mathcal{W}$ by

$$a\left((\mathbf{w}, \xi, p), (\mathbf{z}, \zeta, q)\right) := \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{w}) : \boldsymbol{\varepsilon}(\mathbf{z}) + \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}}\mathbf{w}\cdot\mathbf{z} + \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}}}\nabla p\cdot\nabla q$$
$$+ \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}} c^2} pq + \int_{\Gamma_{\mathrm{I}}} \xi q - \int_{\Gamma_{\mathrm{I}}} p\zeta.$$

Notice that a is $\mathcal{W}$-elliptic. Let b be the bilinear and continuous form defined on $\mathcal{H}_0\times\mathcal{W}$ by

$$b\left((\mathbf{f}, \eta, g), (\mathbf{z}, \zeta, q)\right) := \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}}\mathbf{f}\cdot\mathbf{z} + \int_{\Gamma_{\mathrm{I}}} \eta q + \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}} c^2} gq.$$

Since a is elliptic, b is continuous, and $\mathcal{W} \hookrightarrow \mathcal{H}_\epsilon \forall \epsilon \in [0, \frac{1}{2})$, then Lax-Milgram Lemma allows us to define the bounded linear operator

$$\mathbf{T} : \mathcal{H}_\epsilon \longrightarrow \mathcal{H}_\epsilon,$$

given by $\mathbf{T}(\mathbf{f}, \eta, g) = (\tilde{\mathbf{w}}, \tilde{\xi}, \tilde{p}) \in \mathcal{W}$ such that

(4.1)
$$a\left((\tilde{\mathbf{w}}, \tilde{\xi}, \tilde{p}), (\mathbf{z}, \zeta, q)\right) = b\left((\mathbf{f}, \eta, g), (\mathbf{z}, \zeta, q)\right) \qquad \forall (\mathbf{z}, \zeta, q) \in \mathcal{W}.$$

Then,

$$\left\|(\tilde{\mathbf{w}}, \tilde{\xi}, \tilde{p})\right\|_{\mathcal{W}} \leq C\|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_0} \leq C\|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon}. \tag{4.2}$$

Clearly, the eigenvalue problem for $\mathbf{T}$ is equivalent to the spectral Problem (2.9) in the sense that $(\mu, (\mathbf{w}, \mathbf{w} \cdot \mathbf{n}, p))$ is an eigenpair of $\mathbf{T}$ with $\mu > 0$, if and only if $(\lambda, (p, \mathbf{w}))$ is a solution of (2.9) and $\lambda = \frac{1}{\mu} - 1$.

From now on, let μ be a fixed eigenvalue of the operator $\mathbf{T}$ and $\mathcal{S}$ its corresponding associated eigenspace.

The following lemma gives an a-priori estimate for the solutions of Problem (4.1) and the eigenfunctions of the operator $\mathbf{T}$:

Lemma 4.1 *Let $s \in (\frac{1}{2}, 1]$ and $t \in (0, 1]$ be as in Theorem 2.2. For each $\epsilon \in (0, s - \frac{1}{2})$, if $(\mathbf{f}, \eta, g) \in \mathcal{H}_\epsilon$ and $(\tilde{\mathbf{w}}, \tilde{\xi}, \tilde{p}) = \mathbf{T}(\mathbf{f}, \eta, g)$, then $\tilde{\mathbf{w}} \in \mathrm{H}^{1+t}(\Omega_{\mathrm{S}})^2$ with*

$$\|\tilde{\mathbf{w}}\|_{1+t,\Omega_{\mathrm{S}}} \leq C\|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon},$$

where C is a strictly positive constant independent of ϵ, whereas $\tilde{p} \in \mathrm{H}^{\frac{3}{2}+\epsilon}(\Omega_{\mathrm{F}})$ with

$$\|\tilde{p}\|_{\frac{3}{2}+\epsilon,\Omega_{\mathrm{F}}} \leq C_\epsilon\|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon}.$$

Furthermore, if $(\mathbf{f}, \eta, g) \in \mathcal{S}$, then $\tilde{p} \in \mathrm{H}^{1+s}(\Omega_{\mathrm{F}})$ and

$$\|\tilde{p}\|_{1+s,\Omega_{\mathrm{F}}} \leq C\|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon},$$

with C independent of ϵ.

Proof. By testing (4.1) with adequate smooth functions we obtain:

$$-\mathbf{div}\left[\boldsymbol{\sigma}(\tilde{\mathbf{w}})\right] + \rho_{\mathrm{S}}\tilde{\mathbf{w}} = \rho_{\mathrm{S}}\mathbf{f} \qquad \text{in } \Omega_{\mathrm{S}}, \tag{4.3}$$

$$\boldsymbol{\sigma}(\tilde{\mathbf{w}})\mathbf{n} + \tilde{p}\mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{I}}, \tag{4.4}$$

$$\boldsymbol{\sigma}(\tilde{\mathbf{w}})\mathbf{n}_{\mathrm{S}} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{N}}, \tag{4.5}$$

$$\tilde{\mathbf{w}} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{4.6}$$

$$-c^2\Delta\tilde{p} + \tilde{p} = g \qquad \text{in } \Omega_{\mathrm{F}}, \tag{4.7}$$

$$\frac{\partial\tilde{p}}{\partial n} + \rho_{\mathrm{F}}\tilde{\mathbf{w}} \cdot \mathbf{n} = \rho_{\mathrm{F}}\eta \qquad \text{on } \Gamma_{\mathrm{I}}. \tag{4.8}$$

Equations (4.3)–(4.6) define a problem of linear elastostatics with a prescribed traction $-\tilde{p}\mathbf{n}$ on Γ_{I}. Since $\tilde{p} \in \mathrm{H}^1(\Omega_{\mathrm{F}})$, from the trace theorem and the standard a priori estimate for this problem (see [16]), we have that $\tilde{\mathbf{w}} \in \mathrm{H}^{1+t}(\Omega_{\mathrm{S}})^2$ and

$$\|\tilde{\mathbf{w}}\|_{1+t,\Omega_{\mathrm{S}}} \leq C\left(\|\mathbf{f}\|_{0,\Omega_{\mathrm{S}}} + \|\tilde{p}\|_{1,\Omega_{\mathrm{F}}}\right).$$

On the other hand, equations (4.7) and (4.8) define a Neumann problem with boundary condition $-\rho_{\mathrm{F}}\tilde{\mathbf{w}}\cdot\mathbf{n}+\rho_{\mathrm{F}}\eta$. Since $\tilde{\mathbf{w}} \in \mathrm{H}^1(\Omega_{\mathrm{S}})^2$ and $\eta \in \mathrm{H}^{\epsilon}(\Gamma_{\mathrm{I}})$ with $\epsilon < s - \frac{1}{2}$, then we have that $\tilde{p} \in \mathrm{H}^{\frac{3}{2}+\epsilon}(\Omega_{\mathrm{F}})$ and

$$\|\tilde{p}\|_{\frac{3}{2}+\epsilon,\Omega_{\mathrm{F}}} \leq C_{\epsilon}\left(\|g\|_{0,\Omega_{\mathrm{F}}} + \|\tilde{\mathbf{w}}\|_{1,\Omega_{\mathrm{S}}} + \|\eta\|_{\epsilon,\Gamma_{\mathrm{I}}}\right),$$

because of the a priori estimate for this problem (see [11]). Thus, from the previous inequalities and (4.2), we conclude the proof for any $(\mathbf{f}, \eta, g) \in \mathcal{H}_{\epsilon}$.

For $(\mathbf{f}, \eta, g) \in \mathcal{S}$, the lemma follows from Lemma 2.1 and Theorem 2.2. Anyway, we include its proof for the sake of completeness. In this case, $\eta = \mathbf{f}\cdot\mathbf{n} = \mu\tilde{\mathbf{w}}\cdot\mathbf{n}$, and $\tilde{\mathbf{w}} \in \mathrm{H}^1(\Omega_{\mathrm{S}})^2$. Then, the standard error estimate for the Neumann problem (4.7)–(4.8) (see [16]) yields $\tilde{p} \in \mathrm{H}^{1+s}(\Omega_{\mathrm{F}})$ and

$$\|\tilde{p}\|_{1+s,\Omega_{\mathrm{F}}} \leq C\left(\|g\|_{0,\Omega_{\mathrm{F}}} + \|\tilde{\mathbf{w}}\|_{1,\Omega_{\mathrm{S}}}\right) \leq \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_{\epsilon}}.$$

Thus, we conclude the lemma. □

Now we introduce the non-conforming finite element space

$$\mathcal{Y}_h := \mathcal{L}_{h\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})^2 \times \mathcal{L}_h(\Gamma_{\mathrm{I}}) \times \mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}}),$$

where

$$\mathcal{L}_h(\Gamma_{\mathrm{I}}) := \left\{\xi_h \in \mathrm{H}^1(\Gamma_{\mathrm{I}}) : \ \xi_h|_{\ell} \in \mathcal{P}_1(\ell)\ \forall \ell \in \mathcal{E}_h^{\mathrm{I}}\right\}.$$

Then, the discrete analogue of $\mathcal{W}$ is

$$\mathcal{W}_h := \left\{(\mathbf{w}_h, \xi_h, p_h) \in \mathcal{Y}_h : \ \xi_h = \mathbf{w}_h\cdot\mathbf{n} \text{ on } \Gamma_{\mathrm{I}}\right\},$$

with its norm defined by

$$\|(\mathbf{w}_h, \xi_h, p_h)\|_{\mathcal{W}_h}^2 := \|\mathbf{w}_h\|_{1,\Omega_{\mathrm{S}}}^2 + \sum_{T\in\mathcal{T}_h^{\mathrm{F}}} \|p_h\|_{1,T}^2.$$

Notice that $\|\cdot\|_{\mathcal{W}_h}$ is also well defined on $\mathcal{W}$ and $\|(\mathbf{w}, \xi, p)\|_{\mathcal{W}_h}=\|(\mathbf{w}, \xi, p)\|_{\mathcal{W}}$ $\forall(\mathbf{w}, \xi, p) \in \mathcal{W}$.

Let a_h be the bilinear and continuous form defined on $(\mathcal{W}+\mathcal{W}_h)\times(\mathcal{W}+\mathcal{W}_h)$ by

$$a_h\left((\mathbf{w},\xi,p),(\mathbf{z},\zeta,q)\right) := \int_{\Omega_S}\boldsymbol{\sigma}(\mathbf{w}):\boldsymbol{\varepsilon}(\mathbf{z}) + \int_{\Omega_S}\rho_S\mathbf{w}\cdot\mathbf{z} + \sum_{T\in\mathcal{T}_h^F}\int_T\frac{1}{\rho_F}\Pi_T\nabla p\cdot\Pi_T\nabla q + \int_{\Omega_F}\frac{1}{\rho_F c^2}Pp\,Pq + \int_{\Gamma_I}\xi\,P_{\Gamma_I}q - \int_{\Gamma_I}\zeta\,P_{\Gamma_I}p.$$

This bilinear form is $\mathcal{W}_h$-elliptic, uniformly on h, as shown in Lemma 5.1 (to make the presentation clearer, we postpone the proof of this and other technical results to Section 5 below).

Let b_h be the bilinear and continuous form defined on $\mathcal{H}_0\times\mathcal{W}_h$ by

$$b_h\left((\mathbf{f},\eta,g),(\mathbf{z},\zeta,q)\right) := \int_{\Omega_S}\rho_S\mathbf{f}\cdot\mathbf{z} + \int_{\Gamma_I}\eta\,P_{\Gamma_I}q + \int_{\Omega_F}\frac{1}{\rho_F c^2}g\,Pq.$$

Notice that both bilinear forms, a_h and b_h, are continuous, but not necessarily uniformly continuous on h for general regular meshes, because of the terms involving integrals on Γ_I.

Now we are in order to define the discrete analogue of $\mathbf{T}$. For $\epsilon\in[0,\frac{1}{2})$, since $\mathcal{W}_h\hookrightarrow\mathcal{H}_\epsilon$, let

$$\mathbf{T}_h:\mathcal{H}_\epsilon\longrightarrow\mathcal{H}_\epsilon,$$

be defined by $\mathbf{T}_h(\mathbf{f},\eta,g)=(\tilde{\mathbf{w}}_h,\tilde{\xi}_h,\tilde{p}_h)\in\mathcal{W}_h$ such that

$$a_h\Big((\tilde{\mathbf{w}}_h,\tilde{\xi}_h,\tilde{p}_h),(\mathbf{z}_h,\zeta_h,q_h)\Big) = b_h\left((\mathbf{f},\eta,g),(\mathbf{z}_h,\zeta_h,q_h)\right) \quad \forall(\mathbf{z}_h,\zeta_h,q_h)\in\mathcal{W}_h. \tag{4.9}$$

Once more, the eigenvalue problem for $\mathbf{T}_h$ is equivalent to the spectral Problem (3.3) in the sense that $(\mu_h,(\mathbf{w}_h,\mathbf{w}_h\cdot\mathbf{n},p_h))$ is an eigenpair of $\mathbf{T}_h$ with $\mu_h>0$, if and only if $(\lambda_h,(p_h,\mathbf{w}_h))$ is a solution of (3.3) and $\lambda_h=\frac{1}{\mu_h}-1$.

4.2 Convergence

From now on let $\epsilon\in(0,s-\frac{1}{2})$, with $s\in(\frac{1}{2},1]$ as in Theorem 2.2, be an arbitrarily small number. Throughout the rest of the paper, for $(\mathbf{f},\eta,g)\in\mathcal{H}_\epsilon$, we denote

$$(\tilde{\mathbf{w}},\tilde{\mathbf{w}}\cdot\mathbf{n},\tilde{p}) := \mathbf{T}(\mathbf{f},\eta,g)\in\mathcal{W}, \qquad (\tilde{\mathbf{w}}_h,\tilde{\mathbf{w}}_h\cdot\mathbf{n},\tilde{p}_h) := \mathbf{T}_h(\mathbf{f},\eta,g)\in\mathcal{W}_h,$$

and

$$\mathbf{e}_{\tilde{\mathbf{w}}} := \tilde{\mathbf{w}}-\tilde{\mathbf{w}}_h, \qquad e_{\tilde{\mathbf{w}}\cdot\mathbf{n}} := \tilde{\mathbf{w}}\cdot\mathbf{n}-\tilde{\mathbf{w}}_h\cdot\mathbf{n} = \mathbf{e}_{\tilde{\mathbf{w}}}\cdot\mathbf{n}, \qquad e_{\tilde{p}} := \tilde{p}-\tilde{p}_h,$$

the corresponding error terms. As a first step we will estimate $\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{W}_h}$.

Let $\tilde{\mathbf{w}}^{\mathrm{I}}$ and $\tilde{p}^{\mathrm{I}}$ be the conforming piecewise linear Lagrange interpolants of $\tilde{\mathbf{w}}$ and $\tilde{p}$, respectively. Notice that $(\tilde{\mathbf{w}}^{\mathrm{I}}, \tilde{\mathbf{w}}^{\mathrm{I}}\cdot\mathbf{n}, \tilde{p}^{\mathrm{I}}) \in \mathcal{W}\cap\mathcal{W}_h$. Because of Lemma 4.1, $\tilde{\mathbf{w}} \in \mathrm{H}^{1+t}(\Omega_{\mathrm{S}})^2$ and $\tilde{p} \in \mathrm{H}^{\frac{3}{2}+\epsilon}(\Omega_{\mathrm{F}})$. Then, the standard error estimates for the Lagrange interpolant yield:

$$\left\|\tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{\mathrm{I}}\right\|_{1,\Omega_{\mathrm{S}}} \leq Ch^t \|\tilde{\mathbf{w}}\|_{1+t,\Omega_{\mathrm{S}}} \leq Ch^t \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon},$$
$$\left\|\tilde{p} - \tilde{p}^{\mathrm{I}}\right\|_{1,\Omega_{\mathrm{F}}} \leq Ch^{\frac{1}{2}+\epsilon} \|\tilde{p}\|_{\frac{3}{2}+\epsilon,\Omega_{\mathrm{F}}} \leq C_\epsilon h^{\frac{1}{2}+\epsilon} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon}.$$

Consequently,

$$\left\|\left(\tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{\mathrm{I}}, (\tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{\mathrm{I}})\cdot\mathbf{n}, \tilde{p} - \tilde{p}^{\mathrm{I}}\right)\right\|_{\mathcal{W}_h} \leq C_\epsilon h^{r_\epsilon} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon}, \tag{4.10}$$

with $r_\epsilon := \min\{\frac{1}{2}+\epsilon, t\}$. When the source term is an eigenfunction, the order of the approximation is larger. Indeed, if $(\mathbf{f}, \eta, g) \in \mathcal{S}$, using again Lemma 4.1 we have that $\tilde{p} \in \mathrm{H}^{1+s}(\Omega_{\mathrm{F}})$ and $\left\|\tilde{p} - \tilde{p}^{\mathrm{I}}\right\|_{1,\Omega_{\mathrm{F}}} \leq Ch^s \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon}$. Thus, we have in this case

$$\left\|\left(\tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{\mathrm{I}}, (\tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{\mathrm{I}})\cdot\mathbf{n}, \tilde{p} - \tilde{p}^{\mathrm{I}}\right)\right\|_{\mathcal{W}_h} \leq Ch^{r} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon}, \tag{4.11}$$

with $r := \min\{s, t\}$ as in Theorem 2.4.

The following lemma provides similar estimates for $\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{W}_h}$:

Lemma 4.2 *Let $r := \min\{s, t\}$ be as in Theorem 2.4. Then, there exists a strictly positive constant C_ϵ such that*

$$\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{W}_h} \leq C_\epsilon h^{r_\epsilon} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon},$$

with $r_\epsilon := \min\{\frac{1}{2}+\epsilon, t\}$. Furthermore, if $(\mathbf{f}, \eta, g) \in \mathcal{S}$, then

$$\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{W}_h} \leq Ch^{r} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon},$$

with C independent of ϵ.

Proof. Let $(\mathbf{z}_h, \zeta_h, q_h) := (\tilde{\mathbf{w}}^{\mathrm{I}} - \tilde{\mathbf{w}}_h, (\tilde{\mathbf{w}}^{\mathrm{I}} - \tilde{\mathbf{w}}_h)\cdot\mathbf{n}, \tilde{p}^{\mathrm{I}} - \tilde{p}_h)$. Clearly,

$$\begin{aligned}\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{W}_h} \leq{}& \left\|(\tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{\mathrm{I}}, (\tilde{\mathbf{w}} - \tilde{\mathbf{w}}^{\mathrm{I}})\cdot\mathbf{n}, \tilde{p} - \tilde{p}^{\mathrm{I}})\right\|_{\mathcal{W}_h}\\ &+\|(\mathbf{z}_h, \zeta_h, q_h)\|_{\mathcal{W}_h}.\end{aligned} \tag{4.12}$$

The first term on the right hand side above is bounded by (4.10). For the second one we use that a_h is $\mathcal{W}_h$-elliptic uniformly on h (see Lemma 5.1 below) and we obtain

$$\begin{aligned}&\alpha\|(\mathbf{z}_h, \zeta_h, q_h)\|^2_{\mathcal{W}_h}\\ &\quad\leq a_h\left((\tilde{\mathbf{w}}^{\mathrm{I}} - \tilde{\mathbf{w}}, (\tilde{\mathbf{w}}^{\mathrm{I}} - \tilde{\mathbf{w}})\cdot\mathbf{n}, \tilde{p}^{\mathrm{I}} - \tilde{p}), (\mathbf{z}_h, \zeta_h, q_h)\right)\\ &\qquad+a_h\left((\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}), (\mathbf{z}_h, \zeta_h, q_h)\right).\end{aligned} \tag{4.13}$$

In spite of the fact that in general a_h is not uniformly continuous on h, we prove in Lemma 5.3 (see Section 5 below) that

$$
\tag{4.14} a_h\left((\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}, (\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}})\cdot\mathbf{n}, \tilde{p}^{\mathrm{I}}-\tilde{p}), (\mathbf{z}_h,\zeta_h,q_h)\right)
\le \begin{cases} C_\epsilon h^{r_\epsilon}\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h} & \text{for } (\mathbf{f},\eta,g)\in\mathcal{H}_\epsilon,\\ Ch^{r}\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h} & \text{for } (\mathbf{f},\eta,g)\in\mathcal{S}.\end{cases}
$$

On the other hand, for the second term in the right hand side of (4.13) we write

$$
a_h\left((\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}), (\mathbf{z}_h,\zeta_h,q_h)\right) = a_h\left((\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}), (\mathbf{z}_h,\zeta_h,0)\right) + a_h\left((\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}), (\mathbf{0},0,q_h)\right).
$$

Then, taking into account that $(\mathbf{z}_h,\zeta_h,0)\in\mathcal{W}\cap\mathcal{W}_h$, (4.1), and (4.9), we have

$$
\begin{aligned}
&a_h\left((\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}), (\mathbf{z}_h,\zeta_h,0)\right)\\
&\quad = a\left((\tilde{\mathbf{w}},\tilde{\mathbf{w}}\cdot\mathbf{n},\tilde{p}),(\mathbf{z}_h,\zeta_h,0)\right) - a_h\left((\tilde{\mathbf{w}}_h,\tilde{\mathbf{w}}_h\cdot\mathbf{n},\tilde{p}_h),(\mathbf{z}_h,\zeta_h,0)\right)\\
&\qquad - \int_{\Gamma_{\mathrm{I}}} P_{\Gamma_{\mathrm{I}}}\tilde{p}\,\mathbf{z}_h\cdot\mathbf{n} + \int_{\Gamma_{\mathrm{I}}}\tilde{p}\,\mathbf{z}_h\cdot\mathbf{n}\\
&\quad = \int_{\Gamma_{\mathrm{I}}}(\tilde{p}-P_{\Gamma_{\mathrm{I}}}\tilde{p})\,\mathbf{z}_h\cdot\mathbf{n} = \int_{\Gamma_{\mathrm{I}}}(\tilde{p}-P_{\Gamma_{\mathrm{I}}}\tilde{p})(\mathbf{z}_h\cdot\mathbf{n}-P_{\Gamma_{\mathrm{I}}}\mathbf{z}_h\cdot\mathbf{n})\\
&\quad \le Ch\|\tilde{p}\|_{1,\Omega_{\mathrm{F}}}\|\mathbf{z}_h\|_{1,\Omega_{\mathrm{S}}},
\end{aligned}
$$

where the last inequality follows from the standard error estimates for $P_{\Gamma_{\mathrm{I}}}$. Therefore, from (4.2) we obtain

$$
\tag{4.15} a_h\left((\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}), (\mathbf{z}_h,\zeta_h,0)\right) \le Ch\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h}.
$$

Now, for $(\mathbf{0},0,q_h)\in\mathcal{W}_h$, from the definitions of $\mathbf{e}_{\tilde{\mathbf{w}}}$, $e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}$, and $e_{\tilde{p}}$, and (4.9), we have

$$
\begin{aligned}
&a_h\left((\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}), (\mathbf{0},0,q_h)\right)\\
&\quad = \sum_{T\in\mathcal{T}_h^{\mathrm{F}}}\int_T \frac{1}{\rho_{\mathrm{F}}}\nabla\tilde{p}\cdot\nabla q_h + \int_{\Omega_{\mathrm{F}}}\frac{1}{\rho_{\mathrm{F}}c^2}\tilde{p}q_h + \int_{\Gamma_{\mathrm{I}}}\tilde{\mathbf{w}}\cdot\mathbf{n}\,P_{\Gamma_{\mathrm{I}}}q_h\\
&\qquad - \sum_{T\in\mathcal{T}_h^{\mathrm{F}}}\int_T\frac{1}{\rho_{\mathrm{F}}}\nabla\tilde{p}\cdot\left[\nabla q_h - \Pi_T\nabla q_h\right]\\
&\qquad - \int_{\Omega_{\mathrm{F}}}\frac{1}{\rho_{\mathrm{F}}c^2}\tilde{p}(q_h - Pq_h) - \int_{\Gamma_{\mathrm{I}}}\eta\,P_{\Gamma_{\mathrm{I}}}q_h - \int_{\Omega_{\mathrm{F}}}\frac{1}{\rho_{\mathrm{F}}c^2}g\,Pq_h.
\end{aligned}
$$

Then, by integrating by parts and using that $\tilde{p} \in \mathrm{H}^{\frac{3}{2}+\epsilon}(\Omega_{\mathrm{F}})$, (4.7), and (4.8), we obtain

$$\begin{aligned} & a_h\left((\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}), (\mathbf{0}, 0, q_h)\right) \\ & \quad = \frac{1}{\rho_{\mathrm{F}}} \sum_{\ell \in \mathcal{E}_h^{\mathrm{o}}} \int_\ell \frac{\partial \tilde{p}}{\partial n_\ell} [\![q_h]\!]_\ell - \frac{1}{\rho_{\mathrm{F}}} \sum_{T \in \mathcal{T}_h^{\mathrm{F}}} \int_T \left(\nabla \tilde{p} - \Pi_T \nabla \tilde{p}\right) \cdot \nabla q_h \\ & \qquad - \frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} \tilde{p}\,(q_h - P q_h) - \int_{\Gamma_{\mathrm{I}}} \tilde{\mathbf{w}} \cdot \mathbf{n} \left(q_h - P_{\Gamma_{\mathrm{I}}} q_h\right) \\ & \qquad + \frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} g\,(q_h - P q_h) + \int_{\Gamma_{\mathrm{I}}} \eta \left(q_h - P_{\Gamma_{\mathrm{I}}} q_h\right). \end{aligned}$$

All the terms on the right hand side above are bounded by using Lemmas 5.4–5.8 (see Section 5 below). Hence, we obtain

$$\begin{aligned} & a_h\left((\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}), (\mathbf{0}, 0, q_h)\right) \\ & \quad \le \begin{cases} C_\epsilon h^{\frac{1}{2}+\epsilon} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \|(\mathbf{z}_h, \zeta_h, q_h)\|_{\mathcal{W}_h} & \text{for } (\mathbf{f}, \eta, g) \in \mathcal{H}_\epsilon, \\ C h^s \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \|(\mathbf{z}_h, \zeta_h, q_h)\|_{\mathcal{W}_h} & \text{for } (\mathbf{f}, \eta, g) \in \mathcal{S}. \end{cases} \end{aligned} \tag{4.16}$$

Therefore, (4.13), (4.14), (4.15), and (4.16) yield

$$\|(\mathbf{z}_h, \zeta_h, q_h)\|_{\mathcal{W}_h} \le \begin{cases} C_\epsilon h^{r_\epsilon} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} & \text{for } (\mathbf{f}, \eta, g) \in \mathcal{H}_\epsilon, \\ C h^r \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} & \text{for } (\mathbf{f}, \eta, g) \in \mathcal{S}. \end{cases}$$

Finally, combining these with estimates (4.12), (4.10), and (4.11), we conclude the lemma. □

The following lemma shows that the operators $\mathbf{T}_h$ converge to $\mathbf{T}$ in norm as h goes to 0:

Lemma 4.3 *There holds*

$$\|\mathbf{T}_h - \mathbf{T}\|_{\mathcal{L}(\mathcal{H}_\epsilon, \mathcal{H}_\epsilon)} \to 0 \qquad \text{as } h \to 0.$$

Proof. It is a direct consequence of Lemma 4.2, since $\mathcal{W} + \mathcal{W}_h \hookrightarrow \mathcal{H}_\epsilon$ uniformly on h. □

Remark 4.1 In spite of the fact that the operators $\mathbf{T}$ and $\mathbf{T}_h$ are well defined for $\epsilon = 0$, the technique used to prove this lemma requires $\epsilon > 0$. In fact, we have used that $\tilde{p} \in \mathrm{H}^{\frac{3}{2}+\epsilon}(\Omega_{\mathrm{F}})$ for $\epsilon > 0$ in the proof of Lemma 4.2 to give a meaning to the integrals $\int_\ell \frac{\partial \tilde{p}}{\partial n_\ell} [\![q_h]\!]_\ell$ on each edge ℓ.

4.3 Duality arguments

Our next goal is to prove a higher order estimate for $\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_\epsilon}$ when $(\mathbf{f}, \eta, g) \in \mathcal{S}$. To do this we use a duality argument based on the following auxiliary problem:

$$-\operatorname{\mathbf{div}}\left[\boldsymbol{\sigma}(\mathbf{w}_*)\right] + \rho_{\mathrm{S}}\mathbf{w}_* = \rho_{\mathrm{S}}\mathbf{e}_{\tilde{\mathbf{w}}} \quad \text{in } \Omega_{\mathrm{S}}, \tag{4.17}$$

$$-\boldsymbol{\sigma}(\mathbf{w}_*)\mathbf{n} + p_*\mathbf{n} = e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}\mathbf{n} \quad \text{on } \Gamma_{\mathrm{I}}, \tag{4.18}$$

$$\boldsymbol{\sigma}(\mathbf{w}_*)\mathbf{n}_{\mathrm{S}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{N}}, \tag{4.19}$$

$$\mathbf{w}_* = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}, \tag{4.20}$$

$$-c^2\Delta p_* + p_* = e_{\tilde{p}} \quad \text{in } \Omega_{\mathrm{F}}, \tag{4.21}$$

$$\frac{\partial p_*}{\partial n} - \rho_{\mathrm{F}}\mathbf{w}_* \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{\mathrm{I}}. \tag{4.22}$$

The following is an equivalent variational formulation of the problem above:

Find $(\mathbf{w}_*, \xi_*, p_*) \in \mathcal{W}$ *such that*

$$a\left((\mathbf{z}, \zeta, q), (\mathbf{w}_*, \xi_*, p_*)\right) = \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}}\mathbf{e}_{\tilde{\mathbf{w}}} \cdot \mathbf{z} + \int_{\Gamma_{\mathrm{I}}} e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}\zeta + \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}}c^2} e_{\tilde{p}}q \quad \forall (\mathbf{z}, \zeta, q) \in \mathcal{W}. \tag{4.23}$$

Since a is $\mathcal{W}$-elliptic, because of Lax-Milgram Lemma, this problem attains a unique solution satisfying

$$\|(\mathbf{w}_*, \xi_*, p_*)\|_{\mathcal{W}} \leq C\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0}. \tag{4.24}$$

Throughout the rest of the paper, $(\mathbf{w}_*, \xi_*, p_*)$ will denote the solution of Problem (4.23).

The following lemma gives a regularity result for $(\mathbf{w}_*, \xi_*, p_*)$:

Lemma 4.4 *Let* $s \in (\frac{1}{2}, 1]$ *and* $t \in (0, 1]$ *be as in Theorem 2.2. Then, there exists a strictly positive constant* C *such that* $\mathbf{w}_* \in \mathrm{H}^{1+r'}(\Omega_{\mathrm{S}})^2$, $p_* \in \mathrm{H}^{1+s}(\Omega_{\mathrm{F}})$, *and*

$$\|\mathbf{w}_*\|_{1+r',\Omega_{\mathrm{S}}} + \|p_*\|_{1+s,\Omega_{\mathrm{F}}} \leq C\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0},$$

with $r' := \min\{\frac{1}{2}, t\}$.

Proof. By using suitable test functions in (4.23) it is simple to show that $(\mathbf{w}_*, \xi_*, p_*)$ satisfies (4.17)–(4.22). Equations (4.21) and (4.22) define a classical Neumann problem with boundary condition $\rho_{\mathrm{F}}\mathbf{w}_* \cdot \mathbf{n}$. Since $\mathbf{w}_* \in$

$\mathrm{H}^1(\Omega_S)^2$, from the trace theorem and the usual a priori estimate for this problem (see [16]) we have that $p_* \in \mathrm{H}^{1+s}(\Omega_F)$ and

$$\|p_*\|_{1+s,\Omega_F} \leq C\left(\|e_{\tilde{p}}\|_{0,\Omega_F} + \|\mathbf{w}_*\|_{1,\Omega_S}\right) \leq C\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0},$$

the latter because of (4.24). On the other hand, equations (4.17)–(4.20) define a linear elastostatics problem with prescribed traction $(p_* - e_{\tilde{\mathbf{w}}\cdot\mathbf{n}})\,\mathbf{n} \in \mathrm{L}^2(\Gamma_I)$. (Indeed, this prescribed traction is more regular, but we are seeking an estimate involving $\|e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}\|_{\mathrm{L}^2(\Gamma_I)}$.) Thus, $\mathbf{w}_* \in \mathrm{H}^{1+r'}(\Omega_S)^2$ for $r' = \min\{\frac{1}{2}, t\}$ (see [16]) and

$$\|\mathbf{w}_*\|_{1+r',\Omega_S} \leq C\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0}. \qquad \square$$

Now we are able to prove a higher order estimate for $\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0}$ when $(\mathbf{f}, \eta, g) \in \mathcal{S}$:

Lemma 4.5 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0} \leq C h^{r+r'} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon},$$

where C *is a constant independent of* ϵ, $r := \min\{s, t\}$ *as in Theorem 2.4 and* $r' := \min\{\frac{1}{2}, t\}$ *as in the previous lemma.*

Proof. For $(\mathbf{f}, \eta, g) \in \mathcal{S}$, by using equations (4.17), (4.18), and (4.21), we have

$$\begin{aligned}
&\int_{\Omega_S} \rho_S |\mathbf{e}_{\tilde{\mathbf{w}}}|^2 + \int_{\Gamma_I} |e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}|^2 + \int_{\Omega_F} \frac{1}{\rho_F c^2} |e_{\tilde{p}}|^2 \\
&\quad = \int_{\Omega_S} \mathbf{e}_{\tilde{\mathbf{w}}} \cdot \left\{-\mathbf{div}\,[\boldsymbol{\sigma}(\mathbf{w}_*)] + \rho_S \mathbf{w}_*\right\} + \int_{\Gamma_I} \mathbf{e}_{\tilde{\mathbf{w}}} \cdot [-\boldsymbol{\sigma}(\mathbf{w}_*)\mathbf{n} + p_* \mathbf{n}] \\
&\qquad + \int_{\Omega_F} e_{\tilde{p}} \left(-\frac{1}{\rho_F} \Delta p_* + \frac{1}{\rho_F c^2} p_*\right) \\
&\quad = \int_{\Omega_S} \boldsymbol{\varepsilon}(\mathbf{e}_{\tilde{\mathbf{w}}}) : \boldsymbol{\sigma}(\mathbf{w}_*) + \int_{\Omega_S} \rho_S \mathbf{e}_{\tilde{\mathbf{w}}} \cdot \mathbf{w}_* + \int_{\Gamma_I} e_{\tilde{\mathbf{w}}\cdot\mathbf{n}} p_* \\
&\qquad + \sum_{T \in \mathcal{T}_h^F} \int_T \frac{1}{\rho_F} \nabla e_{\tilde{p}} \cdot \nabla p_* - \sum_{\ell \in \mathcal{E}_h^o} \int_\ell \frac{1}{\rho_F} [\![e_{\tilde{p}}]\!]_\ell \frac{\partial p_*}{\partial n_\ell} - \int_{\Gamma_I} e_{\tilde{p}} \mathbf{w}_* \cdot \mathbf{n} \\
&\qquad + \int_{\Omega_F} \frac{1}{\rho_F c^2} e_{\tilde{p}} p_*,
\end{aligned}$$

where we have used integration by parts and equations (4.19) and (4.22) for the second equality.

Let $\mathbf{w}_*^{\mathrm{I}}$ and p_*^{I} be the piecewise linear Lagrange interpolants of $\mathbf{w}_*$ and p_*, respectively. Since $(\mathbf{w}_*^{\mathrm{I}}, \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n}, p_*^{\mathrm{I}}) \in \mathcal{W} \cap \mathcal{W}_h$, we can use it as a test function in (4.1) and (4.9). Thus, we obtain the following residual equation:

$$\begin{aligned}
&\int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{e}_{\tilde{\mathbf{w}}}) : \boldsymbol{\varepsilon}(\mathbf{w}_*^{\mathrm{I}}) + \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} \mathbf{e}_{\tilde{\mathbf{w}}} \cdot \mathbf{w}_*^{\mathrm{I}} + \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}}} \nabla e_{\tilde{p}} \cdot \nabla p_*^{\mathrm{I}} \\
&+ \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}} c^2} e_{\tilde{p}} p_*^{\mathrm{I}} + \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}} c^2} \tilde{p}_h \left(p_*^{\mathrm{I}} - P p_*^{\mathrm{I}}\right) + \int_{\Gamma_{\mathrm{I}}} e_{\tilde{\mathbf{w}}\cdot\mathbf{n}} p_*^{\mathrm{I}} \\
&+ \int_{\Gamma_{\mathrm{I}}} \tilde{\mathbf{w}}_h \cdot \mathbf{n} \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}}\right) - \int_{\Gamma_{\mathrm{I}}} e_{\tilde{p}} \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n} - \int_{\Gamma_{\mathrm{I}}} \left(\tilde{p}_h - P_{\Gamma_{\mathrm{I}}} \tilde{p}_h\right) \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n} \\
&\quad = \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}} c^2} (g - Pg) p_*^{\mathrm{I}} + \int_{\Gamma_{\mathrm{I}}} \eta \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}}\right).
\end{aligned}$$

Therefore, subtracting the last two equations we obtain

$$\begin{aligned}
&\int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} |\mathbf{e}_{\tilde{\mathbf{w}}}|^2 + \int_{\Gamma_{\mathrm{I}}} |e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}|^2 + \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}} c^2} \left|e_{\tilde{p}}\right|^2 \\
&\quad = \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{e}_{\tilde{\mathbf{w}}}) : \boldsymbol{\varepsilon}\left(\mathbf{w}_* - \mathbf{w}_*^{\mathrm{I}}\right) + \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} \mathbf{e}_{\tilde{\mathbf{w}}} \cdot \left(\mathbf{w}_* - \mathbf{w}_*^{\mathrm{I}}\right) \\
&\qquad + \int_{\Gamma_{\mathrm{I}}} e_{\tilde{\mathbf{w}}\cdot\mathbf{n}} \left(p_* - p_*^{\mathrm{I}}\right) + \frac{1}{\rho_{\mathrm{F}}} \sum_{T \in \mathcal{T}_h^{\mathrm{F}}} \int_T \nabla e_{\tilde{p}} \cdot \nabla \left(p_* - p_*^{\mathrm{I}}\right) \\
&\qquad + \frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} e_{\tilde{p}} \left(p_* - p_*^{\mathrm{I}}\right) - \int_{\Gamma_{\mathrm{I}}} e_{\tilde{p}} \left(\mathbf{w}_* - \mathbf{w}_*^{\mathrm{I}}\right) \cdot \mathbf{n} \\
&\qquad - \frac{1}{\rho_{\mathrm{F}}} \sum_{\ell \in \mathcal{E}_h^{\mathrm{o}}} \int_\ell [\![e_{\tilde{p}}]\!]_\ell \frac{\partial p_*}{\partial n_\ell} - \frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} \tilde{p}_h \left(p_*^{\mathrm{I}} - P p_*^{\mathrm{I}}\right) \\
&\qquad - \int_{\Gamma_{\mathrm{I}}} \tilde{\mathbf{w}}_h \cdot \mathbf{n} \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}}\right) + \int_{\Gamma_{\mathrm{I}}} \left(\tilde{p}_h - P_{\Gamma_{\mathrm{I}}} \tilde{p}_h\right) \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n} \\
&\qquad + \frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} (g - Pg) p_*^{\mathrm{I}} + \int_{\Gamma_{\mathrm{I}}} \eta \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}}\right).
\end{aligned}$$

All the terms on the right hand side above are bounded by using Lemmas 5.9–5.18 (see Section 5 below). Hence, we obtain

$$\begin{aligned}
\left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{H}_0}^2 &\le C \left(\int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} |\mathbf{e}_{\tilde{\mathbf{w}}}|^2 + \int_{\Gamma_{\mathrm{I}}} |e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}|^2 + \int_{\Omega_{\mathrm{F}}} \frac{1}{\rho_{\mathrm{F}} c^2} \left|e_{\tilde{p}}\right|^2 \right) \\
&\le C h^{r+r'} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{H}_0},
\end{aligned}$$

which allows us to conclude the proof. □

Now we are able to prove the claimed higher order estimate in $\mathcal{H}_\epsilon$:

Lemma 4.6 *The following estimate holds:*

$$\left\| (\mathbf{T} - \mathbf{T}_h)\,|_{\mathcal{S}} \right\|_{\mathcal{L}(\mathcal{H}_\epsilon, \mathcal{H}_\epsilon)} \le C h^{r + r'_\epsilon},$$

where C is a constant independent of ϵ and $r'_\epsilon := (1 - 2\epsilon) r'$, with r and r' as in Lemma 4.5.

Proof. For $(\mathbf{f}, \eta, g) \in \mathcal{S}$, from Lemma 4.2 we have

$$\left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{H}_{1/2}} \le \left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{W}_h} \le C h^r \left\| (\mathbf{f}, \eta, g) \right\|_{\mathcal{H}_\epsilon}$$

and, because of Lemma 4.5,

$$\left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{H}_0} \le C h^{r + r'} \left\| (\mathbf{f}, \eta, g) \right\|_{\mathcal{H}_\epsilon}.$$

Thus, the lemma follows from the Interpolation Theorem of Lions and Peetre (see, for instance, Theorem 1.4 in [15]). □

4.4 Spectral approximation results and higher order estimate for the post-processed pressure

Now we are in order to apply the standard abstract spectral approximation theory (see, for instance, [3]). Let m denote the multiplicity of the eigenvalue $\mu > 0$ of $\mathbf{T}$ and $\mathcal{S}$ the corresponding eigenspace as above. Since $\|\mathbf{T} - \mathbf{T}_h\|_{\mathcal{H}_\epsilon} \to 0$ as $h \to 0$, then there exist m eigenvalues of $\mathbf{T}_h$, $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ (repeated accordingly to their respective multiplicities) converging to μ (see [3]). Let $\mathcal{S}_h$ be the direct sum of the corresponding associated eigenspaces.

We recall the definition of the gap $\widehat{\delta}$ between $\mathcal{S}$ and $\mathcal{S}_h$:

$$\widehat{\delta}(\mathcal{S}, \mathcal{S}_h) := \max\left\{ \delta(\mathcal{S}, \mathcal{S}_h), \delta(\mathcal{S}_h, \mathcal{S}) \right\},$$

with

$$\delta(\mathcal{S}, \mathcal{S}_h) := \sup_{\substack{(\mathbf{w}, \xi, p) \in \mathcal{S} \\ \|(\mathbf{w}, \xi, p)\|_{\mathcal{H}_\epsilon} = 1}} \left[\inf_{(\mathbf{w}_h, \xi_h, p_h) \in \mathcal{S}_h} \left\| (\mathbf{w} - \mathbf{w}_h, \xi - \xi_h, p - p_h) \right\|_{\mathcal{H}_\epsilon} \right],$$

and $\delta(\mathcal{S}_h, \mathcal{S})$ analogously defined.

The following theorem is an immediate consequence of Theorem 7.1 in [3] and Lemmas 4.3 and 4.6:

Theorem 4.2 *For $\epsilon \in (0, s - \frac{1}{2})$, with $s \in (\frac{1}{2}, 1]$ as in Theorem 2.2, there holds*

$$\widehat{\delta}(\mathcal{S}_h, \mathcal{S}) \le C h^{r + r'_\epsilon},$$

with r and r'_ϵ as in Lemma 4.6, and C a strictly positive constant not depending on ϵ.

As a consequence of this theorem, we can prove a higher order error estimate for the post-processed pressure p_h^{L}, which is the main result of this paper:

Theorem 4.3 *Let* $(\lambda_h, (\mathbf{u}_h, \mathbf{w}_h))$ *be a solution of Problem (2.10) such that* $\lambda_h \to \lambda$ *as* $h \to 0$. *Let* p_h^{L} *be the post-processed pressure defined by (3.1)–(3.2). Then, there exists a solution* $(\lambda, (\mathbf{u}, \mathbf{w}))$ *of Problem (2.8) and a strictly positive constant* C *such that*

$$\left\| p - p_h^{\mathrm{L}} \right\|_{0,\Omega_{\mathrm{F}}} \leq C h^{r+r'} \left(\|p\|_{0,\Omega_{\mathrm{F}}} + \|\mathbf{w}\|_{0,\Omega_{\mathrm{S}}} \right),$$

where $p = -\rho_{\mathrm{F}} c^2 \operatorname{div} \mathbf{u}$ *is the corresponding fluid pressure,* $r = \min\{s, t\}$, *and* $r' = \min\{\frac{1}{2}, t\}$, *with* $s \in (\frac{1}{2}, 1]$ *and* $t \in (0, 1]$ *as in Theorem 2.2.*

Proof. For $(\lambda_h, (\mathbf{u}_h, \mathbf{w}_h))$ a solution of Problem (2.10), let $(\lambda_h, (p_h, \mathbf{w}_h))$ be the solution of Problem (3.3) as in Lemma 3.1. From Theorem 4.2, there exists a solution $(\lambda, (p, \mathbf{w}))$ of Problem (2.9) such that

$$\begin{aligned} \|p - p_h\|_{0,\Omega_{\mathrm{F}}} + \|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega_{\mathrm{S}}} &\leq C h^{r+r'_\epsilon} \|(\mathbf{w}, \mathbf{w}\cdot\mathbf{n}, p)\|_{\mathcal{H}_\epsilon} \\ &\leq C h^{r+r'_\epsilon} \left(\|p\|_{0,\Omega_{\mathrm{F}}} + \|\mathbf{w}\|_{1,\Omega_{\mathrm{S}}} \right) \end{aligned} \tag{4.25}$$

$\forall \epsilon \in (0, s - \frac{1}{2})$, with C independent of ϵ and $r'_\epsilon = (1 - 2\epsilon) r'$. On the other hand, from Lemma 3.2, we have

$$\begin{aligned} \left\| p - p_h^{\mathrm{L}} \right\|_{0,\Omega_{\mathrm{F}}} &\leq \|p - p_h\|_{0,\Omega_{\mathrm{F}}} + \left\| p_h - p_h^{\mathrm{L}} \right\|_{0,\Omega_{\mathrm{F}}} \\ &\leq \|p - p_h\|_{0,\Omega_{\mathrm{F}}} + C h^2 \|p_h\|_{0,\Omega_{\mathrm{F}}} \\ &\leq \|p - p_h\|_{0,\Omega_{\mathrm{F}}} + C h^2 \left(\|p - p_h\|_{0,\Omega_{\mathrm{F}}} + \|p\|_{0,\Omega_{\mathrm{F}}} \right) \\ &\leq C \left(\|p - p_h\|_{0,\Omega_{\mathrm{F}}} + h^2 \|p\|_{0,\Omega_{\mathrm{F}}} \right). \end{aligned}$$

Thus, from these two estimates, Lemma 2.1, and Theorem 2.2, we obtain $\forall \epsilon \in (0, s - \frac{1}{2})$,

$$\left\| p - p_h^{\mathrm{L}} \right\|_{0,\Omega_{\mathrm{F}}} \leq C h^{r+r'_\epsilon} \left(\|p\|_{0,\Omega_{\mathrm{F}}} + \|\mathbf{w}\|_{0,\Omega_{\mathrm{S}}} \right).$$

Therefore, the theorem follows by passing to the limit as $\epsilon \to 0$. □

Remark 4.2 As a by-product of (4.25) in the previous proof, we have also proved a higher order error estimate in $\mathrm{L}^2(\Omega_{\mathrm{S}})$ norm for the solid displacements $\mathbf{w}_h$ obtained from Problem (2.10).

5 Lemmata

This section contains statements and proofs of some technical lemmas used above. We preserve the notation of the previous section.

Lemma 5.1 *There exists a strictly positive constant α, independent of h, such that*

$$a_h\left((\mathbf{z}_h,\zeta_h,q_h),(\mathbf{z}_h,\zeta_h,q_h)\right)\geq\alpha\|(\mathbf{z}_h,\zeta_h,q_h)\|^2_{\mathcal{W}_h}\qquad\forall(\mathbf{z}_h,\zeta_h,q_h)\in\mathcal{W}_h.$$

Proof. Clearly, it is enough to show that there exists a positive constant α_1, independent of h, such that

$$\alpha_1\|q_h\|^2_{1,\Omega_{\mathrm{F}}}\leq\sum_{T\in\mathcal{T}_h^{\mathrm{F}}}\int_T\left|\Pi_T\boldsymbol{\nabla}q_h\right|^2+\sum_{T\in\mathcal{T}_h^{\mathrm{F}}}\int_T|Pq_h|^2\quad\forall q_h\in\mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}}).$$

Now, for any $T\in\Omega_{\mathrm{F}}$, a scaling argument applied to $\|q_h\|_{0,T}$ leads to

$$\|q_h\|^2_{1,T}\leq C\left[\|\boldsymbol{\nabla}q_h\|^2_{0,T}+\frac{1}{|T|}\left(\int_T q_h\right)^2\right],$$

with C only depending on the regularity of T. Since

$$\frac{1}{|T|}\left(\int_T q_h\right)^2=\int_T|Pq_h|^2,$$

we only need to prove that $\|\boldsymbol{\nabla}q_h\|^2_{0,T}\leq C\left\|\Pi_T\boldsymbol{\nabla}q_h\right\|^2_{0,T}$.

Now, since $q_h\in\mathcal{CR}_h^{\mathrm{b}}(\Omega_{\mathrm{F}})$, we can write $q_h|_T=q_{\mathrm{L}}+\alpha_T b_T$, with q_{L} a linear function and b_T the basis bubble function introduced in Remark 3.4. Straightforward computations yield

$$\|\boldsymbol{\nabla}q_h\|^2_{0,T}=\left\|\boldsymbol{\nabla}q_{\mathrm{L}}\right\|^2_{0,T}+\alpha_T^2\left\|\boldsymbol{\nabla}b_T\right\|^2_{0,T}$$

and

$$\begin{aligned}\left\|\Pi_T\boldsymbol{\nabla}q_h\right\|^2_{0,T}&=\left\|\Pi_T\boldsymbol{\nabla}q_{\mathrm{L}}\right\|^2_{0,T}+\alpha_T^2\left\|\Pi_T\boldsymbol{\nabla}b_T\right\|^2_{0,T}\\&=\left\|\boldsymbol{\nabla}q_{\mathrm{L}}\right\|^2_{0,T}+\alpha_T^2\left\|\Pi_T\boldsymbol{\nabla}b_T\right\|^2_{0,T},\end{aligned}$$

the latter because $\boldsymbol{\nabla}q_{\mathrm{L}}\in\mathcal{RT}_0(T)$. Thus, we only need to prove that

$$\left\|\boldsymbol{\nabla}b_T\right\|^2_{0,T}\leq C\left\|\Pi_T\boldsymbol{\nabla}b_T\right\|^2_{0,T}.$$

Scaling arguments show that

$$\left\|\boldsymbol{\nabla}b_T\right\|^2_{0,T}\leq C.$$

On the other hand, using that $\mathcal{RT}_0(T) = \text{span}\{(1,0),(0,1),(x,y)\}$, we obtain

$$\Pi_T \nabla b_T(x,y) = \frac{-9|T|}{10\int_T |(x-x_T, y-y_T)|^2}(x-x_T, y-y_T) \qquad \forall (x,y) \in T,$$

where (x_T, y_T) is the barycenter of T. Hence, since $\int_T |(x-x_T, y-y_T)|^2 = \frac{|T|}{3}\sum_{\ell\subset\partial T}|\ell|^2$, we have

$$\|\Pi_T \nabla b_T\|_{0,T}^2 = \left(\frac{9}{10}\right)^2 \frac{|T|^2}{\int_T |(x-x_T, y-y_T)|^2} = \frac{2.43\,|T|}{\sum_{\ell\subset\partial T}|\ell|^2} \geq C,$$

and we conclude the lemma. □

In the proofs of the following lemmas we will use several times the following local trace inequality:

Lemma 5.2 *Let T be a triangle and ℓ one of its edges. There exists a positive constant C only depending on the minimum angle of T, such that, if $q \in \mathrm{H}^\nu(T)$ with $\nu \in (\frac{1}{2}, 1]$, then*

$$\|q\|_{0,\ell} \leq C\left(|\ell|^{-\frac{1}{2}}\|q\|_{0,T} + |\ell|^{\nu-\frac{1}{2}}|q|_{\nu,T}\right).$$

Proof. Since the inequality holds for the reference triangle $\widehat{T}$ of vertices $(0,0)$, $(1,0)$, and $(0,1)$ (see [16]), a change of variable leads to

$$\frac{1}{|\ell|}\|q\|_{0,\ell}^2 \leq \widehat{C}\left(\frac{1}{|T|}\|q\|_{0,T}^2 + \frac{h_T^{2+2\nu}}{|T|^2}|q|_{\nu,T}^2\right),$$

where h_T is the diameter T and $\widehat{C}$ only depends on the reference triangle $\widehat{T}$. Thus, the lemma follows. □

Lemma 5.3 *The following inequality holds*

$$a_h\left((\tilde{\mathbf{w}}^{\mathrm{I}} - \tilde{\mathbf{w}}, (\tilde{\mathbf{w}}^{\mathrm{I}} - \tilde{\mathbf{w}})\cdot\mathbf{n}, \tilde{p}^{\mathrm{I}} - \tilde{p}), (\mathbf{z}_h, \zeta_h, q_h)\right) \leq \begin{cases} C_\epsilon h^{r_\epsilon}\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h} & \textit{for } (\mathbf{f},\eta,g) \in \mathcal{H}_\epsilon, \\ C h^{r}\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h} & \textit{for } (\mathbf{f},\eta,g) \in \mathcal{S}. \end{cases}$$

Proof. From the definition of a_h we have

$$
\begin{aligned}
&a_h\left((\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}, (\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}})\cdot\mathbf{n}, \tilde{p}^{\mathrm{I}}-\tilde{p}), (\mathbf{z}_h,\zeta_h,q_h)\right)\\
&\quad=\int_{\Omega_{\mathrm{S}}}\boldsymbol{\sigma}(\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}):\boldsymbol{\varepsilon}(\mathbf{z}_h)+\int_{\Omega_{\mathrm{S}}}\rho_{\mathrm{S}}\left(\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}\right)\cdot\mathbf{z}_h\\
&\qquad+\sum_{T\in\mathcal{T}_h^{\mathrm{F}}}\int_T\frac{1}{\rho_{\mathrm{F}}}\nabla\left(\tilde{p}^{\mathrm{I}}-\tilde{p}\right)\cdot\Pi_T\nabla q_h\\
&\qquad+\int_{\Omega_{\mathrm{F}}}\frac{1}{\rho_{\mathrm{F}}c^2}\left(\tilde{p}^{\mathrm{I}}-\tilde{p}\right)Pq_h-\int_{\Gamma_{\mathrm{I}}}\zeta_h\,P_{\Gamma_{\mathrm{I}}}\left(\tilde{p}^{\mathrm{I}}-\tilde{p}\right)\\
&\qquad+\int_{\Gamma_{\mathrm{I}}}\left(\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}\right)\cdot\mathbf{n}\,P_{\Gamma_{\mathrm{I}}}q_h\\
&\quad\le C\left\|(\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}, (\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}})\cdot\mathbf{n}, \tilde{p}^{\mathrm{I}}-\tilde{p})\right\|_{\mathcal{W}_h}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h}\\
&\qquad+\int_{\Gamma_{\mathrm{I}}}\left(\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}\right)\cdot\mathbf{n}\,P_{\Gamma_{\mathrm{I}}}q_h.
\end{aligned}
$$

For the first term on the right hand side of the inequality above we have

$$
\begin{aligned}
&\left\|(\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}, (\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}})\cdot\mathbf{n}, \tilde{p}^{\mathrm{I}}-\tilde{p})\right\|_{\mathcal{W}_h}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h}\\
&\quad\le C_\epsilon h^{r_\epsilon}\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h}
\end{aligned}
$$

if $(\mathbf{f},\eta,g)\in\mathcal{H}_\epsilon$ (from (4.10)), and a similar inequality with $C_\epsilon h^{r_\epsilon}$ substituted by Ch^r if $(\mathbf{f},\eta,g)\in\mathcal{S}$ (from (4.11)).

To estimate the second term, for each edge $\ell\subset\Gamma_{\mathrm{I}}$, let $T_{\mathrm{F}}\in\mathcal{T}_h^{\mathrm{F}}$ and $T_{\mathrm{S}}\in\mathcal{T}_h^{\mathrm{S}}$ be such that $T_{\mathrm{F}}\cap T_{\mathrm{S}}=\ell$. For $\tilde{\mathbf{w}}\in\mathrm{H}^{1+t}(\Omega_{\mathrm{S}})^2$, Lemma 5.2 applied to $(\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}})$ and q_h, both with $\nu=1$, and standard error estimates for the Lagrange interpolant $\tilde{\mathbf{w}}^{\mathrm{I}}$, yield

$$
\begin{aligned}
\int_\ell\left(\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}\right)\cdot\mathbf{n}\,P_\ell q_h&\le\left\|\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}}\right\|_{0,\ell}\|q_h\|_{0,\ell}\\
&\le C\left(|\ell|^{\frac{1}{2}+t}\|\tilde{\mathbf{w}}\|_{1+t,T_{\mathrm{S}}}\right)\left(|\ell|^{-\frac{1}{2}}\|q_h\|_{1,T_{\mathrm{F}}}\right).
\end{aligned}
$$

Then, summing up on all the edges $\ell\subset\Gamma_{\mathrm{I}}$ and using Lemma 4.1 we have

$$
\int_{\Gamma_{\mathrm{I}}}(\tilde{\mathbf{w}}^{\mathrm{I}}-\tilde{\mathbf{w}})\cdot\mathbf{n}\,P_{\Gamma_{\mathrm{I}}}q_h\le Ch^t\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|\mathbf{z},\zeta,q)\|_{\mathcal{W}_h}.
$$

Since both, $r_\epsilon<t$ and $r<t$, then we conclude the proof. □

Lemma 5.4 *The following estimate holds*

$$
\begin{aligned}
&\left|\frac{1}{\rho_{\mathrm{F}}}\sum_{\ell\in\mathcal{E}_h^{\mathrm{o}}}\int_\ell\frac{\partial\tilde{p}}{\partial n_\ell}[\![q_h]\!]_\ell\right|\\
&\quad\le\begin{cases}C_\epsilon h^{\frac{1}{2}+\epsilon}\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h} & \textit{for } (\mathbf{f},\eta,g)\in\mathcal{H}_\epsilon,\\ Ch^s\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h} & \textit{for } (\mathbf{f},\eta,g)\in\mathcal{S}.\end{cases}
\end{aligned}
$$

Proof. For $\ell \in \mathcal{E}_h^{\mathrm{o}}$, let $T_1, T_2 \in \mathcal{T}_h^{\mathrm{F}}$ be such that $T_1 \cap T_2 = \ell$. Since $[\![q_h]\!]_\ell$ is a linear function vanishing at the midpoint of ℓ, we have

$$\begin{aligned}
\left|\int_\ell \frac{\partial \tilde{p}}{\partial n_\ell}[\![q_h]\!]_\ell\right| &= \left|\int_\ell \left[\frac{\partial \tilde{p}}{\partial n_\ell} - P_\ell\left(\frac{\partial \tilde{p}}{\partial n_\ell}\right)\right][\![q_h]\!]_\ell\right| \\
&= \left|\int_\ell \left[\frac{\partial \tilde{p}}{\partial n_\ell} - P_\ell\left(\frac{\partial \tilde{p}}{\partial n_\ell}\right)\right](q_h|_{T_1})\right. \\
&\quad \left. - \int_\ell \left[\frac{\partial \tilde{p}}{\partial n_\ell} - P_\ell\left(\frac{\partial \tilde{p}}{\partial n_\ell}\right)\right](q_h|_{T_2})\right| \\
&\le \sum_{i=1,2} \left|\int_\ell \left[\frac{\partial \tilde{p}}{\partial n_\ell} - P_\ell\left(\frac{\partial \tilde{p}}{\partial n_\ell}\right)\right][(q_h|_{T_i}) - P_\ell(q_h|_{T_i})]\right| \\
&\le \sum_{i=1,2} \|\nabla \tilde{p}\cdot \mathbf{n}_\ell - P(\nabla \tilde{p}\cdot \mathbf{n}_\ell)\|_{0,\ell}\, \|(q_h|_{T_i}) - P(q_h|_{T_i})\|_{0,\ell}.
\end{aligned}$$

For $(\mathbf{f}, \eta, g) \in \mathcal{H}_\epsilon$, by using Lemma 5.2 applied to $[\nabla \tilde{p}\cdot \mathbf{n}_\ell - P(\nabla \tilde{p}\cdot \mathbf{n}_\ell)]$ with $\nu = \frac{1}{2} + \epsilon$, and to $[(q_h|_{T_i}) - P(q_h|_{T_i})]$, $i = 1, 2$, with $\nu = 1$, from standard estimates for the projection P, we obtain

$$\left|\int_\ell \frac{\partial \tilde{p}}{\partial n_\ell}[\![q_h]\!]_\ell\right| \le C \sum_{i=1,2} \left(|\ell|^\epsilon \|\nabla \tilde{p}\|_{\frac{1}{2}+\epsilon, T_i}\right)\left(|\ell|^{\frac{1}{2}} \|\nabla q_h\|_{0,T_i}\right).$$

Thus, summing up on all the edges $\ell \in \mathcal{E}_h^{\mathrm{o}}$ and using Lemma 4.1, we conclude the lemma for $(\mathbf{f}, \eta, g) \in \mathcal{H}_\epsilon$. The case $(\mathbf{f}, \eta, g) \in \mathcal{S}$ can be dealt with analogously. □

Lemma 5.5 *The following estimate holds*

$$\begin{aligned}
&\left|\frac{1}{\rho_{\mathrm{F}}} \sum_{T \in \mathcal{T}_h^{\mathrm{F}}} \int_T \left(\nabla \tilde{p} - \Pi_T \nabla \tilde{p}\right)\cdot \nabla q_h\right| \\
&\quad \le \begin{cases} C_\epsilon h^{\frac{1}{2}+\epsilon} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \|(\mathbf{z}_h, \zeta_h, q_h)\|_{\mathcal{W}_h} & \textit{for } (\mathbf{f}, \eta, g) \in \mathcal{H}_\epsilon, \\ C h^s \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \|(\mathbf{z}_h, \zeta_h, q_h)\|_{\mathcal{W}_h} & \textit{for } (\mathbf{f}, \eta, g) \in \mathcal{S}. \end{cases}
\end{aligned}$$

Proof. It is an immediate consequence of standard error estimates for Π_T and Lemma 4.1. □

Lemma 5.6 *The following estimates hold*

$$\left|\frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} \tilde{p}(q_h - P q_h)\right| \le C h \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \|(\mathbf{z}_h, \zeta_h, q_h)\|_{\mathcal{W}_h}$$

and

$$\left|\frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} g(q_h - P q_h)\right| \le C h \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \|(\mathbf{z}_h, \zeta_h, q_h)\|_{\mathcal{W}_h}.$$

Proof. They follow immediately from the standard error estimate for P and (4.2). □

Lemma 5.7 *The following estimate holds*

$$\left|\int_{\Gamma_{\mathrm{I}}} \tilde{\mathbf{w}}\cdot\mathbf{n}\left(q_h - P_{\Gamma_{\mathrm{I}}} q_h\right)\right| \le Ch\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h}.$$

Proof. We have

$$\left|\int_{\Gamma_{\mathrm{I}}} \tilde{\mathbf{w}}\cdot\mathbf{n}\left(q_h - P_{\Gamma_{\mathrm{I}}} q_h\right)\right| \le \left\|\tilde{\mathbf{w}}\cdot\mathbf{n} - P_{\Gamma_{\mathrm{I}}}(\tilde{\mathbf{w}}\cdot\mathbf{n})\right\|_{0,\Gamma_{\mathrm{I}}}\left\|q_h - P_{\Gamma_{\mathrm{I}}} q_h\right\|_{0,\Gamma_{\mathrm{I}}}.$$

By using standard error estimates for $P_{\Gamma_{\mathrm{I}}}$ and the trace theorem, we bound the first term on the right hand side as follows

$$\left\|\tilde{\mathbf{w}}\cdot\mathbf{n} - P_{\Gamma_{\mathrm{I}}}(\tilde{\mathbf{w}}\cdot\mathbf{n})\right\|_{0,\Gamma_{\mathrm{I}}} \le Ch^{\min\left\{1,\frac{1}{2}+t\right\}}\|\tilde{\mathbf{w}}\|_{1+t,\Omega_{\mathrm{S}}}.$$

On the other hand, for each edge $\ell\subset\Gamma_{\mathrm{I}}$, let $T_\ell\in\mathcal{T}_h^{\mathrm{F}}$ be such that $\ell\subset\partial T_\ell$. By using Lemma 5.2 applied to $(q_h - Pq_h)$ with $\nu=1$ and the error estimate for P, we have

$$\left\|q_h - P_{\Gamma_{\mathrm{I}}} q_h\right\|_{0,\ell} \le \|q_h - Pq_h\|_{0,\ell} \le C|\ell|^{\frac{1}{2}}\|\nabla q_h\|_{0,T_\ell}.$$

Thus, summing up on all the edges $\ell\subset\Gamma_{\mathrm{I}}$ and using Lemma 4.1, since $\min\{\frac{3}{2},1+t\}>1$, we conclude the proof. □

Lemma 5.8 *The following estimate holds*

$$\left|\int_{\Gamma_{\mathrm{I}}} \eta(q_h - P_{\Gamma_{\mathrm{I}}} q_h)\right| \le \begin{cases} Ch^{\frac{1}{2}+\epsilon}\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h} & \textit{for } (\mathbf{f},\eta,g)\in\mathcal{H}_\epsilon,\\ Ch\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon}\|(\mathbf{z}_h,\zeta_h,q_h)\|_{\mathcal{W}_h} & \textit{for } (\mathbf{f},\eta,g)\in\mathcal{S}.\end{cases}$$

Proof. For $(\mathbf{f},\eta,g)\in\mathcal{S}$, $\eta=\mathbf{f}\cdot\mathbf{n}=\mu\tilde{\mathbf{w}}\cdot\mathbf{n}\in\prod_{j=1}^{J}\mathrm{H}^{\frac{1}{2}}(\Gamma_j)$, with

$$\left(\sum_{j=1}^{J}\|\eta\|^2_{\frac{1}{2},\Gamma_j}\right)^{\frac{1}{2}} \le C\|\tilde{\mathbf{w}}\|_{1,\Omega_{\mathrm{S}}} \le C\|(\mathbf{f},\eta,g)\|_{\mathcal{H}_\epsilon},$$

the latter because of (4.2). Then, we proceed in this case as in the proof of the previous lemma, with $\tilde{\mathbf{w}}\cdot\mathbf{n}$ substituted by η. On the other hand, for $(\mathbf{f},\eta,g)\in\mathcal{H}_\epsilon$, $\eta\in\mathrm{H}^\epsilon(\Gamma_{\mathrm{I}})$ and we modify the proof by using that $\left\|\eta - P_{\Gamma_{\mathrm{I}}}\eta\right\|_{0,\ell}\le Ch^\epsilon\|\eta\|_{\epsilon,\ell}$ in this case. □

Lemma 5.9 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left| \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}(\mathbf{e}_{\tilde{\mathbf{w}}}) : \boldsymbol{\varepsilon}(\mathbf{w}_* - \mathbf{w}_*^{\mathrm{I}}) + \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} \mathbf{e}_{\tilde{\mathbf{w}}} \cdot \left(\mathbf{w}_* - \mathbf{w}_*^{\mathrm{I}}\right) \right| \\ \leq C h^{r+r'} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0}.$$

Proof. The inequality follows immediately from standard approximation properties of the Lagrange interpolant of $\mathbf{w}_*$, and Lemmas 4.4 and 4.2. □

Lemma 5.10 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left| \int_{\Gamma_{\mathrm{I}}} e_{\tilde{\mathbf{w}}\cdot\mathbf{n}} \left(p_* - p_*^{\mathrm{I}}\right) \right| \leq C h^{r+1} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0}.$$

Proof. Using the trace theorem and standard approximation properties of the Lagrange interpolant of $p_*|_{\Gamma_{\mathrm{I}}}$ we have

$$\left| \int_{\Gamma_{\mathrm{I}}} e_{\tilde{\mathbf{w}}\cdot\mathbf{n}} \left(p_* - p_*^{\mathrm{I}}\right) \right| \leq C h \|p_*\|_{\frac{3}{2},\Omega_{\mathrm{F}}} \|\mathbf{e}_{\tilde{\mathbf{w}}}\|_{1,\Omega_{\mathrm{S}}}.$$

Then, the result follows from Lemmas 4.4 and 4.2. □

Lemma 5.11 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left| \frac{1}{\rho_{\mathrm{F}}} \sum_{T \in \mathcal{T}_h^{\mathrm{F}}} \int_T \nabla e_{\tilde{p}} \cdot \nabla \left(p_* - p_*^{\mathrm{I}}\right) + \frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} e_{\tilde{p}} \left(p_* - p_*^{\mathrm{I}}\right) \right| \\ \leq C h^{r+s} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0}.$$

Proof. It is a direct application of the standard approximation properties of the Lagrange interpolant of p_* and Lemmas 4.4 and 4.2. □

Lemma 5.12 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left| \int_{\Gamma_{\mathrm{I}}} e_{\tilde{p}} \left(\mathbf{w}_* - \mathbf{w}_*^{\mathrm{I}}\right) \cdot \mathbf{n} \right| \leq C h^{r+r'} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0}.$$

Proof. For each edge $\ell \subset \Gamma_{\mathrm{I}}$, let $T_{\mathrm{F}} \in \mathcal{T}_h^{\mathrm{F}}$ and $T_{\mathrm{S}} \in \mathcal{T}_h^{\mathrm{S}}$ be such that $T_{\mathrm{F}} \cap T_{\mathrm{S}} = \ell$. Lemma 5.2 applied to $e_{\tilde{p}}$ and $(\mathbf{w}_* - \mathbf{w}_*^{\mathrm{I}})$, both with $\nu = 1$, and the standard approximation properties of the Lagrange interpolant $\mathbf{w}_*^{\mathrm{I}}$ yield

$$\begin{aligned} \left| \int_{\ell} e_{\tilde{p}} \left(\mathbf{w}_* - \mathbf{w}_*^{\mathrm{I}}\right) \cdot \mathbf{n} \right| &\leq \left\|e_{\tilde{p}}\right\|_{0,\ell} \left\|\mathbf{w}_* - \mathbf{w}_*^{\mathrm{I}}\right\|_{0,\ell} \\ &\leq C \left(|\ell|^{-\frac{1}{2}} \left\|e_{\tilde{p}}\right\|_{1,T_{\mathrm{F}}}\right) \left(|\ell|^{\frac{1}{2}+r'} \|\mathbf{w}_*\|_{1+r',T_{\mathrm{S}}}\right). \end{aligned}$$

Thus, summing up on all the edges $\ell \subset \Gamma_{\mathrm{I}}$ and using Lemmas 4.4 and 4.2, we conclude the proof. □

Lemma 5.13 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left| \frac{1}{\rho_{\mathrm{F}}} \sum_{\ell \in \mathcal{E}_h^{\mathrm{o}}} \int_\ell [\![e_{\tilde{p}}]\!]_\ell \frac{\partial p_*}{\partial n_\ell} \right| \leq C h^{r+s} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{H}_0}.$$

Proof. First we observe that $\tilde{p}$ is continuous because of Lemma 4.4. Consequently, we have $\int_\ell [\![e_{\tilde{p}}]\!]_\ell = -\int_\ell [\![p_h]\!]_\ell = 0$. Then, proceeding as in the proof of Lemma 5.4, we obtain

$$\left| \int_\ell [\![e_{\tilde{p}}]\!]_\ell \frac{\partial p_*}{\partial n_\ell} \right| \leq C \sum_{i=1,2} \left(|\ell|^{s-\frac{1}{2}} \|\boldsymbol{\nabla} p_*\|_{s,T_i} \right) \left(|\ell|^{\frac{1}{2}} \left\| \boldsymbol{\nabla} e_{\tilde{p}} \right\|_{0,T_i} \right),$$

where $T_1, T_2 \in \mathcal{T}_h^{\mathrm{F}}$ are such that $T_1 \cap T_2 = \ell$. Thus, summing up on all the edges $\ell \in \mathcal{E}_h^{\mathrm{o}}$ and using Lemma 4.4 and 4.2 we conclude the proof. □

Lemma 5.14 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left| \frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} \tilde{p}_h \left(p_*^{\mathrm{I}} - P p_*^{\mathrm{I}} \right) \right| \leq C h^2 \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{H}_0}.$$

Proof. Since $\tilde{p}_h|_T \in \mathrm{H}^1(T)\ \forall T \in \mathcal{T}_h^{\mathrm{F}}$ and $p_* \in \mathrm{H}^{1+s}(\Omega_{\mathrm{F}})$, we have

$$\begin{aligned} \int_T \tilde{p}_h \left(p_*^{\mathrm{I}} - P p_*^{\mathrm{I}} \right) = \int_T (\tilde{p}_h - P\tilde{p}_h) \left(p_*^{\mathrm{I}} - P p_*^{\mathrm{I}} \right) &\leq C h^2 \|\boldsymbol{\nabla} \tilde{p}_h\|_{0,T} \left\| \boldsymbol{\nabla} p_*^{\mathrm{I}} \right\|_{0,T} \\ &\leq C h^2 \left(\|\boldsymbol{\nabla} \tilde{p}\|_{0,T} + \left\| \boldsymbol{\nabla} e_{\tilde{p}} \right\|_{0,T} \right) \|p_*\|_{1+s,T}, \end{aligned}$$

where we have used standard error estimates for P and the Lagrange interpolant p_*^{I}, and the definition of $e_{\tilde{p}}$. Thus, after summing up on all the triangles $T \in \mathcal{T}_h^{\mathrm{F}}$, the lemma follows from (4.2) and Lemmas 4.4 and 4.2. □

Lemma 5.15 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left| \int_{\Gamma_{\mathrm{I}}} \tilde{\mathbf{w}}_h \cdot \mathbf{n} \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}} \right) \right| \leq C h^{1+r} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{H}_0}.$$

Proof. We have

$$\int_{\Gamma_{\mathrm{I}}} \tilde{\mathbf{w}}_h \cdot \mathbf{n} \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}} \right) = \int_{\Gamma_{\mathrm{I}}} \tilde{\mathbf{w}} \cdot \mathbf{n} \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}} \right) - \int_{\Gamma_{\mathrm{I}}} e_{\tilde{\mathbf{w}}\cdot\mathbf{n}} \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}} \right).$$

The first term on the right hand side above can be treated as in the proof of Lemma 5.7. Thus, by using Lemma 4.1, we obtain

(5.1)
$$\left| \int_{\Gamma_{\mathrm{I}}} \tilde{\mathbf{w}} \cdot \mathbf{n} \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}} \right) \right| \leq C h^{\min\left\{1, \frac{1}{2}+t\right\}} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\| p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}} \right\|_{0,\Gamma_{\mathrm{I}}}.$$

For the second term we apply Lemma 4.2 to obtain

$$\left|\int_{\Gamma_{\mathrm{I}}} e_{\tilde{\mathbf{w}}\cdot\mathbf{n}} \left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}}\right)\right| \leq Ch^r \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\| p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}} \right\|_{0,\Gamma_{\mathrm{I}}}.$$

Now, using standard error estimates for the projection $P_{\Gamma_{\mathrm{I}}}$ and the Lagrange interpolant p_*^{I}, and Lemma 4.4, we have

(5.2)

$$\begin{aligned}\left\| p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}} \right\|_{0,\Gamma_{\mathrm{I}}} &\leq \left\| p_*^{\mathrm{I}} - p_* \right\|_{0,\Gamma_{\mathrm{I}}} + \left\| p_* - P_{\Gamma_{\mathrm{I}}} p_* \right\|_{0,\Gamma_{\mathrm{I}}} + \left\| P_{\Gamma_{\mathrm{I}}} (p_* - p_*^{\mathrm{I}}) \right\|_{0,\Gamma_{\mathrm{I}}} \\ &\leq Ch \|p_*\|_{\frac{3}{2},\Omega_{\mathrm{F}}} \leq Ch \left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}\right\|_{\mathcal{H}_0}.\end{aligned}$$

Thus, since $r \leq \min\{1, \frac{1}{2} + t\}$, we conclude the lemma. □

Lemma 5.16 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$, *the following estimate holds*

$$\left|\int_{\Gamma_{\mathrm{I}}} \left(\tilde{p}_h - P_{\Gamma_{\mathrm{I}}} \tilde{p}_h\right) \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n}\right| \leq Ch^{1+r'} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0}.$$

Proof. Firstly, we have

$$\int_{\Gamma_{\mathrm{I}}} \left(\tilde{p}_h - P_{\Gamma_{\mathrm{I}}} \tilde{p}_h\right) \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n} \leq \left\| \tilde{p}_h - P_{\Gamma_{\mathrm{I}}} \tilde{p}_h \right\|_{0,\Gamma_{\mathrm{I}}} \left\| \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n} - P_{\Gamma_{\mathrm{I}}} \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n} \right\|_{0,\Gamma_{\mathrm{I}}}.$$

The first term on the right hand side above is bounded as $(q_h - P_{\Gamma_{\mathrm{I}}} q_h)$ in the proof of Lemma 5.7. Then, using the definition of $e_{\tilde{p}}$ we obtain

$$\begin{aligned}\left\| \tilde{p}_h - P_{\Gamma_{\mathrm{I}}} \tilde{p}_h \right\|_{0,\Gamma_{\mathrm{I}}} &\leq Ch^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h^{\mathrm{F}}} \|\tilde{p}_h\|_{1,T}^2 \right)^{\frac{1}{2}} \\ &\leq Ch^{\frac{1}{2}} \left[\|\tilde{p}\|_{1,\Omega_{\mathrm{F}}} + \left(\sum_{T \in \mathcal{T}_h^{\mathrm{F}}} \|e_{\tilde{p}}\|_{1,T}^2 \right)^{\frac{1}{2}} \right] \\ &\leq Ch^{\frac{1}{2}} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon},\end{aligned}$$

the latter because of (4.2) and Lemma 4.2. On the other hand, the second term is bounded as $\left(p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}}\right)$ in Lemma 5.15 and we obtain

$$\left\| \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n} - P_{\Gamma_{\mathrm{I}}} \mathbf{w}_*^{\mathrm{I}} \cdot \mathbf{n} \right\|_{0,\Gamma_{\mathrm{I}}} \leq Ch^{\frac{1}{2}+r'} \|\mathbf{w}_*\|_{1+r',\Omega_{\mathrm{S}}} \leq Ch^{\frac{1}{2}+r'} \left\|(\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}})\right\|_{\mathcal{H}_0}.$$

Thus, we conclude the lemma. □

Lemma 5.17 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left| \frac{1}{\rho_{\mathrm{F}} c^2} \int_{\Omega_{\mathrm{F}}} (g - Pg) p_*^{\mathrm{I}} \right| \leq Ch^2 \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{H}_0}.$$

Proof. We have

$$\int_{\Omega_{\mathrm{F}}} (g - Pg) p_*^{\mathrm{I}} \leq \|g - Pg\|_{0,\Omega_{\mathrm{F}}} \left\| p_*^{\mathrm{I}} - Pp_*^{\mathrm{I}} \right\|_{0,\Omega_{\mathrm{F}}} \leq Ch^2 \|g\|_{1,\Omega_{\mathrm{F}}} \left\| p_*^{\mathrm{I}} \right\|_{1,\Omega_{\mathrm{F}}}$$
$$\leq Ch^2 \|g\|_{1,\Omega_{\mathrm{F}}} \|p_*\|_{1+s,\Omega_{\mathrm{F}}},$$

where we have used standard error estimates for P and the Lagrange interpolant. Now, since $g = \mu \tilde{p}$, for $(\mathbf{f}, \eta, g) \in \mathcal{S}$, then we conclude the proof from the estimate (4.2) and Lemma 4.4. □

Lemma 5.18 *For* $(\mathbf{f}, \eta, g) \in \mathcal{S}$*, the following estimate holds*

$$\left| \int_{\Gamma_{\mathrm{I}}} \eta (p_*^{\mathrm{I}} - P_{\Gamma_{\mathrm{I}}} p_*^{\mathrm{I}}) \right| \leq Ch^{\frac{3}{2}+r'} \|(\mathbf{f}, \eta, g)\|_{\mathcal{H}_\epsilon} \left\| (\mathbf{e}_{\tilde{\mathbf{w}}}, e_{\tilde{\mathbf{w}}\cdot\mathbf{n}}, e_{\tilde{p}}) \right\|_{\mathcal{H}_0}.$$

Proof. Let $(\mathbf{f}, \eta, g) \in \mathcal{S}$. Then, $\eta = \mathbf{f} \cdot \mathbf{n} = \mu \tilde{\mathbf{w}} \cdot \mathbf{n}$. Thus, we proceed as in the proof of Lemma 5.15, and the result follows from estimates (5.1) and (5.2), and the fact that $r' = \min\{\frac{1}{2}, t\}$.

Acknowledgement. The authors want to express their gratitude to Rafael Muñoz-Sola from Universidade de Santiago de Compostela, Spain, for very helpful discussions.

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