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Algebras with implication and fusion: a different point of view

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Abstract. This work uses well-known results on tensor products of lattices and semilattices developed by Fraser and Grätzer et al., and the duality for bounded distributive lattices introduced by Cignoli et al., in order to develop dual categorical equivalences involving bounded distributive lattices with fusion and implication, respectively. We show that these equivalences are essentially those developed by Cabrer and Celani as part of the PhD thesis of the former.

1. Introduction

In recent years, many varieties of algebras associated with many-valued logics have been introduced. Most of these algebras are commutative, integral, bounded distributive residuated lattices ([14]), as for example, the variety of MV-algebras ([6]), the variety of MTL-algebras ([7]), the variety of $ITML$-algebras ([10]), the variety of bounded implicative lattices ([16]), and the variety of WH-algebras ([5]). All these algebras are bounded distributive lattices with two additional binary operations (a fusion and an implication) satisfying special additional conditions.

During the decades of the 70s and 80s, many authors studied tensor products of join-semilattices with zero ([11, 12]; see also [13] and references therein). In 1991, Cignoli et al. [8] proposed a representation theory for the category of bounded distributive lattices and zero-preserving join-homomorphisms.

Independently of [8], Cabrer and Celani introduced in [4] and [3] a categorical duality for algebras with fusion and implications. In [2], the author compares both dualities, looking at the latter as an extension of the former.

The aim of this paper is to develop the duality of Cabrer and Celani directly from [8]. This will be done on the basis of the following known results:

(i) the tensor product of join-semilattices defines a monoidal structure on the category of join-semilattices with zero;

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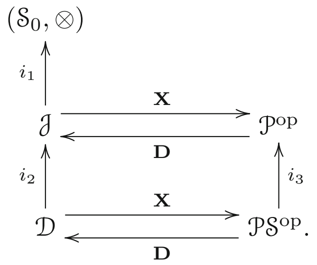
(ii) for any pair of bounded distributive lattices L and M, their tensor product as join-semilattices with zero is a lattice isomorphic to the coproduct of L and M in the category of bounded distributive lattices;
(iii) the duality introduced in [8] extends Priestley duality.

These facts can be summarized in the following diagram of categories and functors;



Functor i_1 is the inclusion of the category of bounded distributive lattices and zero-preserving join-homomorphisms into the category of join-semilattices with zero; i_2 is the inclusion of the category of bounded distributive lattices into the category of bounded distributive lattices and zero-preserving join-homomorphisms, and i_3 is the inclusion of the category of Priestley spaces into the category of spaces defined in [8]. Both i_2 and i_3 are surjective on objects and i_1 is full. Hence, a monoidal structure is inherited by $\mathcal{J}$. Priestley duality is clearly monoidal, when we consider coproduct in the category of bounded distributive lattices and product in the category of Priestley spaces. The main point to be observed is that the duality of Cignoli et al. is also monoidal, when we consider the monoidal structure inherited by $\mathcal{J}$ and the categorical product in $\mathcal{P}$, as we shall show further on.

Since $\mathcal{J}$ is monoidal, fusions and implications can be seen as objects of appropriate subcategories of the category of morphisms of $\mathcal{J}$. Hence, they pass through the duality of Cignoli et al. to certain relations on Priestley spaces.

The paper is structured as follows. In Section 2, we recall some basic results and definitions about monoidal categories. In Section 3, the results from [8] relevant for the present work are recalled. In Section 4, we recall from [11, 12, 13] the results about tensor product of semilattices we shall need further on. In Section 5, appropriate subcategories of the category of morphisms of $\mathcal{J}$ are introduced, in order to represent in $\mathcal{J}$ fusions and implications. In Section 6, we study the monoidal structure of the duality of [8]. In Section 7, we see how fusions and implications pass through the duality. Section 8 is by far the longest and most technical, and is devoted to the comparison of our results with those of Cabrer and Celani. In the last section, we exemplify with the properties of associativity and commutativity of fusion, how diagramatic properties of the operations translate through the duality to certain conditions on the associated relations.

2. Monoidal categories

In this section, we shall recall from [15] the basic definitions and results on monoidal categories used in this article.

A *monoidal category* $\mathcal{M}$ is one with a bifunctor $\square\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ that is associative up to natural isomorphisms $\alpha\colon a\square(b\square c)\to(a\square b)\square c$, and equipped with an object e, which is a unit up to natural isomorphisms $\lambda\colon e\square a\to a$ and $\rho\colon a\square e\to a$. These arrows must also make the following diagrams commute:

$$\begin{array}{ccccc} a\square(b\square(c\square d)) & \xrightarrow{\alpha} & (a\square b)\square(c\square d) & \xrightarrow{\alpha} & ((a\square b)\square c)\square d \\ \downarrow{\scriptstyle 1\square\alpha} & & & & \downarrow{\scriptstyle \alpha\square 1} \\ a\square((b\square c)\square d) & & \xrightarrow{\qquad\alpha\qquad} & & (a\square(b\square c))\square d\ , \end{array} \tag{2.1}$$

$$\begin{array}{ccc} a\square(e\square b) & \xrightarrow{\qquad\alpha\qquad} & (a\square e)\square b\ , \\ {\scriptstyle 1\square\lambda}\searrow & & \swarrow{\scriptstyle \rho\square 1} \\ & a\square b & \end{array} \qquad e\square e \underset{\rho}{\overset{\lambda}{\rightrightarrows}} e\ . \tag{2.2}$$

Note that the commutativity of the second diagram in (2.2) is equivalent to equality of the morphisms ρ and λ.

A monoidal category $(\mathcal{M},\square,e)$ is said to be *symmetric* if there are natural isomorphisms $\sigma\colon a\square b\to b\square a$ making appropriate diagrams commute.

The category of all vector spaces over a fixed field F with the usual tensor product $\otimes_F$ as $\square$ and the one-dimensional vector space F as unit is a standard example of a (symmetric) monoidal category. Other standard examples of (symmetric) monoidal categories are given by any category $\mathcal{C}$ having finite (co)products, with the categorical (co)product as $\square$ and the (initial) terminal object as e.

A (*strong*) *monoidal functor* between monoidal categories $\mathcal{M}=(\mathcal{M},\square,e)$ and $\mathcal{M}'=(\mathcal{M}',\square',e')$ is a functor $F\colon\mathcal{M}\to\mathcal{M}'$, together with natural isomorphisms $\eta\colon F(a)\square F(b)\to F(a\square b)$ and an isomorphism $\theta\colon e'\to F(e)$ in $\mathcal{M}'$, making the following diagrams in $\mathcal{M}'$ commute:

$$\begin{array}{ccc} F(a)\square'(F(b)\square'F(c)) & \xrightarrow{\alpha'} & (F(a)\square'F(b))\square'F(c) \\ \downarrow{\scriptstyle 1\square'\eta} & & \downarrow{\scriptstyle \eta\square'1} \\ F(a)\square'F(b\square c) & & F(a\square b)\square'F(c) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ F((a\square b)\square c) & \xrightarrow{F(\alpha)} & F(a\square(b\square c))\ , \end{array} \tag{2.3}$$

$$\begin{array}{ccc} F(a)\Box' e' & \xrightarrow{\rho'} & F(a) \\ {\scriptstyle 1\Box'\theta}\downarrow & & \uparrow{\scriptstyle F(\rho)} \\ F(a)\Box' F(e) & \xrightarrow{\eta} & F(a\Box e)\ , \end{array} \qquad \begin{array}{ccc} e'\Box' F(a) & \xrightarrow{\lambda'} & F(a) \\ {\scriptstyle \theta\Box' 1}\downarrow & & \uparrow{\scriptstyle F(\lambda)} \\ F(e)\Box' F(a) & \xrightarrow{\eta} & F(e\Box a)\ . \end{array} \tag{2.4}$$

The composition of monoidal functors is monoidal.

A *monoid* in a monoidal category $\mathcal{M}$ is an object m of $\mathcal{M}$ together with two arrows $\mu\colon m\Box m \to m$ and $\nu\colon e \to m$ such that the following diagrams commute:

$$\begin{array}{ccccc} m\Box(m\Box m) & & \xrightarrow{\quad\alpha\quad} & & (m\Box m)\Box m \\ {\scriptstyle 1\Box\mu}\downarrow & & & & \downarrow{\scriptstyle \mu\Box 1} \\ m\Box m & \xrightarrow{\mu} & m & \xleftarrow{\mu} & m\Box m\ , \end{array} \qquad \begin{array}{ccccc} e\Box m & \xrightarrow{\nu\Box 1} & m\Box m & \xleftarrow{1\Box\nu} & m\Box e\ . \\ & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\rho} & \\ & & m & & \end{array} \tag{2.5}$$

For example, a monoid in the category of F-vector spaces is an F-algebra, and a monoid in the category of sets, with the cartesian product as $\Box$, is a monoid (in the usual sense).

A morphism $f\colon (m,\mu,\nu) \to (m',\mu',\nu')$ of monoids in $\mathcal{M}$ is given by an arrow $f\colon m \to m'$ in $\mathcal{M}$ such that $f\mu = \mu'(f\Box f)\colon m\Box m \to m$ and $f\nu = \nu'\colon e' \to m'$. With these arrows, monoids in $\mathcal{M}$ constitute a category, $Mon_{\mathcal{M}}$.

Similarly, we can define a *comonoid* in a monoidal category $\mathcal{M}$ as an object c of $\mathcal{M}$ together with two arrows $\delta\colon c \to c\Box c$ and $\varepsilon\colon c \to e$ such that the duals of the diagrams (2.5) commute.

3. Categories $\mathcal{J}$ and $\mathcal{P}$

Let us start by fixing some notation. We write $\mathcal{D}$ for the category of bounded distributive lattices and $\mathcal{PS}$ for the category of Priestley spaces. We write $\mathbf{X}$ and $\mathbf{D}$ for the functors

$$\mathbf{X} : \mathcal{D} \leftrightarrows \mathcal{PS}^{\mathrm{op}} : \mathbf{D}$$

that realize Priestley duality. For $L \in \mathcal{D}$ and $X \in \mathcal{PS}$, we write φ_L and ϵ_X for the isomorphisms $\varphi_L\colon L \to \mathbf{D}(\mathbf{X}(L))$ and $\epsilon_X\colon X \to \mathbf{X}(\mathbf{D}(X))$ given by the duality.

Let $L, M \in \mathcal{D}$. By a join-homomorphism from L into M we understand a mapping $j\colon L \to M$ such that $j(0) = 0$ and $j(a \vee b) = j(a) \vee j(b)$. The meet-homomorphisms are defined dually. A map $h\colon L \to M$ is a homomorphism if and only if it is both a join-homomorphism and a meet-homomorphism. The category whose objects are bounded distributive lattices and whose morphisms are join-homomorphisms will be denoted by $\mathcal{J}$. Note that isomorphisms in $\mathcal{D}$ and $\mathcal{J}$ are the same, the one-to-one and onto homomorphisms.

Given a relation $R \subseteq X \times Y$, for each $x \in X$, $R(x)$ will denote the image of $\{x\}$ by R, i.e., $R(x) = \{y \in Y : (x,y) \in R\}$.

If X and Y are Priestley spaces, $R \subseteq X \times Y$, and V is a subset of Y, we define $\mathbf{D}(R)(V) := \{x \in X : R(x) \cap V \neq \emptyset\}$. A relation R is said to be a *Priestley relation* provided the following conditions are satisfied:

(i) For every $x \in X$, $R(x)$ is a closed and decreasing subset of Y.
(ii) For each $U \in \mathbf{D}(Y)$, $\mathbf{D}(R)(U) \in \mathbf{D}(X)$.

We write $\mathcal{P}$ for the category whose objects are Priestley spaces and whose morphisms are Priestley relations. If $j\colon L \to M$ is a morphism in $\mathcal{J}$, then $\mathbf{X}(j) \subseteq \mathbf{X}(M) \times \mathbf{X}(L)$, given by

$$(Q, P) \in \mathbf{X}(j) \quad \text{iff} \quad P \subseteq j^{-1}(Q),$$

is a morphism in $\mathcal{P}$. Conversely, if $R \subseteq X \times Y$ is a morphism in $\mathcal{P}$ then the function $\mathbf{D}(R)\colon \mathbf{D}(Y) \to \mathbf{D}(X)$ is a morphism in $\mathcal{J}$. It was proved in [8] that there exists a dual categorical equivalence between $\mathcal{J}$ and $\mathcal{P}$. We shall refer to this equivalence as CLP duality. A Priestley relation is said to be *functional* in the case that $\mathrm{dom}(R) = X$ and $R(x)$ has a greatest element for each $x \in X$. If $h\colon X \to Y$ is a continuous and monotone function between Priestley spaces, then $R_h = \{(x, y) \in X \times Y : y \leq h(x)\}$ is a functional Priestley relation. If $R \subseteq X \times Y$ is a functional Priestley relation, then we can define the morphism of Priestley spaces $h_R\colon X \to Y$ where $h_R(x)$ is the greatest element of $R(x)$. Moreover, we have that $R = R_{h_R}$ and $h = h_{R_h}$ ([8, Remark 1.3(ii)]). Hence, the category of Priestley spaces may be seen as a subcategory of $\mathcal{P}$.

Since CLP duality extends Priestley's, we shall use the same notation for the former; i.e, we also write $\mathbf{X} : \mathcal{J} \leftrightarrows \mathcal{P}^{\mathrm{op}} : \mathbf{D}$ for the functors that realize the adjunction, and φ and ϵ for the corresponding isomorphisms.

4. Tensor product

Write $\mathcal{S}_0$ for the category of join-semilattices with zero. If A, B, and C are objects in $\mathcal{S}_0$, a function $f\colon A \times B \to C$ is said to be a *bimorphism* if the functions $g_a\colon B \to C$ and $h_b\colon A \to C$ defined by $g_a(b) := f(a, b)$ and $h_b(a) := f(a, b)$ are morphisms in $\mathcal{S}_0$ for all $a \in A$ and $b \in B$. A join-semilattice with zero T is a *tensor product* in $\mathcal{S}_0$ of A and B if there is a bimorphism $f\colon A \times B \to T$ such that for any join-semilattice with zero C and any bimorphism $g\colon A \times B \to C$ there is a unique morphism $h\colon T \to C$ in $\mathcal{S}_0$ satisfying $g = hf$. Note that T is generated by $f(A \times B)$. The tensor product of A and B in the category $\mathcal{S}_0$ is often written as $A \otimes B$ and the image of (a, b) under the canonical bimorphism $f\colon A \times B \to A \otimes B$ as $a \otimes b$. Note that if $A, B \in \mathcal{D}$ and $h\colon A \to B$ is a function, then h is a morphism in $\mathcal{S}_0$ iff h is a morphism in $\mathcal{J}$.

Let A and B be bounded distributive lattices. It is well known that in this case, $A \otimes B$ is a bounded distributive lattice ([12, Lemma 3.15]) and it is isomorphic as a lattice to the coproduct in $\mathcal{D}$ of A and B, $A * B$. If $F_1 \in \mathbf{X}(A)$

and $F_2 \in \mathbf{X}(B)$, we write $F_{1,2}$ for the set

$$\Big\{ \bigvee_{i=1}^{n} a_i \otimes b_i \in A \otimes B : \bigvee_{i=1}^{n} a_i \otimes b_i \geq a \otimes b, \text{ for some } a \in F_1 \text{ and } b \in F_2 \Big\}.$$

Conversely, if $F \in \mathbf{X}(A \otimes B)$ we define

$$F_1 = \{a \in A : a \otimes b \in F \text{ for some } b \in B\},$$
$$F_2 = \{b \in B : a \otimes b \in F \text{ for some } a \in A\}.$$

Similarly to the case of join-semilattices studied in [11] (see Theorem 3.5), it can be seen that the following theorem holds.

Theorem 4.1. *Let $A, B \in \mathcal{D}$. Then $\eta \colon \mathbf{X}(A) \times \mathbf{X}(B) \to \mathbf{X}(A \otimes B)$, given by $\eta(F_1, F_2) = F_{1,2}$, is an isomorphism of Priestley spaces and the function η^{-1} is given by $\eta^{-1}(F) = (F_1, F_2)$.*

The following lemma is a part of the folklore of Priestley spaces.

Lemma 4.2.

(i) *Let X be a Priestley space, and let A be an open upset of X. If $x \notin A$, then there exists $U \in \mathbf{D}(X)$ such that $x \in U$ and $U \cap A^c = \emptyset$.*

(ii) *If X and Y are Priestley spaces, then for every $A \in \mathbf{D}(X \times Y)$, there exist $U_1, \ldots, U_n \in \mathbf{D}(X)$ and $V_1, \ldots, V_n \in \mathbf{D}(Y)$ such that $A = \bigcup_{i=1}^{n}(U_i \times V_i)$.*

Finally the following result will be central in what follows.

Theorem 4.3. *Let X and Y be Priestley spaces.*

The function $j \colon \mathbf{D}(X) \otimes \mathbf{D}(Y) \to \mathbf{D}(X \times Y)$ given by

$$j(\bigvee_{i=1}^{n} U_i \otimes V_i) = \bigcup_{i=1}^{n} (U_i \times V_i)$$

is an isomorphism in $\mathcal{D}$.

Proof. First note that $j \in \mathcal{J}$. We define $k \colon \mathbf{D}(X \times Y) \to \mathbf{D}(X) \otimes \mathbf{D}(Y)$ by $k(\bigcup_{i=1}^{n} U_i \times V_i) = \bigvee_{i=1}^{n}(U_i \otimes V_i)$ (correctness of the definition follows from item (ii) of Lemma 4.2), so $k \in \mathcal{J}$. Moreover, $kj = Id_{\mathbf{D}(X) \otimes \mathbf{D}(Y)}$ and $jk = Id_{\mathbf{D}(X \times Y)}$, so j and k are isomorphisms in $\mathcal{J}$. Thus, they are isomorphisms in $\mathcal{D}$. $\square$

The universal property of the tensor product [13] allows us to claim that $(\mathcal{S}_0, \otimes, 2)$ is monoidal. Here, 2 is the total order in two elements. Since $\mathcal{J}$ is a full subcategory of $\mathcal{S}_0$ and $A \otimes B$ is in $\mathcal{J}$ whenever A and B are, $\mathcal{J}$ inherits the monoidal structure from $\mathcal{S}_0$. Hence, $(\mathcal{J}, \otimes, 2)$ is monoidal (in fact, symmetric monoidal).

5. Some categories of interest

Let us start this section by characterizing the monoids in $\mathcal{J}$.

Recall from [4] that a *bounded distributive lattice with fusion* (or DLF for short) is an algebra $(L, \wedge, \vee, \circ, 0, 1)$, where $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, $\circ$ is a binary operation on L, and the following conditions hold:

(a) $a \circ 0 = 0 \circ a = 0$,
(b) $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$,
(c) $(a \vee b) \circ c = (a \circ c) \vee (a \circ c)$.

Lemma 5.1. *Let $L \in \mathcal{J}$ and let $\circ$ be a binary operation. The following are equivalent:*

(i) $(L, \circ)$ *is a DLF.*
(ii) *There is a unique morphism $f\colon L \otimes L \to L$ in $\mathcal{J}$ such that $f(a \otimes b) = a \circ b$.*

Proof. Suppose that $(L, \circ)$ is a DLF and let $f\colon L \times L \to L$ be the function given by $f(a, b) = a \circ b$.

Since $(a \mapsto a \circ b)\colon L \to L$ and $(b \mapsto a \circ b)\colon L \to L$ are join-homomorphisms, by (a), (b) and (c) above, f is a bimorphism. Hence, by the universal property of the tensor product, there exists an unique morphism $\bar{f}\colon L \otimes L \to L$ in $\mathcal{J}$ such that $\bar{f}(a \otimes b) = a \circ b$.

Conversely, suppose there is a morphism $f\colon L \otimes L \to L$ in $\mathcal{J}$ and define $a \circ b := f(a \otimes b)$ for any $a, b \in L$. Then, straightforward computations show that $\circ$ satisfies (a), (b) and (c) in the definition of a DLF. □

Let $f\colon L \otimes L \to L$ be a morphism in $\mathcal{J}$. The proof of the above lemma shows that we can associate a DLF with f by defining on L a binary operation $a \circ b := f(a \otimes b)$. What properties must have $\circ$ in order to get a monoid in $\mathcal{J}$?

The commutativity of the first diagram in (2.5) reads as

$$f(1 \otimes f) = f(f \otimes 1)\alpha\colon L \otimes (L \otimes L) \to L.$$

Applying both sides of this equality to a basic tensor $a \otimes (b \otimes c)$, we get $f(a \otimes f(b \otimes c)) = f(f(a \otimes b) \otimes c)$. Writing it in terms of $\circ$, we have that

$$a \circ (b \circ c) = (a \circ b) \circ c.$$

Hence, the commutativity of this diagram just says that $\circ$ is associative.

For the second diagram in (2.5), note that a map $\nu\colon 2 \to L$ in $\mathcal{J}$ is completely determined by $\nu(1) \in L$. Then we can think of ν as a constant $e = \nu(1) \in L$. Hence, the commutativity of the second diagram in (2.5) said that for every $a \in L$,

$$e \circ a = a = a \circ e.$$

It follows that we have proved the following result.

Proposition 5.2. *A triple (L, f, ν) is a monoid in $\mathcal{J}$ if and only if the algebra $(L, \vee, \wedge, \circ, 0, 1, e)$ is a DLF in which the binary operation $\circ$, defined as above, and the constant $e = \nu(1)$ satisfy*

(d) $a \circ (b \circ c) = (a \circ b) \circ c$, *and*
(e) $e \circ a = a = a \circ e$.

Although $Mon_{\mathcal{J}}$ is an interesting category, it is not general enough for the purpose of representing most algebras of interest in non-classical logic. In this work, we shall consider the following categories.

We write $\mathcal{DB}$ for the category whose objects are the morphisms of $\mathcal{J}$ of the form $f\colon L \otimes M \to N$, for L, M and N in $\mathcal{J}$, and is such that

$$h \in Mor_{\mathcal{DB}}(f\colon L \otimes M \to N, g\colon L' \otimes M' \to N')$$

is a triple $h = (h_1, h_2, h_3)$ of arrows in $\mathcal{D}$ that makes the following diagram commute,

$$\begin{array}{ccc} L \otimes M & \xrightarrow{f} & N \\ {\scriptstyle h_1 \otimes h_2}\downarrow & & \downarrow{\scriptstyle h_3} \\ L' \otimes M' & \xrightarrow{g} & N' \end{array} \quad . \tag{5.1}$$

Remark 5.3. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories, and let $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be functors. Let us recall from [1] the definition of the *comma category* $F \downarrow G$. This is defined as the category whose objects are triples (A, f, B), with $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $f\colon FA \to GB \in \mathcal{C}$, and whose morphisms from (A, f, B) to $(A'.f', B')$ are pairs $(a, b) \in Mor_{\mathcal{A}}(A, A') \times Mor_{\mathcal{B}}(B, B')$ such that the square

$$\begin{array}{ccc} FA & \xrightarrow{f} & GB \\ {\scriptstyle Fa}\downarrow & & \downarrow{\scriptstyle Gb} \\ FA' & \xrightarrow{f'} & GB' \end{array} \tag{5.2}$$

commutes. Composition is defined componentwise.

Note that $\mathcal{DB}$ could also be presented as the comma category $(- \otimes - \downarrow i_2)$, where $- \otimes -\colon \mathcal{D} \times \mathcal{D} \to \mathcal{J}$ is the semilattice tensor product and $i_2\colon \mathcal{D} \to \mathcal{J}$ is the embedding of the category of distributive lattices in $\mathcal{J}$.

We write $\mathcal{DF}$ for the full subcategory of $\mathcal{DB}$ whose objects are of the form $f\colon L \otimes L \to L$, for $L \in \mathcal{J}$.

It follows from Lemma 5.1 that category $\mathcal{DF}$ is categorically equivalent to the category of DLFs; so in the future, we shall identify both categories.

Remark 5.4. Let us observe that since every object L in $\mathcal{J}$ is a lattice, we implicitly have for every L, a monoid $(L, \inf, 1)$ in $\mathcal{J}$ given on basic tensors by $\inf(a \otimes b) = a \wedge b$.

We write $\mathcal{DI}$ for the full subcategory of $\mathcal{DB}$ whose objects are of the form $i\colon L \otimes L^{\mathrm{op}} \to L^{\mathrm{op}}$, for $L \in \mathcal{J}$.

Recall from [4] that a *bounded distributive lattice with implication* (or DLI for short) is an algebra $(L, \wedge, \vee, \to, 0, 1)$ where $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, $\to$ is a binary operation on L, and the following conditions hold:

(a) $a \to 1 = 0 \to a = 1$,
(b) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$,
(c) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.

It can be seen that the category $\mathcal{DI}$ is categorically equivalent to the category of DLIs; so in the future, we shall identify both categories.

Remark 5.5. To be more general, we could have taken categories whose objects are the operations of the form $f\colon L_1^{\varepsilon_1} \otimes \cdots \otimes L_n^{\varepsilon_n} \to L_0^{\varepsilon_0}$, where $L_i^{\varepsilon_i}$ represents either L_i^{op} or L_i. This allows us to represent operations on L of the sort studied in [17].

For the sake of simplicity, in this article, we restrict ourselves to the categories $\mathcal{DF}$ and $\mathcal{DI}$.

6. CLP duality

Let us consider the adjoint pair $\mathbf{X} : \mathcal{J} \leftrightarrows \mathcal{P}^{\mathrm{op}} : \mathbf{D}$ that realizes CLP duality. Write $\{*\}$ for the space in $\mathcal{P}$ with just one point and 2 for the total order on two elements $\{0 < 1\}$. Straightforward computations show that $\mathbf{D}(\{*\}) = 2$ and $\mathbf{X}(2) = \{*\}$.

By Theorems 4.1 and 4.3, we also have isomorphisms

$$\eta\colon \mathbf{X}(A) \times \mathbf{X}(B) \to \mathbf{X}(A \otimes B) \quad \text{and} \quad j\colon \mathbf{D}(X) \otimes \mathbf{D}(Y) \to \mathbf{D}(X \times Y).$$

Strictly speaking, we have written $\eta\colon \mathbf{X}(A) \times \mathbf{X}(B) \to \mathbf{X}(A \otimes B)$ to indicate a relation $\eta \subseteq (\mathbf{X}(A) \times \mathbf{X}(B)) \times \mathbf{X}(A \otimes B)$ since it is an arrow of $\mathcal{P}$. When no confusion can arise, we shall write $R\colon X \to Y$ to indicate a relation $R \subseteq X \times Y$. To be consistent with this notation, we write SR for the composition of relations $R\colon X \to Y$ and $S\colon Y \to Z$. Note that $SR \subseteq X \times Z$.

The facts that $\mathbf{X} : \mathcal{J} \leftrightarrows \mathcal{P}^{\mathrm{op}} : \mathbf{D}$ acts on objects as Priestley duality does and that there is an isomorphism between $A \otimes B$ and $A * B$ imply the commutativity of diagrams (2.3) and (2.4). Hence, CLP is a monoidal dual equivalence; i.e, the monoidal structures of $\mathcal{J}$ and $\mathcal{P}$ are preserved by $\mathbf{D}$ and $\mathbf{X}$. In what follows, we shall take advantage of this fact to develop representation theories for the operations on lattices introduced in Section 5.

7. Dualizing operations

Let $f\colon L \otimes M \to N$ be an object of $\mathcal{DB}$. Then f goes through $\mathbf{X}$ to a relation $\mathbf{X}(f) \subseteq \mathbf{X}(N) \times \mathbf{X}(L \otimes M)$. Let $\eta\colon \mathbf{X}(L) \times \mathbf{X}(M) \to \mathbf{X}(L \otimes M)$ be the map given in Theorem 4.1. Composing with the natural isomorphism $\eta^{-1} \in \mathcal{P}$, we get the relation $\underline{\mathbf{X}(f)} \subseteq \mathbf{X}(N) \times (\mathbf{X}(L) \times \mathbf{X}(M))$, which is also in $\mathcal{P}$. Note that for $P \in \mathbf{X}(N)$ and $Q \in \mathbf{X}(L) \times \mathbf{X}(M)$, we have that

$$(P, Q) \in \underline{\mathbf{X}(f)} \text{ iff } (P, \eta(Q)) \in \mathbf{X}(f). \tag{7.1}$$

On the other hand, $h = (h_1, h_2, h_3)\colon (f\colon L\otimes M \to N) \to (g\colon L'\otimes M' \to N')$ is a morphism in $\mathcal{DB}$ if and only if diagram (5.1) in $\mathcal{J}$ commutes, and this happens if and only if the following diagram in $\mathcal{P}$ commutes:

$$\begin{array}{ccc} \mathbf{X}(N') & \xrightarrow{\mathbf{X}(h_3)} & \mathbf{X}(N) \\ {\scriptstyle \mathbf{X}(g)}\downarrow & & \downarrow{\scriptstyle \mathbf{X}(f)} \\ \mathbf{X}(L'\otimes N') & \xrightarrow{\mathbf{X}(h_1\otimes h_2)} & \mathbf{X}(L\otimes M)\ . \end{array} \tag{7.2}$$

Since $\mathbf{X}$ is monoidal, diagram (7.2) commutes if and only if the following diagram does:

$$\begin{array}{ccc} \mathbf{X}(N') & \xrightarrow{\mathbf{X}(h_3)} & \mathbf{X}(N) \\ {\scriptstyle \underline{\mathbf{X}(g)}}\downarrow & & \downarrow{\scriptstyle \underline{\mathbf{X}(f)}} \\ \mathbf{X}(L')\times\mathbf{X}(N') & \xrightarrow{\mathbf{X}(h_1)\times\mathbf{X}(h_2)} & \mathbf{X}(L)\times\mathbf{X}(M)\ . \end{array} \tag{7.3}$$

The above observation suggests the following definition.

Definition 7.1. We write $\mathcal{PB}$ for the category whose object are relations $R \subseteq X \times (Y \times Z) \in \mathcal{P}$, and whose morphisms from $R \subseteq X \times (Y \times Z)$ to $S \subseteq X' \times (Y' \times Z')$ are the triples of Priestley spaces morphisms $\alpha := (\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_1\colon X \to X'$, $\alpha_2\colon Y \to Y'$, and $\alpha_3\colon Z \to Z'$, making the following diagram commute in $\mathcal{P}$:

$$\begin{array}{ccc} X & \xrightarrow{\alpha_1} & X' \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle S} \\ Y\times Z & \xrightarrow{\alpha_2\times\alpha_3} & Y'\times Z'. \end{array} \tag{7.4}$$

Note that $\mathcal{PB}$ could also be presented as the comma category $(i_{\mathcal{P}} \downarrow -\times-)$, where $-\times-\colon \mathcal{P}\times\mathcal{P} \to \mathcal{P}$ is the product in $\mathcal{P}$, seen as a functor, and $i_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{P}$ is the identity functor.

The commutativity of diagram (7.4) can be also stated as in the following remark.

Remark 7.2. For every $x \in X'$, $U \in \mathbf{D}(Y)$, and $V \in \mathbf{D}(Z)$,

$$R(x) \cap (\alpha_2^{-1}(U) \times \alpha_3^{-1}(V)) \neq \emptyset \text{ iff } S(\alpha_1(x)) \cap (U \times V) \neq \emptyset. \tag{7.5}$$

Indeed, straightforward computations show that condition (7.5) is equivalent to the commutativity of the following diagram in $\mathcal{J}$:

$$\begin{array}{ccc} \mathbf{D}(Y'\times Z') & \xrightarrow{\mathbf{D}(\alpha_2\times\alpha_3)} & \mathbf{D}(Y\times Z) \\ {\scriptstyle \mathbf{D}(S)}\downarrow & & \downarrow{\scriptstyle \mathbf{D}(R)} \\ \mathbf{D}(X') & \xrightarrow{\mathbf{D}(\alpha_1)} & \mathbf{D}(X)\ . \end{array}$$

On the other hand, the commutativity of previous diagram is equivalent to the commutativity of diagram (7.4).

The following result is immediate from the definition of $\mathcal{PB}$.

Theorem 7.3. *There is a dual categorical equivalence between categories* $\mathcal{DB}$ *and* $\mathcal{PB}$.

We shall still use the letters $\mathbf{X}$ and $\mathbf{D}$ to denote the functors that realize the duality of Theorem 7.3.

We have already mentioned that since each object L of $\mathcal{J}$ is a lattice, also the dual lattice L^{op} is an object of $\mathcal{J}$. Then both $\mathbf{X}(L)$ and $\mathbf{X}(L^{\mathrm{op}})$ are objects of $\mathcal{P}$. Straightforward computations show that as ordered topological spaces, $\mathbf{X}(L^{\mathrm{op}}) \cong \mathbf{X}(L)^{\mathrm{op}}$, where $\mathbf{X}(L)^{\mathrm{op}}$ and $\mathbf{X}(L)$ are equal as topological spaces, but the order of $\mathbf{X}(L)^{\mathrm{op}}$ is the dual (or opposite) to the order of $\mathbf{X}(L)$. Let us write $\beta\colon \mathbf{X}(L)^{\mathrm{op}} \to \mathbf{X}(L^{\mathrm{op}})$ for the morphism of Priestley spaces given by $\beta(F) := F^c$, the set theoretical complement of F.

Now we shall define two subcategories of $\mathcal{PB}$.

Definition 7.4. We write $\mathcal{PF}$ for the subcategory of $\mathcal{PB}$ whose object are relations $R \subseteq X \times (X \times X) \in \mathcal{P}$, and whose morphisms are of the form $\alpha := (\alpha, \alpha, \alpha)$.

We write $\mathcal{PI}$ for the subcategory of $\mathcal{PB}$ whose object are relations $R \subseteq X^{\mathrm{op}} \times (X \times X^{\mathrm{op}}) \in \mathcal{P}$, and whose morphisms are of the form $\alpha := (\alpha, \alpha, \alpha)$.

Note that the duality of Theorem 7.3 restricts to dualities between categories $\mathcal{DF}$ and $\mathcal{PF}$ and between categories $\mathcal{DI}$ and $\mathcal{PI}$.

Also observe that, associated with any object $R \subseteq X \times (X \times X) \in \mathcal{PF}$, we have a relational Priestley space (X, R), in the sense of [17], and similarly, to any $R \subseteq X^{\mathrm{op}} \times (X \times X^{\mathrm{op}}) \in \mathcal{PI}$. We call *relational spaces with fusion* (RSF for short) and *relational spaces with implication* (RSI for short) the just mentioned ordered topological spaces associated with the objects of $\mathcal{PF}$ and $\mathcal{PI}$, respectively.

8. Relation with other representation theories

In this section, we shall give explicit descriptions of the ternary relations that appeared in the previous section, and compare the equivalences mentioned before with those developed in [2], [4], and [3].

Let $L \in \mathcal{J}$; let $R \subseteq \mathbf{X}(L^{\mathrm{op}}) \times \mathbf{X}(L \otimes L^{\mathrm{op}})$ be a morphism in $\mathcal{P}$, and let $\underline{R}$ be defined as in (7.1).

Define the function

$$\sigma\colon \mathbf{X}(L)^{\mathrm{op}} \times (\mathbf{X}(L) \times \mathbf{X}(L)^{\mathrm{op}}) \to \mathbf{X}(L^{\mathrm{op}}) \times (\mathbf{X}(L) \times \mathbf{X}(L^{\mathrm{op}}))$$

by the condition $\sigma(O, (P, Q)) = (O^c, (P, Q^c))$, and consider the relation $\overline{R} \subseteq (\mathbf{X}(L)^{\mathrm{op}} \times (\mathbf{X}(L) \times \mathbf{X}(L)^{\mathrm{op}})$ defined in the following way:

$$(O, (P, Q)) \in \overline{R} \text{ iff } \sigma(O, (P, Q)) \in \underline{R} \text{ iff } (O^c, (P, Q^c)) \in \underline{R}.$$

Observe that if R is a morphism in $\mathcal{P}$, then $\overline{R}$ is a morphism in $\mathcal{P}$.

Let $R \subseteq X^{\mathrm{op}} \times (X \times X^{\mathrm{op}}) \in \mathcal{P}$. We write $\widetilde{R}\colon \mathbf{D}(X) \otimes \mathbf{D}(X^{\mathrm{op}}) \to \mathbf{D}(X^{\mathrm{op}})$ for the morphism in $\mathcal{J}$ given by

$$\widetilde{R}(\bigvee_{i=1}^{n} U_i \otimes V_i) = \mathbf{D}(R)(j(\bigvee_{i=1}^{n} U_i \otimes V_i)).$$

Finally, define $\zeta_R\colon \mathbf{D}(X) \otimes \mathbf{D}(X)^{\mathrm{op}} \to \mathbf{D}(X)^{\mathrm{op}}$ by

$$\zeta_R(\bigvee_{i=1}^{n} U_i \otimes V_i) = (\widetilde{R}(\bigvee_{i=1}^{n} U_i \otimes V_i^c))^c.$$

A direct computation proves that ζ_R is a morphism in $\mathcal{J}$. For $U, V \in \mathbf{D}(X)$, consider the binary relation in $\mathbf{D}(X)$: $(U \Rightarrow_R V) := \zeta_R(U \otimes V)$. By definition of $\Rightarrow_R$, we have that

$$(U \Rightarrow_R V) = \{x \in X : R(x) \cap (U \times V^c) = \emptyset\}.$$

The following result can then be checked.

Proposition 8.1. *If (X, R) is an RSI, then $(\mathbf{D}(X), \Rightarrow_R)$ is a DLI.*

Another equivalence for the category of DLIs was introduced in [2] and [4]. We shall recall, for completeness, the basic definitions and relevant results of the mentioned articles.

Let $(L, \to)$ be a DLI. If F and G are filters of L, we define

$$F \to G = \{x \in L : (\exists (f, g) \in F \times G : f \leq g \to x)\}.$$

Let $T_L \subseteq \mathbf{X}(L) \times \mathbf{X}(L) \times \mathbf{X}(L)$ be the ternary relation defined as

$$(P, Q, D) \in T_L \quad \text{iff} \quad P \to Q \subseteq D.$$

Definition 8.2. If X is a Priestley space, T is a ternary relation in X, and $U, V \in \mathbf{D}(X)$, then we define the set

$$U \to_T V = \{x \in X : \forall y \forall z((x, y, z) \in T, y \in U) \Rightarrow z \in V\}.$$

Let X be a Priestley space, and let T be a ternary relation in X. A structure (X, T) is a *DLI*-space provided that the following conditions hold.

(I) For every $U, V \in \mathbf{D}(X)$, $U \to_T V \in \mathbf{D}(X)$.

(II) For every $x, y, z \in X$, if $\epsilon_X(x) \to_T \epsilon_X(y) \subseteq \epsilon_X(z)$, then $(x, y, z) \in T$.

Note that $U \in \epsilon_X(x) \to_T \epsilon_X(y)$ iff there exists $(V, W) \in (\epsilon_X(x) \times \epsilon_X(y))$ such that $V \subseteq W \to_T U$.

An i-morphism between the *DLI*-spaces $(X_1, \leq, T_1)$ and $(X_2, \leq, T_2)$ is a function $g\colon X_1 \to X_2$ that satisfies the following conditions:

(i1) g is continuous and monotone.

(i2) If $(x, y, z) \in T_1$, then $(g(x), g(y), g(z)) \in T_2$.

(i3) If $(g(x), y', z') \in T_2$, then there exist $y, z \in X_1$ such that $(x, y, z) \in T_1$, $y' \leq g(y)$, and $g(z) \leq z'$.

We write $\mathcal{DLI}$ for the above-mentioned category.

There is a dual categorical equivalence between this category and the category of DLIs. Since $\mathcal{PI}$ is also dually equivalent to the category of DLIs, it follows that $\mathcal{DLI}$ and $\mathcal{PI}$ are equivalent. Let us make this equivalence explicit.

Let X be a Priestley space. If $R \subseteq X^{\mathrm{op}} \times (X \times X^{\mathrm{op}})$, we define $R' \subseteq X \times X \times X$ by $(x, y, z) \in R'$ if and only if $(x,(y,z)) \in R$. Conversely, if $R' \subseteq X \times X \times X$, we define $R \subseteq X^{\mathrm{op}} \times (X \times X^{\mathrm{op}})$ in the same way.

Let us now state some useful technical lemmata.

Lemma 8.3. *Let R be an object of $\mathcal{PB}$, and let*

$$\eta\colon \mathbf{X}(\mathbf{D}(Y)) \times \mathbf{X}(\mathbf{D}(Z)) \to \mathbf{X}(\mathbf{D}(Y) \otimes \mathbf{D}(Z))$$

be the map given in Theorem 4.1. Then the following conditions hold:

(i) $\bigvee_{i=1}^{n}(U_i \otimes V_i) \in \eta(\epsilon_Y(y), \epsilon_Z(z))$ *iff* $\bigcup_{i=1}^{n}(U_i \times V_i) \in \epsilon_{Y\times Z}(y,z)$;

(ii) $\epsilon_X(x)\underline{(\mathbf{X}(\widetilde{R})}(\epsilon_Y(y), \epsilon_Z(z))$ *iff* $\epsilon_X(x)\mathbf{X}\mathbf{D}(R)\epsilon_{Y\times Z}(y,z)$;

(iii) $\epsilon_X(x)\underline{\mathbf{X}(\widetilde{R})}(\epsilon_Y(y), \epsilon_Z(z))$ *iff* $xR(y,z)$.

Proof. (i): We have that $\bigvee_{i=1}^{n}(U_i \otimes V_i) \in \eta(\epsilon_Y(y), \epsilon_Z(z))$ iff there exists k, with $1 \le k \le n$, such that $U_k \otimes V_k \in \eta(\epsilon_Y(y), \epsilon_Z(z))$. This happens iff $U_k \in \epsilon_Y(y)$ and $V_k \in \epsilon_Z(z)$, which is equivalent to $U_k \times V_k \in \epsilon_{Y\times Z}(y,z)$. The latter is equivalent to $\bigcup_{i=1}^{n}(U_i \times V_i) \in \epsilon_{Y\times Z}(y,z)$.

(ii): We have $\epsilon_X(x)\underline{\mathbf{D}(\widetilde{R})}(\epsilon_Y(y), \epsilon_Z(z))$ iff $\epsilon_X(x_1)\mathbf{D}\mathbf{X}(\widetilde{R})\eta(\epsilon_Y(y), \epsilon_Z(z))$, which is equivalent to $\eta(\epsilon_Y(y), \epsilon_Z(z)) \subseteq [\widetilde{R}]^{-1}(\epsilon_X(x))$, which is equivalent to $\bigvee_{i=1}^{n}(U_i \otimes V_i) \in \eta(\epsilon_Y(y), \epsilon_Z(z))$ (with $U_i, V_i \in \mathbf{D}(X)$ for $i = 1, \dots, n$) implying that $\widetilde{R}(\bigvee_{i=1}^{n}(U_i \otimes V_i)) \in \epsilon_X(x)$. By item (i), this is equivalent to the statement that if $\bigcup_{i=1}^{n}(U_i \times V_i) \in \epsilon_{Y\times Z}(y,z)$, then $\mathbf{D}(R)(\bigcup_{i=1}^{n}(U_i \times V_i)) \in \epsilon_X(x)$, which is equivalent to $\epsilon_{Y\times Z}(y,z) \subseteq (\mathbf{D}(R))^{-1}(\epsilon_X)$, which is equivalent to $\epsilon_X(x)\mathbf{X}\mathbf{D}(R)\epsilon_{Y\times Z}(y,z)$.

(iii): This follows from item (ii) and item (iv) of [8, Lemma 1.5]. $\square$

Similarly, we can get the following result.

Lemma 8.4. *Let (X, R) be an RSI. Then for every $x, y, z \in X$, we have that $\epsilon_X(x)\overline{\mathbf{X}(\zeta_R)}(\epsilon_X(y), \epsilon_X(z))$ if and only if $xR(y,z)$.*

Lemma 8.5. *If R is an object of $\mathcal{PI}$ and $\epsilon_X(x) \to_{R'} \epsilon_X(y) \subseteq \epsilon_X(z)$, then*

$$(\epsilon_X(x), (\epsilon_X(y), \epsilon_X(z))) \in \overline{(\mathbf{X}(\zeta_R))}.$$

Proof. Note that for every $x, y, z \in X$, we have that $(\epsilon_X(x), (\epsilon_X(y), \epsilon_X(z))) \in \overline{(\mathbf{X}(\zeta_R))}$ if and only if for every $\bigcup_{i=1}^{n}(A_i \times B_i^c) \in \mathbf{D}(X \times X^{\mathrm{op}})$ with $A_i, B_i \in \mathbf{D}(X)$ for $i = 1, \dots, n$, we have that if $(y,z) \in \bigcup_{i=1}^{n}(A_i \times B_i^c)$, then $x \in \mathbf{D}(R)(\bigcup_{i=1}^{n} A_i \times B_i^c)$.

Let $(y,z) \in \bigvee_{i=1}^{n}(A_i \times B_i^c)$, so there exists k, $1 \le k \le n$, such that $y \in A_k$ and $z \in B_k^c$. Suppose that $x \notin \mathbf{D}(R)(\bigvee_{i=1}^{n} A_i \times B_i^c)$, so $R(x) \cap (\bigcup_{i=1}^{n} A_i \times B_i^c) = \emptyset$. In particular, we have that $R'(x) \cap (A_k \times B_k^c) = \emptyset$. Then $x \in A_k \to_{R'} B_k$. Thus, $x \in A_k \to_{R'} B_k \subseteq A_k \to_{R'} B_k$ and $y \in A_k$. So, $B_k \in \epsilon_X(x) \to_{R'} \epsilon_X(y)$, and by hypothesis, we conclude that $z \in B_k$, which is a contradiction. The first observation of this proof then implies the desired result. $\square$

We can now state one of the main results of this section.

Theorem 8.6. *Let X be a Priestley space. Then $R \subseteq X \times (X \times X)$ is in $\mathcal{PI}$ if and only if (X, R') is a DLI-space.*

Proof. For the necessity, take $U, V \in \mathbf{D}(X)$. An easy computation proves that

$$(U \to_{R'} V)^c = \{x \in X : \forall y \forall z((x, y, z) \in R, y \in U) \Rightarrow z \in V\}^c = (U \Rightarrow_R V)^c.$$

Since $R \in \mathcal{PI}$, we have, using Proposition 8.1, that $U \Rightarrow_R V \in \mathbf{D}(X)$, and then $U \to_{R'} V \in \mathbf{D}(X)$.

Let $x, y, z \in X$ such that $\epsilon_X(x) \to_{R'} \epsilon_X(y) \subseteq \epsilon_X(z)$. By Lemma 8.5, we have that $(\epsilon_X(x), (\epsilon_X(y), \epsilon_X(z))) \in \overline{\mathbf{X}(\zeta_R)}$. By Lemma 8.4, we have that $(x, (y, z)) \in R$, so $(x, y, z) \in R'$.

To show sufficiency, we have to see that R is a morphism in the category $\mathcal{P}$. Let $U, V \in \mathbf{D}(X)$. Then $(U \Rightarrow_R V) = U \to_{R'} V \in \mathbf{D}(X)$. It is enough to show that for every $x \in X$, $R(x)$ is a closed downset. Take $(y, z) \in R(x)$ and $(a, b) \leq (y, z)$. Hence, $a \leq y$ and $z \leq b$. Suppose that $(a, b) \notin R(x)$, so $(x, a, b) \notin R'$. Thus, by hypothesis, we conclude that $\epsilon_X(x) \to_{R'} \epsilon_X(a) \nsubseteq \epsilon_X(b)$, and as a consequence, there exists $U \in \mathbf{D}(X)$ such that $U \in \epsilon_X(x) \to_{R'} \epsilon_X(a)$ and $b \notin U$. Then there exist $V, W \in \mathbf{D}(X)$ such that $V \subseteq W \to_{R'} U$, $x \in V$, and $a \in W$. In particular, $x \in W \to_{R'} U$ and $R'(x) \cap (W \times U^c) = \emptyset$; but $(a, b) \in R'(x) \cap (W \times U^c)$, which is a contradiction. Thus, $R(x)$ is a downset.

Let us now see that $R(x)$ is closed. Take $(y, z) \notin R(x)$. Hence, $(x, y, z) \notin R'$. Thus, we have, by hypothesis, that $\epsilon_X(x) \to_{R'} \epsilon_X(y) \nsubseteq \epsilon_X(z)$. Then there exists $U \in \mathbf{D}(X)$ such that $U \in \epsilon_X(x) \to_{R'} \epsilon_X(y)$ and $z \notin U$. There exits $V, W \in \mathbf{D}(X)$ such that $V \subseteq W \to_{R'} U$, $x \in V$, and $y \in W$.

Let us prove that $(y, z) \in W \times U^c \subseteq R'(x)^c$. Take $(a, b) \in W \times U^c$. Since $x \in V$, we have that $x \in W \to_{R'} U$ and, therefore, $R'(x) \cap (W \times U^c) = \emptyset$. Using that $(a, b) \in W \times U^c$, we conclude that $(a, b) \notin R'(x)$. Therefore, $R(x) = R'(x)$ is closed since $R'(x)^c$ is open. □

For the proof of the following proposition, the characterization of morphisms in $\mathcal{PB}$ given in Remark 7.2 will be useful.

Proposition 8.7. *Let $g \colon X \to Y$ be a morphism of Priestley spaces. Then there exist relations R and S such that $g \colon (X, R) \to (Y, S)$ is a morphism in $\mathcal{PI}$ if and only if there exist relations R' and S' such that $g \colon (X, R') \to (Y, S')$ is an i-morphism.*

Proof. For the necessity, take $(x, y, z) \in R'$, and hence $(y, z) \in R(x)$. Suppose that $(g(x), g(y), g(z)) \notin S'$, and hence $(g(y), g(z)) \notin S(g(x))$. By (i) and (ii) of Lemma 4.2, there exist $U, V \in \mathbf{D}(Y)$ such that $(g(y), g(z)) \in U \times V^c$ with $(U \times V^c) \cap S(g(x)) = \emptyset$. Since $g \in \mathcal{PI}$, we have $(g^{-1}(U) \times g^{-1}(V^c)) \cap R(x) = \emptyset$. Since $(y, z) \in (g^{-1}(U) \times g^{-1}(V^c)) \cap R(x)$, we get a contradiction. Then condition $(i2)$ in the definition of i-morphism holds.

Now take $(g(x), y', z') \in S'$, and hence $(y', z') \in S(g(x))$. For every $U, V \in \mathbf{D}(X)$ with $(y', z') \in U \times V^c$, we have $(U \times V^c) \cap S(g(x)) \neq \emptyset$. Then for every

$U, V \in \mathbf{D}(X)$ with $(y', z') \in U \times V^c$, we have $(g^{-1}(U) \times g^{-1}(V^c)) \cap R(x) \neq \emptyset$. Consider the family

$$\mathcal{F} = \{(g^{-1}(U) \times g^{-1}(V^c)) \cap R(x) : U, V \in \mathbf{D}(X), (y', z') \in U \times V^c\}.$$

This is a family of closed subsets of $X \times X^{\mathrm{op}}$.

To see $\mathcal{F}$ has the finite intersection property, let $(g^{-1}(U_i) \times g^{-1}(V_i^c)) \cap R(x) \in \mathcal{F}$ for $i = 1, \ldots, n$; then using that

$$\bigcap_{i=1}^{n} ((U_i \times V_i^c) \cap S(g(x))) = (\bigcap_{i=1}^{n} U_i \times \bigcap_{i=1}^{n} V_i^c) \cap S(g(x)) \neq \emptyset,$$

we conclude that $(\bigcap_{i=1}^{n} (g^{-1}(U_i) \times g^{-1}(V_i))) \cap R(x) \neq \emptyset$. Using that $X \times X^{\mathrm{op}}$ is compact, we conclude that $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. Therefore, there exist $y, z \in X$ such $(x, y, z) \in R'$, $g(y) \in U$ for every $U \in \mathbf{D}(X)$ such that $y' \in U$, whereas $g(z) \in V^c$ for every $V \in \mathbf{D}(X)$ such that $z' \in V^c$.

Suppose now that $y' \not\leq g(y)$. Thus, there exists $U \in \mathbf{D}(X)$ such that $y' \in U$ and $g(y) \notin U$; which is a contradiction. Then $y' \leq g(y)$. Similarly, we prove that $g(z) \leq z'$.

To show sufficiency, take $U, V \in \mathbf{D}(X)$ such that $R(x) \cap (g^{-1}(U) \times g^{-1}(V^c)) \neq \emptyset$. Then there exist $y, z \in X$ such that $(x, y, z) \in R'$, $f(y) \in U$ and $g(z) \in V^c$. Since g is an i-morphism, $(g(x), g(y), g(z)) \in S'$, and hence $S(g(x)) \cap (U \times V^c) \neq \emptyset$.

Conversely, let $S(g(x)) \cap (U \times V^c) \neq \emptyset$. Thus, there exist $y', z' \in Y$ such that $(g(x), y', z') \in S'$ with $(y', z') \in U \times V^c$. Since g is an i-morphism, there exist $y, z \in X$ such that $y' \leq g(y)$ and $f(z) \leq z'$. Hence, $(g(y), g(z)) \in U \times V^c$ and as a consequence, $(y, z) \in R(x) \cap (g^{-1}(U) \times g^{-1}(V^c))$. Thus, $R(x) \cap (g^{-1}(U) \times g^{-1}(V^c)) \neq \emptyset$. □

We have already studied implications. Let us now study fusions. Let $(L, \circ)$ be a DLF. If F and G are filters of L, we define

$$F \circ G = \{x \in L : \exists (f, g) \in F \times G, f \circ g \leq x\}.$$

Let $R_L \subseteq \mathbf{X}(L) \times \mathbf{X}(L) \times \mathbf{X}(L)$ be the ternary relation defined as

$$(P, Q, D) \in R_L \text{ iff } P \circ Q \subseteq D.$$

If X is a Priestley space, T is a ternary relation in X, and $U, V \in \mathbf{D}(X)$, then we define the set

$$U \circ_T V = \{x \in X : \exists (y, z) \in U \times V, (x, y, z) \in T\}.$$

Definition 8.8. Let X be a Priestley space, and let T be a ternary relation in X. A structure (X, T) is a *DLF*-space if the following conditions hold:

(I) For every $U, V \in \mathbf{D}(X)$, $U \circ_T V \in \mathbf{D}(X)$.
(II) For every $x, y, z \in X$, if $\epsilon_X(x) \circ_T \epsilon_X(y) \subseteq \epsilon_X(z)$, then $(x, y, z) \in T$.

Note that $U \in \epsilon_X(x) \circ_T \epsilon_X(y)$ iff there exists $(V, W) \in (\epsilon_X(x) \times \epsilon_X(y))$ such that $V \circ_T W \subseteq U$.

An f-morphism between the DLF-spaces $(X_1, \leq, T_1)$ and $(X_2, \leq, T_2)$ is a function $g\colon X_1 \to X_2$ that satisfies the following conditions:

(f1) g is continuous and monotone.
(f2) If $(x, y, z) \in T_1$, then $(g(x), g(y), g(z)) \in T_2$.
(f3) If $(x', y', g(z)) \in T_2$, then there exist $x, y \in X_1$ such that $(x, y, z) \in T_1$, $x' \leq g(x)$, and $y' \leq g(y)$.

The above-mentioned category will be denoted $\mathcal{DLF}$.

Since $\mathcal{PF}$ is also dually equivalent to the category of DLFs, it follows that $\mathcal{DLF}$ and $\mathcal{PF}$ are equivalent. Let us make this equivalence explicit.

Let X be a Priestley space. If $R \subseteq X \times (X \times X)$, we define $R^+ \subseteq X \times X \times X$ by $(x, (y, z)) \in R$ if and only if $(y, z, x) \in R^+$. Conversely, if $R \subseteq X \times X \times X$, we define $R^- \subseteq X \times (X \times X)$ by $(x, (y, z)) \in R^-$ if and only if $(y, z, x) \in R$.

Lemma 8.9. *If R is in $\mathcal{PF}$ and $\epsilon_X(x) \circ_{R^+} \epsilon_X(y) \subseteq \epsilon_X(z)$, then*

$$(\epsilon_X(z), (\epsilon_X(x), \epsilon_X(y))) \in \underline{\mathbf{X}(\,\overline{\mathbf{D}(R)}\,)}.$$

Proof. This is similar to the proof of Lemma 8.5. □

Proposition 8.10. *Let X be a Priestley space.*

(i) *If R is in $\mathcal{PF}$, then (X, R^+) is a DLF-space.*
(ii) *If (X, R) is a DLF-space, then R^- is in $\mathcal{PF}$.*

Proof. This is similar to the proof of Proposition 8.6, using Lemmas 8.9 and 8.3 (iii). □

Proposition 8.11. *Let $g\colon X \to Y$ be a morphism of Priestley spaces.*

(i) *If $g\colon R \to S \in \mathcal{PF}$, then $g\colon (X, R^+) \to (Y, S^+)$ is an f-morphism.*
(ii) *If $g\colon (X, R) \to (Y, S)$ is an f-morphism, then $g\colon R^- \to S^- \in \mathcal{PF}$.*

Proof. This is similar to the proof of Proposition 8.7, using Proposition 8.10 and Lemma 4.2. □

In this section, we have shown that the representation theories for DLFs and DLIs developed in this work are essentially those introduced in [2, 4, 3], where the authors compare their theories with other theories existing in the literature. We refer the interested reader to [2, 4, 3] and the references therein.

9. Some final remarks

By Lemma 5.1, a fusion $\circ$ on a bounded distributive lattice L can be seen as a function $f\colon L \otimes L \to L$ such that $f(a \otimes b) = a \circ b$ for any basic tensor $a \otimes b \in L \otimes L$.

Since CLP is a monoidal duality, properties characterized by the commutativity of certain diagrams (both in $\mathcal{J}$ and $\mathcal{P}$) and the monoidal bifunctor may be translated from one category to the other by the duality in question.

For example, it follows that a fusion f on L is associative if and only if the following diagram commutes:

$$\begin{array}{ccccc} L\otimes(L\otimes L) & & \xrightarrow{\quad\alpha\quad} & & (L\otimes L)\otimes L \\ \Big\downarrow{\scriptstyle 1_L\otimes f} & & & & \Big\downarrow{\scriptstyle f\otimes 1_L} \\ L\otimes L & \xrightarrow{\ f\ } & L & \xleftarrow{\ f\ } & L\otimes L\ , \end{array} \tag{9.1}$$

i.e., when we have the equality $f(1_L\otimes f)=f(f\otimes 1_L)$. Employing the adjunction, diagram (9.1) becomes:

$$\begin{array}{ccccc} \mathbf{X}(L)\times(\mathbf{X}(L)\times\mathbf{X}(L)) & & \xrightarrow{\quad\alpha\quad} & & (\mathbf{X}(L)\times\mathbf{X}(L))\times\mathbf{X}(L) \\ \Big\uparrow{\scriptstyle 1_{\mathbf{X}(L)}\times\underline{\mathbf{X}(f)}} & & & & \Big\uparrow{\scriptstyle \underline{\mathbf{X}(f)}\times 1_{\mathbf{X}(L)}} \\ \mathbf{X}(L)\times\mathbf{X}(L) & \xleftarrow{\ \underline{\mathbf{X}(f)}\ } & \mathbf{X}(L) & \xrightarrow{\ \underline{\mathbf{X}(f)}\ } & \mathbf{X}(L)\times\mathbf{X}(L)\ . \end{array}$$

Hence, we get that a Priestley relation $R\subseteq X\times(X\times X)$ represents an associative fusion if and only if the following diagram commutes:

$$\begin{array}{ccccc} X\times(X\times X) & & \xrightarrow{\quad\alpha\quad} & & (X\times X)\times X \\ \Big\uparrow{\scriptstyle \Delta_X\times R} & & & & \Big\uparrow{\scriptstyle R\times\Delta_X} \\ X\times X & \xleftarrow{\ R\ } & X & \xrightarrow{\ R\ } & X\times X\ , \end{array}$$

i.e., when the equality of relations $(\Delta_X\times R)R=(R\times\Delta_X)R$ holds. Here, Δ_X is the identity relation for X in $\mathcal{P}$, which is given by $(x,y)\in\Delta_X$ iff $x\leq y$.

As another example, we can give a condition on a Priestley relation R so that $\mathbf{D}(R)$ becomes a commutative fusion.

Recall that a fusion $f\colon L\otimes L\to L$ is commutative if and only if $f\tau=f$, where $\tau\colon L\otimes L\to L\otimes L$ is given by

$$\tau(\bigvee_{i=1}^{n}(a_i\otimes b_i))=\bigvee_{i=1}^{n}(b_i\otimes a_i).$$

Note that the definition of τ together with Theorem 2.6 of [11] implies that it is a morphism of $\mathcal{D}$. Thus, its associated Priestley relation is functional [8, Example 1.3 (ii)].

As a consequence, $f\tau=f$ if and only if $\mathbf{X}(f)=\mathbf{X}(f\tau)=\mathbf{X}(\tau)\mathbf{X}(f)$, which is equivalent to the identity

$$\underline{\mathbf{X}(f)}=\underline{\mathbf{X}(\tau)\mathbf{X}(f)}=\eta^{-1}\mathbf{X}(\tau)\mathbf{X}(f)=(\eta^{-1}\mathbf{X}(\tau)\eta)(\eta^{-1}\mathbf{X}(f))=\tau^*\underline{\mathbf{X}(f)},$$

where $\tau^*=\eta^{-1}\mathbf{X}(\tau)\eta$.

Hence, a Priestley relation $R\subseteq X\times(X\times X)$ would represent a commutative fusion on $\mathbf{D}(X)$ if and only if

$$R=\tau^*R. \tag{9.2}$$

Here, $\tau^*\subseteq(X\times X)\times(X\times X)$ is the relation in $\mathcal{P}$ such that $((x,y),(z,w))\in\tau^*$ iff $x\leq w$ and $y\leq z$.

The following lemma is useful in determining whether two given Priestley relations are equal. The proof of this lemma makes use of the Prime Filter Theorem. For the convenience of the reader, we recall this theorem (see [9, p. 9] for the dual of this theorem), before the statement of the lemma.

Theorem 9.1 (Prime Filter Theorem). *Let L be a distributive lattice, let F be a filter in L, and let I be an ideal in L that is disjoint from F. Then there is a prime filter containing F and not intersecting I.*

Lemma 9.2. *Let $g, h\colon L \to M$ be a morphisms in $\mathfrak{J}$. Then $g(x) \leq h(x)$ for every $x \in L$ if and only if $\mathbf{X}(g) \subseteq \mathbf{X}(h)$.*

Proof. Easy computations show the necessity.

To show sufficiency, suppose that there is $x \in L$ such that $g(x) \nleq h(x)$. Then by the Prime Filter Theorem, there exists $Q \in \mathbf{X}(M)$ such that $g(x) \in Q$ and $h(x) \notin Q$. It is clear that $g^{-1}(Q^c)$ is an ideal, and that if $F(\langle x \rangle)$ is the filter generated by $\{x\}$, then $F(\langle x \rangle) \cap g^{-1}(Q^c) = \emptyset$, so by the Prime Filter Theorem again there exists $P \in \mathbf{X}(L)$ such that $x \in P$ and $P \cap g^{-1}(Q^c) = \emptyset$. Thus, $P \subseteq g^{-1}(Q)$. On the other hand, we have that $\mathbf{X}(g) \subseteq \mathbf{X}(h)$, so $P \subseteq h^{-1}(Q)$. Thus, using that $x \in P$, we conclude that $h(x) \in Q$, which is a contradiction. □

The following results follow directly from Lemma 9.2 and from equation (9.2), respectively.

Lemma 9.3.

(a) *If $g, h\colon L \otimes L \to L \in \mathfrak{J}$, then $\mathbf{X}(g) \subseteq \mathbf{X}(h)$ iff $\underline{\mathbf{X}(g)} \subseteq \underline{\mathbf{X}(h)}$.*
(b) *$(P, (Q, Z)) \in \underline{\mathbf{X}(f\tau)}$ iff $(P, (Z, Q)) \in \underline{\mathbf{X}(f)}$.*

Hence, the usual characterization arises [2]:

Proposition 9.4. *The following conditions are equivalent:*

(i) *For every $a, b \in L$, $a \circ b \leq b \circ a$.*
(ii) *$\underline{\mathbf{X}(f)} \subseteq \underline{\mathbf{X}(f\tau)}$.*
(iii) *$\underline{\mathbf{X}(f)} = \underline{\mathbf{X}(f\tau)}$.*

Proof. By Lemma 9.2, we have that equation $a \circ b \leq b \circ a$ holds if and only if for every $x \in L \otimes L$, $f(x) \leq f\tau(x)$. By Lemma 9.3, $\mathbf{X}(f) \subseteq \mathbf{X}(f\tau)$ if and only if $\underline{\mathbf{X}(f)} \subseteq \underline{\mathbf{X}(f\tau)}$. □

10. Conclusions

We have developed directly from [8] the duality of [2, 4, 3]. This was done on the basis of the following well-known results:

(i) The tensor product of join-semilattices defines a monoidal structure on the category of join-semilattices with zero, see Section 4.

(ii) For any pair of bounded distributive lattices L and M, their tensor product as join-semilattices with zero is a lattice isomorphic to the coproduct of L and M in the category of bounded distributive lattices, see Section 4.

(iii) The duality introduced in [8] extends Priestley duality, see Section 6.

More precisely, let $\mathcal{DB}$ be the comma category $(-\otimes- \downarrow i_2)$, where we have that $-\otimes-\colon \mathcal{D}\times\mathcal{D}\to\mathcal{J}$ is the semilattice tensor product and $i_2\colon \mathcal{D}\to\mathcal{J}$ is the embedding of the category of distributive lattices in $\mathcal{J}$. Let $\mathcal{PB}$ be the the comma category $(i_{\mathcal{P}}\downarrow -\times-)$, where $-\times-\colon \mathcal{P}\times\mathcal{P}\to\mathcal{P}$ is the product in $\mathcal{P}$, seen as a functor, and $i_{\mathcal{P}}\colon \mathcal{P}\to\mathcal{P}$ is the identity functor. We have proved in Theorem 7.3 that there is a dual categorical equivalence between $\mathcal{DB}$ and $\mathcal{PB}$. Write $\mathcal{PF}$ for the subcategory of $\mathcal{PB}$ whose objects are relations $R\subseteq X\times(X\times X)\in\mathcal{P}$, and whose morphisms are of the form $\alpha := (\alpha,\alpha,\alpha)$. Write $\mathcal{PI}$ for the subcategory of $\mathcal{PB}$ whose objects are relations $R\subseteq X^{\mathrm{op}}\times(X\times X^{\mathrm{op}})\in\mathcal{P}$, and whose morphisms are of the form $\alpha := (\alpha,\alpha,\alpha)$. The duality of Theorem 7.3 restricts to dualities between categories $\mathcal{DF}$ and $\mathcal{PF}$ and between categories $\mathcal{DI}$ and $\mathcal{PI}$.

Associated with any object $R\subseteq X\times(X\times X)\in\mathcal{PF}$ we have a relational Priestley space (X,R) in the sense of [17], and similarly, with any $R\subseteq X^{\mathrm{op}}\times(X\times X^{\mathrm{op}})\in\mathcal{PI}$. This allows us to give, in Section 8, an alternative description of the representation theory studied in [2].

Since the duality CLP is monoidal, properties characterized by the commutativity of certain diagrams in $\mathcal{J}$ involving the monoidal bifunctor can be translated to commutative diagrams in $\mathcal{P}$. We do this in Section 9.

Some questions still remain open. For example, it would be interesting to know if other Priestley type dualities are monoidal; and if so, try to adapt the formalism of this work to these dualities. It would also be interesting to know if the duality used in this article, and studied in [8], is natural in the sense of [9]; and if it is so, if our representation can be piggybacked from it. This last question was suggested to us by the anonymous referee, but we have no answer for it at the present.

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