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EXACT AND APPROXIMATE INTEGRALS OF SOME CANONICAL SYSTEMS

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BY

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SUMMARY

In the first part of this paper we discuss two cases in which the celebrated Poincaré's theorem is not applicable, but it still holds. In the second part, the formal integral of the restricted problem of three bodies proposed by Contopoulos is discussed. We show that without violating Contopoulos' rule his integral $\sum_0^\infty \Phi_n^* \mu^n$ may be reduced to $\Phi_0^* + \mu\Phi_1^*(q_1, q_2, \sigma)$, if use is made of the integrals of the osculating problem. Moreover, we give a very simple example showing that the approximate integral to the second order in μ, obtained by applying Contopoulos' rule, may be in error of order μ^2.

PART I

1. Let

$$\dot{p}_i = -H_{q_i}, \quad \dot{q}_i = H_{p_i} \qquad (i = 1, 2), \tag{S}$$

be a canonical system of differential equations, and suppose that the conservative Hamiltonian H can be expanded in a series

$$H = H_0 + \mu H_1 + \mu^2 H_2 + \tag{1}$$

convergent for sufficiently small values of $|\mu|$, for all real values of $q = (q_1, q_2)$ and for values of $p = (p_1, p_2)$ in some range D, where we assume that $H_0, H_1,$ are analytic functions of p and q, of which H_0 depends only on p and its Hessian is not zero, while $H_1, H_2,$ are periodic in q, with period 2π.

Let Φ denote a function of p, q and μ which is analytic and single-valued for all real values of q, for sufficiently small values of $|\mu|$, and for values of p which form a domain D, and suppose that Φ is periodic in q, with the period 2π. Under these conditions the function Φ can be expanded as a power series in μ of the form

$$\Phi = \Phi_0 + \mu\Phi_1 + \mu^2\Phi_2 + \tag{2}$$

where $\Phi_0, \Phi_1,$ are analytic functions of p and q, periodic in q. According to Poincaré's theorem no integral of (S) exists (except the energy integral), which is of the form $\Phi = \text{const.}$, provided that in every

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domain δ, however small, contained in D, there are an infinite number of ratios m/n for which not all the corresponding coefficients $a_{m,n}$ in the Fourier series for H_1 vanish when they become secular. (Poincaré [1], Whittaker [2]).

The necessary and sufficient condition that $\Phi = c$ may be an integral of (S) with the Hamiltonian (1), is expressed by the vanishing of the Poisson bracket

$$(\Phi, H) = \sum_{i=1,2} (\Phi_{q_i} H_{p_i} - \Phi_{p_i} H_{q_i})$$

so that we must have

$$(\Phi_0, H_0) + \mu[(\Phi_1, H_0) + (\Phi_0, H_1)] + \quad = 0$$

and therefore

$$(\Phi_0, H_0) = 0 \qquad \text{(A)}$$
$$(\Phi_1, H_0) + (\Phi_0, H_1) = 0,$$

We recall that in the first two steps of the proof of his theorem, Poincaré has shown that if (2) is an integral distinct from (1), we can always suppose that

i) Φ_0 is not a function of H_0 and
ii) Φ_0 cannot involve the variables q_1 and q_2.

2. Let us analyze, in what follows, an attempt of finding integrals of dynamical systems governed by differential equations of the form (S). With the aim of illustrating the process of his version of Poincaré's theorem, Cherry [3] has given, in the twenties, an example which would be interesting "because the series for Φ can be summed in finite terms" With unessential simplifications, this example is as follows: Let $H = H_0 + \mu H_1$ be a given Hamiltonian, where

$$H_0 = p_1 + p_2 - p_1^2 - p_2^2, \qquad H_1 = \exp[i(q_1 + 2q_2)] \qquad (3)$$

Taking $\Phi_0 = p_1 - p_1^2$ we have $(\Phi_0, H_0) = 0$. By means of equations (A) we can calculate in succession Φ_1, Φ_2, finding that the series for Φ is the expansion in powers of μ of

$$\Phi = p_1 - p_1^2 + \frac{1}{5}\mu H_1 - \frac{1}{25}\beta(\sqrt{\gamma^2 - 20\mu H_1} - \gamma)$$

where

$$\beta = 1 - 4p_1 + 2p_2, \qquad \gamma = 3 - 2p_1 - 4p_2$$

Here, the series for H_1 reduces to a single term so that Poincaré's theorem is not applicable. Hence, if the integral $\Phi = c$ found by Cherry should be independent of the integral $H = h$, we would have a very simple case which should not be affected by the celebrated Poincaré's negative result. We will show that this is not the case.

More generally suppose that

$$H_1 = \sum a_{m,n}(p_1, p_2) \exp[i(mq_1 + nq_2)]$$

where $m/n = \lambda$ is a fixed commensurable number.
Let be

$$\Phi = \Phi_0(p_1, p_2) + \mu\Phi_1(p, q) + \quad = c \qquad (4)$$

an integral independent of

$$H = H_0(p_1, p_2) + \mu H_1(p, q) = h \qquad (5)$$

The Jacobian of the functions Φ and H with respect to p_1 and p_2 may be expanded as a power series in μ of the form

$$\frac{\partial(\Phi, H)}{\partial(p_1, p_2)} = \frac{\partial(\Phi_0, H_0)}{\partial(p_1, p_2)} + \mu \left[\frac{\partial(\Phi_1, H_0)}{\partial(p_1, p_2)} + \frac{\partial(\Phi_0, H_1)}{\partial(p_1, p_2)} \right] +$$

Since

$$\frac{\partial(\Phi_0, H_0)}{\partial(p_1, p_2)} \neq 0 \qquad \text{for } (p_1, p_2) \in D_1 \text{ say,}$$

in view of (i), it will be

$$\frac{\partial(\Phi, H)}{\partial(p_1, p_2)} \neq 0 \qquad \text{for } p \in D_1 \text{ and for all } q_1 \text{ and } q_2,$$

at least for sufficiently small values of $|\mu|$. In virtue of the implicit function theorem we can solve the system of equations (4) and (5) for p_1 and p_2, as

$$p_1 = f_1(q_1, q_2, h, c; \mu) \tag{6}$$

$$p_2 = f_2(q_1, q_2, h, c; \mu) \tag{7}$$

Then, by applying the last multiplier theorem we may find the third integral of (S), i.e., the equation of the orbit, as

$$\frac{\partial}{\partial c} \Omega(q_1, q_2, h, c; \mu) = \alpha \qquad \text{(Cf. [2] p. 280)} \tag{8}$$

where Ω is the integral of the total differential

$$d\Omega = p_1 dq_1 + p_2 dq_2 \tag{9}$$

Elimination of q_1 and q_2 from equations (6), (7) and (8) leads to

$$p_2 = \omega(p_1, \sigma), \qquad \sigma = (h, c, \alpha; \mu) \tag{10}$$

so that

$$\dot{p}_2 = \omega'(p_1, \sigma)\dot{p}_1 \; *$$

or, since $mq_1 + nq_2 = n(\lambda q_1 + q_2) \equiv n\zeta$

$$\dot{p}_1 = -\mu \frac{\partial H_1}{\partial q_1} = -\mu\lambda \frac{\partial H_1}{\partial \zeta}; \qquad \dot{p}_2 = -\mu \frac{\partial H_1}{\partial q_2} = -\mu \frac{\partial H_1}{\partial \zeta}, \tag{11}$$

$$(1 - \lambda\omega') \frac{\partial H_1}{\partial \zeta} = 0$$

But this identity implies that $\omega' = 1/\lambda$. Thus

$$p_2 = \frac{1}{\lambda} p_1 + b \qquad (b\text{: a constant independent of } \sigma)$$

* For the Hamiltonian (3) one directly finds from (S) that $\dot{p}_2 = 2\dot{p}_1$.

From equation (9) it follows that

$$d\Omega = \frac{1}{\lambda} p_1(\lambda dq_1 + dq_2) + b dq_2 \tag{12}$$

thus

$$\frac{1}{\lambda}\frac{\partial p_1}{\partial q_1} - \frac{\partial p_1}{\partial q_2} = 0$$

and from here one obtains $p_1 = f(\zeta)$ so that in virtue of (12)

$$\Omega = \Omega^*(\zeta) + bq_2$$

The equation of the orbit is then $\partial\Omega^*/\partial c = \alpha$ which implies that $\zeta =$ const., hence $p_1 =$ const., contradicting equation (11). This contradiction will disprove the existence of the integral (4).

3. Suppose now that H_1 depends on two classes of terms:

$$H_1 = \sum b_n(p_1, p_2) \exp(in\zeta_1) + \sum c_n(p_1, p_2) \exp(in\zeta_2)$$

where

$$\zeta_j = \lambda_j q_1 + q_2, \quad j = 1, 2.$$

If the system (S) corresponding to the Hamilton an $H = H_0 + \mu H_1$ has the independent integrals

$$H(p, q; \mu) = h$$
$$\Phi(p, q; \mu) = c$$

we can find, as before, by means of the equations $p_1 = f_1$, $p_2 = f_2$ and the equation of the orbit $\partial\Omega/\partial c = \alpha$, an expression of the same form as (10) and, since

$$\frac{\partial H_1}{\partial q_1} = \lambda_1 \frac{\partial H_1}{\partial \zeta_1} + \lambda_2 \frac{\partial H_1}{\partial \zeta_2}, \quad \frac{\partial H_1}{\partial q_2} = \frac{\partial H_1}{\partial \zeta_1} + \frac{\partial H_1}{\partial \zeta_2},$$

we must have, in virtue of (S)

$$(1 - \lambda_1\omega')\frac{\partial H_1}{\partial \zeta_1} + (1 - \lambda_2\omega')\frac{\partial H_1}{\partial \zeta_2} \equiv 0 \tag{13}$$

But this identity cannot be verified, which is a contradiction. As an example, we mention the case (quoted by Contopoulos in [4]), when

$$H_1 = f(p_1, p_2) \cos q_1 \cos q_2$$

Identity (13) now reduces to

$$(1 - \omega') \sin(q_1 + q_2) - (1 + \omega') \sin(q_1 - q_2) \equiv 0$$

which cannot be verified.

PART II

4. Now let us consider the formal integral of the restricted problem of three bodies recently proposed by Contopoulos [5]. Let (q_1, q_2) denote the coordinates of the null mass-point referred to a synodic system with the primary $1 - \mu$ as origin. The Hamiltonian H takes the form

$$H \equiv H_0 + \mu H_1 \equiv \frac{1}{2}(p_1^2 + p_2^2) - (q_1 p_2 - q_2 p_1) - \frac{1}{r} + \mu\left(\frac{1}{r} - \frac{1}{\rho} + q_1\right) \tag{14}$$

where

$$r^2 = q_1^2 + q_2^2 \qquad \rho^2 = (q_1 - 1)^2 + q_2^2$$
$$p_1 = \dot{q}_1 - q_2 \qquad p_2 = \dot{q}_2 + q_1$$

For this Hamiltonian a formal integral

$$\Phi^* = \Phi_0^* + \mu\Phi_1^* + \mu^2\Phi_2^* + \tag{15}$$

has been constructed by Contopoulos, step by step, by the relations

$$\Phi_0^* = q_1 p_2 - q_2 p_1$$
$$\Phi_{j+1}^* = \left\{ - \int (\Phi_j^*, H_1) dt \right\}, \qquad j = 0, 1, 2, \qquad \text{(the brackets are ours)} \tag{16}$$

where

$$(\Phi_j^*, H_1) = -\frac{\partial \Phi_j^*}{\partial p_1}\frac{\partial H_1}{\partial q_1} - \frac{\partial \Phi_j^*}{\partial p_2}\frac{\partial H_1}{\partial q_2}$$

"is the Poisson bracket, in which the variables are expressed as trigonometric functions of the time, found by solving the two-body problem in the rotating system; after integration, the resulting trigonometric functions are to be expressed again by means of the original variables"

Let us analyze this result in some detail. For the given Hamiltonian (14) conditions (A) reduce to

$$(\Phi_0, H_0) = 0$$
$$(\Phi_j, H_0) + (\Phi_{j-1}, H_1) = 0, \quad j = 1, 2, \tag{A$_0$}$$

We have

$$(\Phi_0, H_1) = -\frac{\partial \Phi_0}{\partial p_1}\frac{\partial H_1}{\partial q_1} - \frac{\partial \Phi_0}{\partial p_2}\frac{\partial H_1}{\partial q_2} \tag{17}$$

and

$$(\Phi_0, H_0) = \frac{\partial \Phi_0}{\partial q_1}\frac{\partial H_0}{\partial p_1} - \frac{\partial \Phi_0}{\partial p_1}\frac{\partial H_0}{\partial q_1} + \frac{\partial \Phi_0}{\partial q_2}\frac{\partial H_0}{\partial p_2} - \frac{\partial \Phi_0}{\partial p_2}\frac{\partial H_0}{\partial q_2} \tag{18}$$

But from system (S) it follows that

$$\frac{\partial H_0}{\partial p_i} = \dot{q}_i \qquad -\frac{\partial H_0}{\partial q_i} = \dot{p}_i + \mu\frac{\partial H_1}{\partial q_i} \tag{19}$$

Equation (18) can be written as

$$(\Phi_0, H_0) = \frac{d\Phi_0}{dt} + \mu\left(\frac{\partial\Phi_0}{\partial p_1}\frac{\partial H_1}{\partial q_1} + \frac{\partial\Phi_0}{\partial p_2}\frac{\partial H_1}{\partial q_2}\right)$$

Hence, it will be, in view of (17)

$$\frac{d\Phi_0}{dt} - \mu(\Phi_0, H_1) = 0 \quad \text{whenever} \quad (\Phi_0, H_0) = 0$$

From here we obtain the identity

$$\Phi_0 - \mu \int (\Phi_0, H_1)\, dt = K \tag{I}$$

where K is a constant.

The general solution of the first partial differential equation of the system (A_0) can be written as an arbitrary function of integrals of the same form as those of the planar problem of two bodies. These integrals are, the energy integral

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2) - (q_1 p_2 - q_2 p_1) - \frac{1}{r} = h_0$$

and the angular momentum integral

$$q_1 p_2 - q_2 p_1 = c$$

If we choose $\Phi_0 = H_0$ we obtain, in virtue of (17) and (19)

$$(\Phi_0, H_1) = -\frac{\partial H_1}{\partial q_1}\dot{q}_1 - \frac{\partial H_1}{\partial q_2}\dot{q}_2 = -\frac{dH_1}{dt}$$

Identity (I) then gives for the restricted problem of three bodies, the energy integral $H_0 + \mu H_1 = h$. If we take $\Phi_0 = q_1 p_2 - q_2 p_1$ it will be

$$(\Phi_0, H_1) = \frac{\partial H_1}{\partial q_1} q_2 - \frac{\partial H_1}{\partial q_2} q_1 = q_2(1 - \rho^{-3}) = f(q_1, q_2) \text{ say,}$$

so that Id. (I) reduces to the obvious identity

$$q_1 p_2 - q_2 p_1 - \mu \int f(q_1, q_2)\, dt = K_0 \tag{I$_0$}$$

5. For orbits of small eccentricity near the primary of mass $1 - \mu$ this identity (I_0) may be transformed in an approximate integral to the first order in μ and finite time, by applying Contopoulos' rule.

Let (Q, P) be the coordinates and momenta of the corresponding unperturbed problem. We have, to the zero order in μ:

$$\int f(q_1, q_2)\, dt = \int f(Q_1, Q_2)\, dt = -J \text{ say.} \tag{20}$$

Assuming that $r_0^2 = Q_1^2 + Q_2^2 \ll 1$ we obtain the rapidly convergente expansion

$$\rho_0^{-3} \equiv (1 + Q_1^2 + Q_2^2 - 2Q_1)^{-3/2} = 1 + 3Q_1 + 6Q_1^2 - \frac{3}{2}Q_2^2 +$$

Substitution into (20) gives

$$J = \int \left(3Q_1Q_2 + 6Q_1^2Q_2 - \frac{3}{2} Q_2^3 + \quad \right) dt$$

By using the formulae of the problem of two bodies (trigonometric functions of the time), Contopoulos has found for this integral an expression of the form

$$J = \Omega(P_1, P_2, Q_1, Q_2, \sigma) \equiv \Omega(P, Q, \sigma)$$

where $\sigma = (a, e, \tilde{\omega}, t_\pi)$ is the set of osculating elements.

We thus have, to the zero order in μ

$$J = \Omega(p, q, \sigma) = \Phi_1^* \tag{21}$$

Substitution into (I_6) leads to the approximate integral to the first order in μ and finite time

$$\Phi_0^* + \mu\Phi_1^* = q_1p_2 - q_2p_1 + \mu\Omega(p, q, \sigma) \simeq K_1^*$$

If the eccentricity e and the disturbing mass μ are both very small, this approximate integral may be further simplified.

We have, to the zero order in e, assuming that $\tilde{\omega} = t_\pi = 0$:

$$Q_1 = a \cos (n - 1)t \equiv Q_1^{(0)}; \; Q_2 = a \sin (n - 1)t \equiv Q_2^{(0)}$$

Hence we have, to the zero order in e and in μ

$$J = - \int Q_2^{(0)}(1 - (\rho^{(0)})^{-3}) \, dt \equiv J^{(0)}$$

where

$$(\rho^{(0)})^2 = (Q_1^{(0)} - 1)^2 + (Q_2^{(0)})^2 = 1 + a^2 - 2Q_1^{(0)}$$

But

$$J^{(0)} = \frac{1}{n - 1} (Q_1^{(0)} - (\rho^{(0)})^{-1}) + C$$

Thus we obtain the approximate integral

$$q_1p_2 - q_2p_1 + \frac{\mu}{n - 1} \left(q_1 - \frac{1}{\rho} \right) \simeq K_1$$

with error of order μe.

6. Starting from the value of Φ_1^* given by (21), and by following the same way, Contopoulos has found in succession Φ_2^*, Φ_3^*,

But, without violating Contopoulos' rule we can also calculate his integral in the following way.

From the known integrals of the osculating problem, i.e.,

$$\frac{1}{2} (P_1^2 + P_2^2) - (Q_1P_2 - Q_2P_1) - \frac{1}{R} = h_0; \; Q_1P_2 - Q_2P_1 = c_0$$

where

$$R^2 = Q_1^2 + Q_2^2,$$

one obtains for all t

$$P_1 = P_1(Q_1, Q_2), \quad P_2 = P_2(Q_1, Q_2)$$

We then have, to the same order of approximation as in (21)

$$\Omega(P, Q, \sigma) = \Omega(P(Q), Q, \sigma) = \omega(Q_1, Q_2, \sigma) = \omega(q_1, q_2, \sigma) = \Phi_1^*$$

Since Φ_1^* does not depend on p_1 and p_2 it follows that $(\Phi_1^*, H_1) \equiv 0$.

Thus, in view of (16) one finds $\Phi_2^* = \Phi_3 = \quad = 0$ so that Contopoulos' integral (15) reduces to

$$\Phi^* = \Phi_0^* + \mu\Phi_1^* + \quad = q_1p_2 - q_2p_1 + \mu\omega(q_1, q_2, \sigma) = K^*$$

7. Let the solution of system (S) be obtained as power series of the form

$$p_i = P_i + \mu P_i^{(1)} + \mu^2 P_i^{(2)} +$$
$$q_i = Q_i + \mu Q_i^{(1)} + \mu^2 Q_i^{(2)} + \qquad (i = 1, 2),$$

convergent for all t of the finite interval $|t - t_0| \leqq T$ and for all small enough values of $|\mu|$, where (P_i, Q_i) is the solution of (S) corresponding to $\mu = 0$.

Identity (I) gives, after developing the Poissonian (Φ_0, H_1) in power series of μ, to the second order,

$$\Phi_0 - \mu \int (\Phi_0, H_1)_0 dt - \mu^2 \int \left(\frac{\partial}{\partial\mu} (\Phi_0, H_1)\right)_0 dt \simeq K_2 \tag{22}$$

where $(X)_0$ means that X must be evaluated at $\mu = 0$.

From the solution of the osculating problem we may find

$$t = t(P, Q, \sigma), \qquad \sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$$

so that equation (22) becomes

$$\Phi^{(2)} = \Phi_0(p, q) + \mu\Phi_1(P, Q, \sigma) + \mu^2\Phi_2(P, Q, \sigma) \simeq K_2 \tag{23}$$

or else

$$\Phi_0(p, q) + \mu\overline{\Phi}_1(P_1, P_2, \sigma) + \mu^2\overline{\Phi}_2(P, Q, \sigma) \simeq \overline{K}_2 \tag{24}$$

if use is made of the integrals of the unperturbed problem.

Now, since

$$F(p, q, \sigma) = F(P, Q, \sigma) + \mu \left(\frac{\partial F}{\partial\mu}\right)_0 +$$

equation (23) can be written, to the second order in μ as

$$\Phi_0(p, q) + \mu\Phi_1(p, q, \sigma) + \mu^2 \left[\Phi_2(P, Q, \sigma) - \left(\frac{\partial\Phi_1}{\partial\mu}\right)_0\right] \simeq K_2$$

Similarly, equation (24) becomes

$$\Phi_0(p, q) + \mu\overline{\Phi}_1(p_1, p_2, \sigma) + \mu^2 \left[\overline{\Phi}_2(P, Q, \sigma) - \left(\frac{\partial\overline{\Phi}_1}{\partial\mu}\right)_0\right] \simeq \overline{K}_2 \tag{25}$$

Next, we wish to compare this approximate integral to the second order in μ and finite time, with the approximate integral to the same order

$$\Phi_0^*(p, q) + \mu\Phi_1^*(p_1, p_2, \sigma) + \mu^2\Phi_2^*(p, q, \sigma) \simeq K_2^* \tag{26}$$

obtained by applying Contopoulos' rule, being obviously $\Phi_0^* = \Phi_0$ and $\Phi_1^* = \overline{\Phi}_1$.

Let suppose that

$$H_1 = H_1(q_1, q_2); \quad (\Phi_0, H_1) = f(p, q)$$

Putting

$$\pi = f(P, Q); \quad \varphi_1 = \overline{\Phi}_1(P_1, P_2, \sigma); \quad \varphi_2 = \overline{\Phi}_2(P, Q, \sigma)$$

and

$$\Lambda = \frac{\partial\varphi_1}{\partial P_1}\left(\frac{\partial}{\partial\mu}\left(\frac{\partial H_0}{\partial q_1}\right)\right)_0 + \frac{\partial\varphi_1}{\partial P_2}\left(\frac{\partial}{\partial\mu}\left(\frac{\partial H_0}{\partial q_2}\right)\right)_0 - \frac{\partial\pi}{\partial Q_1}\left(\frac{\partial q_1}{\partial\mu}\right)_0 - \frac{\partial\pi}{\partial Q_2}\left(\frac{\partial q_2}{\partial\mu}\right)_0$$

one obtains at once

$$\frac{d}{dt}\left[\varphi_2 - \left(\frac{\partial\overline{\Phi}_1}{\partial\mu}\right)_0\right] = -(\Phi_1^*, H_1)_0 + \Lambda$$

Then if, and only if, $\Lambda \equiv 0$ the approximate integral (25) may be obtained without the perturbation techniques, i.e. by merely applying Contopoulos' rule.

As an example we mention the case when

$$H_0 + \mu H_1 \equiv \frac{1}{4}(5p_1^2 + 5p_2^2 - 6p_1p_2) + \frac{1}{2}(q_1^2 + q_2^2) - \frac{1}{2}\mu(q_1 + q_2)^2$$

We obtain

$$P_1 - P_2 = \sigma_1 \cos(2t + \sigma_2) \qquad Q_1 - Q_2 = 2\sigma_1 \sin(2t + \sigma_2)$$
$$P_1 + P_2 = -\sigma_3 \sin(t + \sigma_4) \qquad Q_1 + Q_2 = \sigma_3 \cos(t + \sigma_4)$$

Taking

$$\Phi_0 = \Phi_0^* = (p_1 + p_2)^2 + (q_1 + q_2)^2$$

one has

$$(\Phi_0, H_0) = 0 \quad \text{and} \quad (\Phi_0, H_1) = 4(p_1 + p_2)(q_1 + q_2)$$

so that

$$\Phi_1^* = -4\left\{\int(p_1 + p_2)(q_1 + q_2)\,dt\right\} = 2\sigma_3^2\left\{\sin^2(t + \sigma_4)\right\}$$

We then have

$$\Phi_1^* = 2(\sigma_3^2 - (q_1 + q_2)^2) \quad \text{or else} \quad \Phi_1^* = 2(p_1 + p_2)^2$$

Taking the first function, it follows that $\Phi_2^* = \Phi_3^* = \quad = 0$.

Then

$$\Phi^* = \Phi_0^* + \mu\Phi_1^* + \quad = (p_1 + p_2)^2 + (1 - 2\mu)(q_1 + q_2)^2 = K^*$$

If we take the second one, we obtain

$$(\Phi_1^*, H_1) = 8(p_1 + p_2)(q_1 + q_2)$$

from which it follows that

$$\Phi_2^* = 4(p_1 + p_2)^2$$

Since

$$\frac{\partial H_0}{\partial q_1} = q_1; \quad \frac{\partial H_0}{\partial q_2} = q_2; \quad \pi = 4(P_1 + P_2)(Q_1 + Q_2); \quad \varphi_1 = 2(P_1 + P_2)^2$$

one has $\Lambda \equiv 0$. We then have the approximate integral

$$\Phi_0^* + \mu\Phi_1^* + \mu^2\Phi_2^* = (p_1 + p_2)^2 + (q_1 + q_2)^2 + 2\mu(p_1 + p_2)^2 + 4\mu^2(p_1 + p_2)^2 \simeq K_2^*$$

with an error of the order μ^3.

8. Unfortunately, we have not been able to show that the same is true for the most important cases in which, as in the restricted problem of three bodies, the Poissonian (Φ_0, H_1) depends only on q_1 and q_2.

Indeed, as we shall see, very simple examples show that the approximate integral (26) obtained by Contopoulos' rule, may be in error of order μ^2.

Let us consider, in fact, the case when

$$H_0 + \mu H_1 \equiv \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{1}{2}\mu(q_1^2 - q_2^2)$$

We have

$$P_1 = \sigma_1 \cos(t + \sigma_2) \qquad Q_1 = \sigma_1 \sin(t + \sigma_2)$$
$$P_2 = -\sigma_3 \sin(t + \sigma_4) \qquad Q_2 = \sigma_3 \cos(t + \sigma_4)$$

Taking

$$\Phi_0^* = q_1 p_2 - q_2 p_1$$

we obtain

$$(\Phi_0^*, H_0) = 0 \quad \text{and} \quad (\Phi_0^*, H_1) = 2q_1 q_2$$

Hence

$$\Phi_1^* = \frac{1}{2}(q_1 p_2 + q_2 p_1) - \sigma_1\sigma_3\left(\tan^{-1}\frac{q_1}{p_1} - \sigma_2\right)\sin(\sigma_2 - \sigma_4)$$

From here it follows that

$$(\Phi_1^*, H_1) = -\sigma_1\sigma_3 \sin(\sigma_2 - \sigma_4)\frac{q_1^2}{p_1^2 + q_1^2}$$

Hence

$$\Phi_2^* = -\left\{\int(\Phi_1^*, H_1)dt\right\} = \frac{1}{2}\left[\sigma_1\sigma_3\left(\tan^{-1}\frac{q_1}{p_1} - \sigma_2\right) - \frac{\sigma_3}{\sigma_1}p_1 q_1\right]\sin(\sigma_2 - \sigma_4)$$

so that (15) becomes, to the second order in μ,

$$\Phi^{*(2)} = \Phi_0^* + \mu\Phi_1^* + \mu^2\Phi_2^*$$

$$= q_1p_2 - q_2p_1 + \frac{1}{2}\mu\left[q_1p_2 + q_2p_1 - 2\sigma_1\sigma_3\left(\tan^{-1}\frac{q_1}{p_1} - \sigma_2\right)\sin(\sigma_2 - \sigma_4)\right]$$

$$+ \frac{1}{2}\mu^2\left[\sigma_1\sigma_3\left(\tan^{-1}\frac{q_1}{p_1} - \sigma_2\right) - \frac{\sigma_3}{\sigma_1}p_1q_1\right]\sin(\sigma_2 - \sigma_4) \cong K_2^* \tag{27}$$

Moreover, from the system of differential equations corresponding to the given Hamiltonian, i.e.,

$$\dot{p}_1 = -(1+\mu)q_1;\ \dot{p}_2 = -(1-\mu)q_2;\ \dot{q}_1 = p_1;\ \dot{q}_2 = p_2$$

it follows that

$$p_1 = \sigma_1\cos(\sqrt{1+\mu}\,t + \sigma_2);\ q_1 = \frac{\sigma_1}{\sqrt{1+\mu}}\sin(\sqrt{1+\mu}\,t + \sigma_2)$$

$$p_2 = -\sigma_3\sin(\sqrt{1-\mu}\,t + \sigma_4);\ q_2 = \frac{\sigma_3}{\sqrt{1-\mu}}\cos(\sqrt{1-\mu}\,t + \sigma_3)$$

By developing in power series, one has

$$p_1 = P_1 - \frac{1}{2}\mu tQ_1 + \frac{1}{8}\mu^2t(Q_1 - tP_1) +$$

$$p_2 = P_2 + \frac{1}{2}\mu tQ_2 + \frac{1}{8}\mu^2t(Q_2 - tP_2) +$$

$$q_1 = Q_1 - \frac{1}{2}\mu(Q_1 - tP_1) + \frac{1}{8}\mu^2(3Q_1 - 3tP_1 - t^2Q_1) +$$

$$q_2 = Q_2 + \frac{1}{2}\mu(Q_2 - tP_2) + \frac{1}{8}\mu^2(3Q_2 - 3tP_2 - t^2Q_2) +$$

Then we have

$$2q_1q_2 = 2Q_1Q_2 - \mu t(Q_1P_2 - Q_2P_1) + \qquad t = \tan^{-1}\frac{Q_1}{P_1} - \sigma_2$$

or, since

$$\int 2Q_1Q_2\,dt = -\frac{1}{2}(Q_1P_2 + Q_2P_1) + \sigma_1\sigma_3\,t\sin(\sigma_2 - \sigma_4)$$

and

$$\int t(Q_2P_1 - Q_1P_2)\,dt = \frac{1}{2}\sigma_1\sigma_3\,t^2\cos(\sigma_2 - \sigma_4)$$

we obtain

$$\int 2q_1q_2\,dt = -\frac{1}{2}\left[Q_1P_2 + Q_2P_1 - 2\sigma_1\sigma_3\left(\tan^{-1}\frac{Q_1}{P_1} - \sigma_2\right)\sin(\sigma_2 - \sigma_4)\right]$$

$$+ \frac{1}{2}\mu\,\sigma_1\sigma_3\left(\tan^{-1}\frac{Q_1}{P_1} - \sigma_2\right)^2\cos(\sigma_2 - \sigma_4)$$

Thus (23) becomes

$$\Phi^{(2)} = q_1p_2 - q_2p_1 + \frac{1}{2}\mu\left[Q_1P_2 + Q_2P_1 - 2\sigma_1\sigma_3\left(\tan^{-1}\frac{Q_1}{P_1} - \sigma_2\right)\sin(\sigma_2 - \sigma_4)\right]$$

$$-\frac{1}{2}\mu^2\,\sigma_1\sigma_3\left(\tan^{-1}\frac{Q_1}{P_1} - \sigma_2\right)^2\cos(\sigma_2 - \sigma_4) \backsimeq K_2 \qquad (28)$$

But we have

$$q_1p_2 + q_2p_1 = Q_1P_2 + Q_2P_1 - \frac{1}{2}\mu\,(Q_1P_2 - Q_2P_1) + $$

and

$$\tan^{-1}\frac{q_1}{p_1} = \tan^{-1}\frac{Q_1}{P_1} - \frac{1}{2}\mu\left(\sigma_1^{-2}P_1Q_1 - \tan^{-1}\frac{Q_1}{P_1} + \sigma_2\right) + $$

If we substitute these values in (28) we find

$$\Phi^{(2)} = q_1p_2 - q_2p_1 + \frac{1}{2}\mu\left[q_1p_2 + q_2p_1 - 2\sigma_1\sigma_3\left(\tan^{-1}\frac{q_1}{p_1} - \sigma_2\right)\sin(\sigma_2 - \sigma_4)\right]$$

$$+ \frac{1}{2}\mu^2\left\{\left[\sigma_1\sigma_3\left(\tan^{-1}\frac{q_1}{p_1} - \sigma_2\right) - \frac{\sigma_3}{\sigma_1}p_1q_1\right]\sin(\sigma_2 - \sigma_4)\right.$$

$$\left. - \sigma_1\sigma_3\left(\tan^{-1}\frac{q_1}{p_1} - \sigma_2\right)^2\cos(\sigma_2 - \sigma_4) + \frac{1}{2}(q_1p_2 - q_2p_1)\right\} \backsimeq K_2$$

Thus the approximate integral (27) is in error of order μ^2.

If we take, for instance, $\sigma_1 = \sigma_3 = 1;\ \sigma_2 = \sigma_4 = 0$ one has, for $t = 0$

$$p_1 = 1;\ p\ = 0;\ q_1 = 0;\ q_2 = 1 + \frac{1}{2}\mu + \frac{3}{8}\mu^2 + $$

and for $t = \dfrac{\pi}{2}$

$$p_1 = -\frac{\pi}{4} + \frac{1}{8}\left(3 - \frac{\pi^2}{4}\right)\mu^2 + \qquad\qquad q_1 = 1 - \frac{\mu}{2} + \frac{1}{8}\left(3 - \frac{\pi^2}{4}\right)\mu^2 + $$

$$p_2 = -1 + \frac{1}{32}\pi^2\mu^2 + \qquad q_2 = \frac{1}{4}\pi\mu + \frac{3}{16}\pi\mu^2 +$$

Thus

$$\Phi^{(2)}(0) = K_2 = -1 - \frac{3}{8}\mu^2 \quad \text{and} \quad \Phi^{(2)}\left(\frac{\pi}{2}\right) = -1 - \frac{3}{8}\mu^2$$

Similarly, one has

$$\Phi^{*(2)}(0) = -1 - \frac{1}{8}\mu^2; \text{ but } \Phi^{*(2)}\left(\frac{\pi}{2}\right) = -1 - \frac{1}{8}\mu^2 + \frac{1}{8}\pi^2\mu^2$$

with an error of second order in μ.

Taking $\sigma_1 = \sqrt{2}$; $\sigma_2 = \frac{\pi}{4}$ $\sigma_3 = 1$; $\sigma_4 = 0$ we obtain, for $t = 0$

$$p_1 = 1;\ p_2 = 0;\ q_1 = 1 - \frac{1}{2}\mu + \frac{3}{8}\mu^2 + \qquad q_2 = 1 + \frac{1}{2}\mu + \frac{3}{8}\mu^2 +$$

and for $t = \frac{\pi}{2}$

$$p_1 = -1 - \frac{1}{4}\pi\mu + \frac{1}{16}\pi\left(1 + \frac{\pi}{2}\right)\mu^2 + \qquad p_2 = -1 + \frac{\pi^2}{32}\mu^2 +$$

$$q_1 = 1 - \frac{1}{2}\left(1 + \frac{\pi}{2}\right)\mu + \frac{1}{8}\left(3 + \frac{3}{2}\pi - \frac{\pi^2}{4}\right)\mu^2 + \qquad q_2 = \frac{1}{4}\pi\mu + \frac{3}{16}\mu^2 +$$

Thus

$$\Phi^{*(2)}(0) = K_2^* = -1 - \frac{3}{8}\mu^2; \text{ but } \Phi^{*(2)}\left(\frac{\pi}{2}\right) = -1 - \frac{3}{8}\mu^2 + \frac{1}{8}(2 + \pi^2)\mu^2$$

with an error of second order in μ.

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