IL NUOVO CIMENTO VOL. 98 A, N. 2 Agosto 1987

Analiticity in Fourth-Order Wave Equations.

C. G. BOLLINI (*)

Centro Brasileiro de Pesquisas Físicas, CNPq/CBPF - Rua Dr. Xavier Sigaud, 150
22290 Rio de Janeiro, RJ, Brasil
Departamento de Física, Facultad de Ciencias C.C. 67
Universidad Nacional de La Plata - La Plata, Argentina
Comisión de Investigaciones Científicas de la Provincia de Buenos Aires - Buenos Aires

J. J. GIAMBIAGI

Centro Brasileiro de Pesquisas Físicas, CNPq/CBPF - Rua Dr. Xavier Sigaud, 150
22290 Rio de Janeiro, RJ, Brasil
Centro Latino Americano de Física, CLAF - Av. Venceslau Braz, 71 (fundos)
22290 Rio de Janeiro, RJ - Brasil

(ricevuto il 13 Aprile 1987)

Summary. — In this paper we present, through a familiar example (δ-function potential in one dimension), the analytic properties of Jost functions associated with fourth-order equations. It is shown how to construct the Jost functions and the two discontinuity matrices associated with the line of singularities. The latter divide the complex k-plane in eight regions of analiticity. One of these matrices is related to the asymptotic behaviour of the scattering state. The other is not. Both are necessary to solve the inverse problem. Besides the usual poles related to bound states there are also other poles associated with total reflexion.

PACS. 11.10. - Field theory.

1. – Introduction.

The possibility of using higher-order equations in particle physics theory has been considered, time and again, but the difficulties found in the process of interpretation are almost unsurmountable and such equations are, therefore, discarded in favour of the second-order ones.

(*) On leave from CBPF.

The reasons are indeed very good. Among them we find that the energy is in general not positive definite. The usual causality relations are not satisfied. The «S matrix» lacks unitarity. The Hilbert space of quantum states has indefinite metric (1)... .

However, sticking to the second-order wave equations does not solve all problems. Notably, gravity theory refuses to be consistently quantized. Furthermore, the consideration of supersymmetry in a space-time with a number of dimensions greater than four (the «Kaluza-Klein» programme) can lead to higher-order wave equations (2,3), so that the dimensionality of space could be related to the order of the equations of motion.

In our opinion, such a relation and the possibility of reducing the degree of divergence through the use of higher-order equations justify the efforts expended in trying to understand and clarify the physical interpretation of the theory.

In (4) we have stated in a brief form the canonical methods necessary to construct the field tensors and the Heisenberg quantization of the fields obeying the higher-order equations. They are equivalent to the results one obtains by using a «Schwinger action integral» method (5). Nevertheless, nothing is said there about the equations and their solutions.

In the present paper, we take a fourth-order stationary «Schrödinger equation» in one dimension and study its solutions, in particular for discontinuous step potentials, square barrier and «δ-function potential».

The motivation is to examine the difficulties in the simplest cases and to learn there how to deal with them in more realistic examples. We discuss the generalized Jost functions (6) related to the problem. We think that these methods and results, little known among physicists will be better understood through the discussion of a very simple example where the physical implications are more clear. For the complete bibliography we refer to (6).

This study shows that the simple structure of the «transition matrix» for the second-order case is here changed into a set of discontinuity matrices. One of them is similar to (and has the same origin as) the second-order one but the others are new elements, not contained in the scattering states, which cannot be ignored for the physical completeness of the theory.

In other words, the knowledge of the usual scattering matrix is not enough for

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the determination of the potential, *i.e.* for the solution of the inverse problem. A fact that is related to the lack of unitarity of the usual naive scattering matrix.

In sect. 2 we introduce the equation and find its solutions for step potential. In sect. 3 we do the same for a δ-function potential. In sect. 4 we introduce and compute the four Jost functions related to the potential of sect. 3. In sect. 5 we define and calculate the «Jost functions» and the discontinuity matrix for the fourth-order case. In sect. 6 we evaluate the isolated singularities. In sect. 7 we discuss the inverse Gelfand-Levitan-Marchenko equations for this problem.

2. – «Schrödinger» equation.

We shall consider the following equation:

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}\varphi + m^3(V-E)\varphi = 0\,. \tag{2.1}$$

We can divide (2.1) by m^4 and consider only adimensional quantities $mx \to x$, $m^{-1}(V-E) \to V-E$. In what follows we take then (2.1) with $m=1$. We begin by solving (2.1) for the

case *a*) $V=0,\ x<0,\ V=\text{const}>0,\ E>V,\ x>0$.

An exponential function $\exp[iKx]$ is a solution of (2.1) if

$$K^4 = E \qquad \text{in } x<0 \tag{2.2}$$

and

$$K^{4\prime} = E-V \qquad \text{in } x>0\,. \tag{2.3}$$

There are then four solutions in each region. Writing $\exp[iKx]$ for a particular solution, say $K=+E^{1/4}$, we have the four solutions

$$\begin{cases} A_1\exp[+iKx]\,, & A_2\exp[-iKx]\,,\\ A_3\exp[+Kx]\,, & A_4\exp[-Kx]\,, \end{cases} \qquad K=E^{1/4},\ x<0, \tag{2.4}$$

and

$$\begin{cases} B_1\exp[+iK'x]\,, & B_2\exp[-iK'x]\,,\\ B_3\exp[+K'x]\,, & B_4\exp[-K'x]\,, \end{cases} \qquad K'=(E-V)^{1/4},\ x>0. \tag{2.5}$$

There are asymptotic and boundary conditions. The latter are dictated by the differential equation (2.1). As we have a fourth-order equation, we must impose continuity of the function and its first three derivatives (provided V has no δ-function singularity).

If we want to describe a situation in which a plane wave is incoming from the left with unit amplitude ($A_1 = 1$) and then is reflected and transmitted (at $x = 0$) as bounded waves ($A_4 = 0$, $B_3 = 0$), with no other plane wave incoming from the right ($B_2 = 0$), we have the solution

$$\text{(2.6)} \qquad \begin{cases} \phi(x) = \exp[iKx] + A_2 \exp[-iKx] + A_3 \exp[Kx] & \text{for } x < 0\,, \\ \phi(x) = B_1 \exp[iK'x] + B_4 \exp[-K'x] & \text{for } x > 0\,, \end{cases}$$

with the boundary conditions

$$\text{(2.7)} \qquad \begin{cases} 1 + A_2 + A_3 = B_1 + B_4\,, \\ iK - iKA_2 + KA_3 = iK' B_1 - K' B_4\,, \\ -K^2 - K^2 A_2 + K^2 A_3 = -K'^2 B_1 + K'^2 B_4\,, \\ -iK^3 + iK^3 A_2 + K^3 A_3 = -iK'^3 B_1 - K'^3 B_4\,. \end{cases}$$

The solution of system (2.7) is ($y = K'/K$)

$$\text{(2.8)} \qquad \begin{cases} A_2 = \dfrac{i(1-y)(1-iy)}{(1+y)(1+iy)}\,, \qquad A_3 = \dfrac{(1+i)(1-y)}{(1+y)}\,, \\ B_1 = \dfrac{2}{y(1+y)}\,, \qquad B_4 = \dfrac{-2(1-y)}{y(1+y)(1+iy)}\,. \end{cases}$$

It is easily checked that to eq. (2.1) there corresponds the conserved current ($V - E$ real)

$$\text{(2.9)} \qquad \begin{cases} j = -i\left(\dfrac{d^3\phi^*}{dx^3}\phi + \dfrac{d\phi^*}{dx}\dfrac{d^2\phi}{dx^2} - \phi^*\dfrac{d^3\phi}{dx^3} - \dfrac{d^2\phi^*}{dx^2}\dfrac{d\phi}{dx}\right), \\ \dfrac{dj}{dx} \equiv 0\,. \end{cases}$$

Using (2.9) for the solution (2.6), we obtain

$$\text{(2.10)} \qquad \begin{cases} j = 4K^3(1 - |A_2|^2) & \text{for } x < 0\,, \\ j = 4K'^3|B_1|^2 & \text{for } x > 0\,. \end{cases}$$

The exponentially decreasing «waves» with coefficients A_3 and B_4 do not contribute to the current.

Obviously, we can define a reflexion coefficient

$$R = |A_2|^2 \tag{2.11}$$

and a transmission coefficient

$$T = \left(\frac{K'}{K}\right)^3 |B_1|^2 . \tag{2.12}$$

With the values given by (2.8) we have

$$R = \frac{(1-y)^2}{(1+y)^2}, \qquad T = \frac{4y}{(1+y)^2} \tag{2.13}$$

and, of course, $R + T = 1$.

Case $b)$ $0 < E < V = \text{const}$ (for $x > 0$), $V = 0$ $(x < 0)$.

The solution to the left of the origin is again given by (2.6) $(x < 0)$. Instead, for

$$x > 0 \quad \text{we take} \quad K' = (V - E)^{1/4} \tag{2.14}$$

and define the four quartic roots of -1 as

$$\varepsilon_1 = \frac{-1+i}{\sqrt{2}}, \quad \varepsilon_2 = \frac{1+i}{\sqrt{2}}, \quad \varepsilon_3 = \frac{1-i}{\sqrt{2}}, \quad \varepsilon_4 = \frac{-1-i}{\sqrt{2}}; \tag{2.15}$$

(2.5) is then replaced by

$$B_i \exp[\varepsilon_i K' x]$$

and the condition of boundedness reduces the solution for $x > 0$ to the form

$$\phi(x) = B_1 \exp[\varepsilon_1 K' x] + B_4 \exp[\varepsilon_4 K' x], \qquad x > 0, \tag{2.16}$$

which describes an exponentially damped wave, so that the transmitted current is zero.

The coefficients A_i and B_i can be found as in (2.7), (2.8); but we prefer to consider here now the special case of an infinite wall $(V \to \infty)$.

In such a limit we easily get

$$\begin{cases} A_2 = \dfrac{1+i}{1-i} + \mathcal{O}\left(\dfrac{1}{y}\right), \quad A_3 = \dfrac{2}{i-1} + \mathcal{O}\left(\dfrac{1}{y}\right), \\ B_1 = 2\,\dfrac{1-i}{1+i}\,\dfrac{1}{y^2} + \mathcal{O}\left(\dfrac{1}{y^3}\right), \quad B_4 = -\dfrac{2}{y^2} + \mathcal{O}\left(\dfrac{1}{y^3}\right). \end{cases} \tag{2.17}$$

We see that on the wall, for $x \to -0$,

$$\psi(0) \to 0, \quad \frac{d\psi}{dx}(0) \to 0, \tag{2.18}$$

while the second derivative tends to a finite value (see (2.16), (2.17)).

3. – The δ-function potential.

By a similar method we can solve the problem of a rectangular potential barrier or the square-well potential, but for our purpose in this note it is better to consider a limiting case, that of the δ-function potential

$$\frac{d^4\varphi}{dx^4} + a\delta(x)\varphi = E\varphi. \tag{3.1}$$

By integration around the origin we deduce the discontinuity of the third derivative

$$\left.\frac{d^3\varphi}{dx^3}\right|_{0^+} - \left.\frac{d^3\varphi}{dx^3}\right|_{0^-} = -a\varphi(0),$$

while the function itself and its first and second derivatives must be continuous at the origin.

If we look for a solution of (3.1), which represents an incident plane wave from the left, and is bounded everywhere, we are led to

$$\begin{cases} \varphi(x) = \exp[iKx] + A\exp[-iKx] + B\exp[Kx], & x < 0, \\ \varphi(x) = C\exp[iKx] + D\exp[-Kx], & x > 0. \end{cases} \tag{3.2}$$

The equations resulting from the condition at $x = 0$ are

$$\begin{cases} 1 + A + B = C + D, \\ i - iA + B = iC - D, \\ -1 - A + B = -C + D, \\ -i + iA + B = -iC - D + \dfrac{a}{K^3}\varphi(0). \end{cases} \tag{3.3}$$

The solution of this system is

$$A = -\frac{ia}{4K^3 - a + ia}, \quad B = iA, \quad C = 1 + A, \quad D = iA = B.$$

The pole of A at $4K^3 = a(1-i)$, which eliminates the first exponential in (3.2), corresponds to a bound state only if the remaining exponential functions decrease for $x \to \pm\infty$. From an analysis of (3.2) we see that this possibility actually occurs only if $a<0$, *i.e.* if the δ-function potential is attractive:

$$K = \left(\frac{|a|}{2\sqrt{2}}\right)^{1/3} \frac{1+i}{\sqrt{2}} \qquad \text{for } a<0 .$$

4. – Jost functions.

We are going to define the «Jost functions» of the problem (see ref. [6] and the references therein) as four linearly independent solutions of the fourth-order equation. These solutions are to be ordered according to the asymptotic behaviour as $x \to \pm\infty$ [6].

Outside the region where the potential is felt (asymptotic regions) we can write (2.1) or (3.1) simply as

$$\frac{d^4\psi}{dx^4} = k^4\psi , \qquad E = k^4 . \tag{4.1}$$

For any complex k there are four solutions

$$\exp[ikx], \qquad \exp[-ikx], \qquad \exp[kx], \qquad \exp[-kx]. \tag{4.2}$$

The behaviour for large x depends on the real part of the exponent.

We take as the first Jost function f_1 that solution of (3.1) which for $x \to -\infty$ has the greatest rate of decrease. The equation fixes the rest of the solution. The second Jost function $f_2(x)$ has for $x \to -\infty$ the exponential with the second greatest rate of decrease. As any admixture with f_1 satisfies also this requirement, this does not fix the solution. As, in principle, f_1 will have the greatest rate of increase for $x \to +\infty$, we are free now to impose for f_2 the extra condition that this solution shall have the second greatest rate of increase for $x \to +\infty$, f_3 will have the next rate for $x \to \pm\infty$ and similarly for f_4.

In order to see more clearly how this procedure works in an actual case, we are now going to take the usual one-dimensional second-order Schrödinger equation and construct the two Jost functions, defined according to the procedures just explained.

The Schrödinger equation reads

$$\left(\frac{1}{i}\right)^2 \frac{d^2\psi}{dx^2} + a\delta(x)\psi = E\psi . \tag{4.3}$$

Asymptotically

$$-\frac{d^2\psi}{dx^2} = k^2\psi . \tag{4.4}$$

The two exponential solutions are

$$\exp[ikx], \qquad \exp[-ikx]. \tag{4.5}$$

Let us now construct the two Jost functions. In the upper half-plane of k, we have

$$\begin{cases} f_1(k, x) = \exp[-ikx], & x<0, \\ A\exp[-ikx] + B\exp[ikx], & x>0, \\ f_2(k, x) = \exp[ikx] + C\exp[-ikx], & x<0, \\ D\exp[ikx], & x>0, \end{cases} \tag{4.6}$$

with

$$A = \frac{2k+ai}{2k}, \quad B = \frac{-ai}{2k}, \quad C = \frac{a}{2ik-a}, \quad D = \frac{2ik}{2ik-a}. \tag{4.7}$$

In the lower half-plane, we have

$$\begin{cases} f_1(k, x) = \exp[ikx], & x<0, \\ A'\exp[ikx] + B'\exp[-ikx], & x>0, \\ f_2(k, x) = \exp[-ikx] + C'\exp[ikx], & x<0, \\ D'\exp[-ikx], & x>0, \\ A' = \dfrac{2k-ia}{2k}, \quad B' = \dfrac{ia}{2k}, \quad C' = \dfrac{ia}{2k-ia}, \quad D' = \dfrac{2k}{2k-ia}. \end{cases} \tag{4.8}$$

The Jost functions are then well defined in the upper half-plane of k (4.6) and in the lower half-plane (4.8), but all along the real axis of k they have both the same type of behaviour at $x = \pm\infty$. In this sense, the real axis appears as a singular axis where we can define two limiting functions according to the way we take the limit coming from above or from below.

For reasons that will become clear later we divide (see [6]) the real axis in two rays, $K>0$, $K<0$, $K = \text{Re}(k)$.

For $R(k)>0$ we take f_1^+ or f_2^+ as the limit from above of (4.6) and f_1^- or f_2^- as the limit from below of (4.8). In the ray $R(k)<0$ we define f_1^+ and f_2^+ as the limit from below of (4.8) and f_1^- f_2^- as the limit from above of (4.6). Note that + or − do not refer to the sign of f but rather to the sense of rotation in the complex k-plane. We call + the clockwise rotation and − the anticlockwise rotation. Explicitly

$$\text{(4.9)}\qquad \begin{cases} f_1^+(K,x) = \exp[-iKx], & x<0, \\ A\exp[-iKx] + B\exp[iKx], & x>0\ (K>0), \\ f_1^+(K,x) = \exp[iKx], & x<0, \\ A'\exp[iKx] + B'\exp[-iKx], & x>0\ (K<0); \end{cases}$$

$$\text{(4.10)}\qquad \begin{cases} f_2^+(K,x) = \exp[iKx] + C\exp[-iKx], & x<0, \\ D\exp[iKx], & x>0\ (K>0), \\ f_2^+(K,x) = \exp[-iKx] + C'\exp[iKx], & x<0, \\ D'\exp[-iKx], & x>0\ (K<0); \end{cases}$$

$$\text{(4.11)}\qquad \begin{cases} f_1^-(K,x) = \exp[iKx], & x<0, \\ A'\exp[iKx] + B'\exp[-iKx], & x>0\ (K>0), \\ f_1^-(K,x) = \exp[-iKx], & x<0, \\ A\exp[-iKx] + B\exp[iKx], & x>0\ (K<0); \end{cases}$$

$$\text{(4.12)}\qquad \begin{cases} f_2^-(K,x) = \exp[-iKx] + C'\exp[iKx], & x<0, \\ D'\exp[-iKx], & x>0\ (K>0), \\ f_2^-(K,x) = \exp[iKx] + C\exp[-iKx], & x<0, \\ D\exp[iKx], & x>0\ (K<0). \end{cases}$$

The functions f_i^+ and f_i^- are solutions of the differential equation (4.3), but, as this is a second-order equation, there must exist a linear relation between the two solutions «+» and the two «−». They are related by a «transition matrix» A:

$$\text{(4.13)}\qquad f^+ = Af^-$$

From (4.9) to (4.12) we can get the matrix A which has the form

$$\text{(4.14)}\qquad A = \begin{pmatrix} 1 & \alpha \\ -\alpha^* & 1-\alpha\alpha^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \tilde{A} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with

$$\alpha = \frac{a}{2i|K| + a} \qquad \text{for any } K. \tag{4.15}$$

The matrix A (or $\tilde{A}$) has only one independent element. To see how this element can be physically measured, we note that (4.13) and (4.14) give

$$f_1^+ = f_2^- + \alpha f_1^- . \tag{4.16}$$

f_1^+ represents a plane wave *in*coming from the left and two plane waves *in*coming and *out*going from the right. Equation (4.16) tells us that such a situation can be achieved by carefully superimposing f_2^- which is an experiment of reflexion and transmission of a wave incoming from the left and f_1^- which is a similar experiment with a wave incoming from the right.

It is then easy to see from (4.16), (4.9), (4.11), (4.12) that α is minus the reflected amplitude

$$\alpha = -C' \tag{4.17}$$

and, of course, the reflexion coefficient is

$$R = |\alpha|^2 .$$

The matrix A can be experimentally determined by measuring amplitude and phase of the reflected wave.

5. – Jost functions for the fourth-order equation.

We construct the Jost functions according to the rules given in sect. 4. For each value of E we have four roots of the equation $E = k^4$, namely $\alpha_i k$, where α_i, $i = 1, 2, 3, 4$, are the four quartic roots of unity

$$(+k,\ -k,\ ik,\ -ik) . \tag{5.1}$$

If we order the exponentials of (4.2) according to the behaviour for large x, there are obvious ambiguities when the real parts of two of them are equal, and this happens when the roots are on four lines given by the real and imaginary axis, and the lines are at 45° degrees with them. This divides the complex k-plane in eight «octants» where the order of the functions is well defined according to the given rules.

When k (complex) is in a given octant, we define α_i such that

$$\text{Re}(\alpha_1 k) > \text{Re}(\alpha_2 k) > \text{Re}(\alpha_3 k) > \text{Re}(\alpha_4 k) . \tag{5.2}$$

We note also that, in general,

(5.3) $$\alpha_3 = -\alpha_2 \quad \text{and} \quad \alpha_4 = -\alpha_1 .$$

The Jost functions for any octant are then

(5.4) $$\begin{cases} f_1(k,\,x) = \exp[\alpha_1 kx]\,, & x<0\,,\\ \exp[\alpha_1 kx] - \dfrac{a}{4k^3}\cdot & \\ \quad\cdot[\alpha_1(\exp[\alpha_1 kx] - \exp[-\alpha_1 kx]) + \alpha_2(\exp[\alpha_2 kx] - \exp[\alpha_2 kx])], & x>0\,;\\ f_2(k,\,x) = \exp[\alpha_2 kx] + \dfrac{a\alpha_1 \exp[\alpha_1 kx]}{4k^3 - a\alpha_1}, & x<0\,,\\ \exp[\alpha_2 kx] - \dfrac{a}{4k^3 - a\alpha_1}[\alpha_2(\exp[\alpha_2 kx] - \exp[-\alpha_2 kx]) - \alpha_1 \exp[-\alpha_1 kx]]\,, & \\ & x>0\,;\\ f_3(k,\,x) = \exp[-\alpha_2 kx] + \dfrac{a(\alpha_1 \exp[\alpha_1 kx] + \alpha_2 \exp[\alpha_2 kx])}{4k^3 - a(\alpha_1+\alpha_2)}, & x<0\,,\\ \exp[-\alpha_2 kx] + \dfrac{a(\alpha_1 \exp[-\alpha_1 kx] + \alpha_2 \exp[-\alpha_2 kx])}{4k^3 - a(\alpha_1+\alpha_2)}, & x>0\,;\\ f_4(k,\,x) = \exp[-\alpha_1 kx] + \dfrac{a}{4k^3 - a\alpha_1}\cdot & \\ \quad\cdot[\alpha_1 \exp[\alpha_1 kx] + \alpha_2(\exp[\alpha_2 kx] - \exp[-\alpha_2 kx])]\,, & x<0\,,\\ \dfrac{4k^3}{4k^3 - a\alpha_1}\exp[-\alpha_1 kx]\,, & x>0\,. \end{cases}$$

It is easy to see that

(5.5) $$\alpha_1 = (-i)^n\,,$$

where $n = 0$ for the first and eight octants, $n = 1$ for the second and third octants, $n = 2$ for the fourth and fifth octants and finally $n = 3$ for the sixth and seventh octants;

(5.6) $$\alpha_2 = (-1)^m i\alpha_1\,,$$

where $m = 0$ for even octants (second, fourth, etc.) and $m = 1$ for odd octants.

As in the second-order case, we can define the «+» and «−» Jost functions on the rays dividing two consecutive octants.

The «plus» functions are those obtained as the limit of (5.4) taken in a clockwise sense to the above-mentioned rays. The «minus» functions are those obtained in the anticlockwise limit.

The values of α_i being those corresponding to the octant from which the respective limit is taken.

Of course, the four plus functions and the four minus functions are solutions of the same fourth-order linear differential equation, so they must be linearly related:

$$f^+ = Af^- . \tag{5.7}$$

Due to the properties of the solutions for values of k that differ by a factor of i, the matrix A has the same value for rays of the same parity. Of course, this can be explicitly verified.

Thus we need only to consider the ray number one for which k is real and positive ($k = K$) and the ray number two for which $k = ((1+i)/\sqrt{2})\,K$ (K = real and positive).

For the first ray the matrix A takes the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & -\alpha^* & 1-\alpha\alpha^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.8}$$

where

$$\alpha = \frac{-ia}{4K^3 - a(1+i)} \tag{5.9}$$

((5.8) and (5.9) are valid for any odd-number ray). For the second ray we have ($k = K((1+i)/\sqrt{2})$)

$$A = \begin{pmatrix} 1 & \beta & 0 & 0 \\ i\gamma^* & 1+i\beta\gamma^* & 0 & 0 \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & i\beta^* & 1+i\gamma\beta^* \end{pmatrix}, \tag{5.10}$$

$$i\beta = \gamma = \frac{-ia}{4K^3\left(\dfrac{-1+i}{\sqrt{2}}\right) - a} \tag{5.11}$$

(the equality $\gamma = i\beta$ only occurs for special cases) and the values (5.10), (5.11) are valid for any even-number ray.

6. - Isolated singularities.

The lines dividing the different octants in which the Jost functions are well defined are not the only singularities. Definition (5.4) shows that the points where

$$4k^3 = a\alpha_1 \tag{6.1}$$

and those for which

$$4k^3 = a(\alpha_1 + \alpha_2) \tag{6.2}$$

require special attention.

Let us first take (6.1). For each value of α_1 there are three values of k for which (6.1) is satisfied. The root that falls in an octant for which α_1 has the chosen value is a singularity of the corresponding Jost function.

We define

$$K = +\left(\frac{|a|}{4}\right)^{1/3}. \tag{6.3}$$

The octant corresponding to the three roots of (6.1) is then determined by the three values

$$\varepsilon = \sqrt[3]{\operatorname{sg} a \cdot \alpha_1}\,, \tag{6.4}$$

i.e.

$$\varepsilon = \sqrt[3]{\alpha_1}\,, \qquad \text{if } a>0\,, \tag{6.5}$$

$$\varepsilon = \sqrt[3]{\alpha_4}\,. \qquad \text{if } a<0\,. \tag{6.6}$$

The interesting roots are

$$\begin{cases} \sqrt[3]{1} = \left(1,\ \exp\left[i\frac{2}{3}\pi\right],\ \exp\left[-i\frac{2}{3}\pi\right]\right), \\ \sqrt[3]{-1} = \left(-1,\ \exp\left[i\frac{\pi}{3}\right],\ \exp\left[-i\frac{\pi}{3}\right]\right), \\ \sqrt[3]{i} = \left(-i,\ \exp\left[i\frac{\pi}{6}\right],\ \exp\left[i\frac{5}{6}\pi\right]\right), \\ \sqrt[3]{-i} = \left(i,\ \exp\left[-i\frac{\pi}{6}\right],\ \exp\left[-i\frac{5}{6}\pi\right]\right). \end{cases} \tag{6.7}$$

One then can see that when $a < 0$ there are no roots for k that correspond to the given value of α_4, while if $a > 0$, one has the following roots:

$$k = K\alpha_1^* \tag{6.8}$$

with K given by (6.3)

$$K = +\left(\frac{|a|}{4}\right)^{1/3}.$$

With (6.1) and (6.8) the Jost functions given by (5.4) have the form (near the pole)

$$\left\{\begin{array}{ll} f_1 = \exp[Kx], & x<0, \\ \exp[-Kx] + 2\sin Kx, & x>0; \\ f_2 = \dfrac{a\alpha_1}{4k^3 - a\alpha_1} f_1, & \\ \dfrac{\alpha_2}{\alpha_1} f_3 = -\exp[+Kx] + 2\sin Kx, & x<0, \\ -\exp[-Kx], & x>0; \\ f_4 = \dfrac{-a\alpha_2}{4k^3 - a\alpha_1} f_3 = \dfrac{-a\alpha_1}{4k^3 - a\alpha_1}\dfrac{\alpha_2}{\alpha_1} f_3 . & \end{array}\right. \tag{6.9}$$

The first Jost function in (6.9) corresponds to a case of total reflexion from the right. Note that, for the value (6.3), the reflected amplitude determined by (5.9) gives $\alpha = 1$. f_3, on the other hand, represents a case of total reflexion from the left. In both cases there are evanescent tales at both sides of the origin.

We then see that, up to $\mathcal{O}(k - K\alpha_1^*)$, the Jost functions are not linearly independent. We have the relation

$$f_i = \frac{\Gamma_{ij}}{k - K\alpha_i^*} f_j, \tag{6.10}$$

where Γ_{ij}, the matrix of residues at the pole, can be computed from (6.9) (only Γ_{21} and Γ_{43} are different from zero).

Let us now consider (6.2):

$$k^3 = \frac{a}{2\sqrt{2}}\,\frac{\alpha_1 + \alpha_2}{\sqrt{2}} = K^3\frac{\alpha_1 + \alpha_2}{\sqrt{2}}\,\mathrm{sg}\,a\,, \tag{6.11}$$

where now

$$K = + \left(\frac{|a|}{2\sqrt{2}} \right)^{1/3}. \tag{6.12}$$

The three roots $\sqrt[3]{(\alpha_1 + \alpha_2)/\sqrt{2}}$ are

$$\begin{cases} \sqrt[3]{\dfrac{1+i}{\sqrt{2}}} = \left(\dfrac{-1+i}{\sqrt{2}}, \exp\left[i\dfrac{\pi}{12}\right], \exp\left[-i\dfrac{7}{12}\pi\right] \right), \\ \sqrt[3]{\dfrac{1-i}{\sqrt{2}}} = \left(-\dfrac{1+i}{\sqrt{2}}, \exp\left[-i\dfrac{\pi}{12}\right], \exp\left[i\dfrac{7}{12}\pi\right] \right), \\ \sqrt[3]{-\dfrac{1+i}{\sqrt{2}}} = \left(\dfrac{1+i}{\sqrt{2}}, \exp\left[i\dfrac{11}{12}\pi\right], \exp\left[-i\dfrac{5}{12}\pi\right] \right), \\ \sqrt[3]{-\dfrac{1+i}{\sqrt{2}}} = \left(\dfrac{1-i}{\sqrt{2}}, \exp\left[-i\dfrac{11}{12}\pi\right], \exp\left[i\dfrac{5}{12}\pi\right] \right). \end{cases} \tag{6.13}$$

When $a > 0$ there are no roots for k that correspond to the given value of $\alpha_1 + \alpha_2$, while if $a < 30$ one has

$$k = K \frac{\alpha_1^* + \alpha_2^*}{\sqrt{2}}, \tag{6.14}$$

with K given by (6.12).

The Jost functions are then given by

$$\begin{cases} f_1 = \exp[\alpha_1 kx], & x<0, \\ \dfrac{\alpha_2}{\alpha_1+\alpha_2}\exp[\alpha_1 kx] + \dfrac{\alpha_1}{\alpha_1+\alpha_2}\exp[-\alpha_1 kx] - \\ \qquad - \dfrac{\alpha_2}{\alpha_1+\alpha_2}(\exp[\alpha_2 kx] - \exp[-\alpha_2 kx]), & x>0; \\ f_2 = \dfrac{\alpha_1}{\alpha_2}\exp[\alpha_1 kx] + \exp[\alpha_2 kx], & x<0, \\ \exp[-\alpha_2 kx] + \dfrac{\alpha_1}{\alpha_2}\exp[-\alpha_1 kx], & x>0; \\ f_3 = \dfrac{a\alpha_2}{4k^3 - a(\alpha_1+\alpha_2)} f_2; \\ f_4 = \dfrac{\alpha_1}{\alpha_2}\exp[\alpha_1 kx] + \exp[-\alpha_1 kx] + \exp[\alpha_2 kx] - \exp[-\alpha_2 kx], & x<0, \\ \exp[-\alpha_1 kx] + \dfrac{\alpha_1}{\alpha_2}\exp[-\alpha_1 kx], & x>0. \end{cases} \tag{6.15}$$

The second Jost function in (6.15) represents a bound state as the exponential functions, due to (6.14), are decreasing towards both sides of the origin.

Again we can write

$$f_i = \frac{\Gamma_{ij}}{k - K\dfrac{\alpha_1^* + \alpha_2^*}{\sqrt{2}}} f_j, \tag{6.16}$$

where now only Γ_{32} is different from zero and its value can be computed from (6.15).

7. – The inverse problem.

In sect. 5 we have defined (following ref. [6]) and classified the four Jost functions according to the asymptotic behaviour for $x \to \pm \infty$ of linearly independent solutions of the wave equation. Those functions are analytic functions of k except for singular lines and points. These properties are general and independent of the potential in (2.1). This is true in particular for the division of the complex k-plane in octants within which the Jost functions are analytic. Different potentials lead to different transition matrices on the singular lines and also to different isolated singularities. We have computed these matrices for the simple case of the δ-function potential in sect. 5 and sect. 6.

The structure of the wave equation determines the general analytic properties of the Jost functions. The potentials determine the singularities of those functions. The residues are proportional to the coupling constant. See (6.9) and (6.15) to verify how the problem of going from the «measured singularities» to the potential, the inverse problem, is more involved than in the 2nd-order case. See [7].

We refer to the literature for this problem [8], which we do not intend to discuss here, only to point out an essential difference. In order to solve the problem, it is necessary to know not only the transition matrix whose elements depend only on the coefficients of the scattering states, but also the transition matrix involving the coefficients of real exponentials. This means that in this case it is not true that all the physics is contained in the scattering states.

8. – Discussion.

We defined the Jost functions associated with the fourth-order wave equation with a δ-function potential, according to the asymptotic behaviour of the

(7) P. Deift, C. Tomei and E. Trubowicz: *Commun. Pure Appl. Math.*, 35, 567 (1982).
(8) I. M. Gel'fand and B. M. Levitan: *Am. Math. Soc., Transl.*, 6, 253 (1955).

corresponding solutions. They are analytic functions in the complex k-plane with rays of discontinuity that divide the complex plane in octants. To each of the rays there corresponds a discontinuity matrix relating the set of Jost functions on both sides. Essentially there are only two such matrices, one for each two consecutive rays. The rest repeats these two by the symmetry properties. The discontinuity matrix on the real positive k-axis (5.8), (5.9) is similar to the usual S-matrix and can be measured asymptotically with the plane-wave state (scattering state). This fact is not true for the other discontinuity matrix, which is related to waves with real exponent amplitudes and is not determined by the scattering plane-wave states. We see that the physical observation of the scattering states, plus the knowledge of the bound states with its residues and the total reflexion state, is not equivalent to the knowledge of the potential.

There are also isolated singularities (poles) of the Jost functions. As in the second-order equations, there are poles associated with bound states (cf. (6.14), (6.15)) but there are also poles associated with total reflexion by the potential (cf. (6.8), (6.9)), a fact which does not appear in the second-order equations.

* * *

The authors are indebted to Mr. N. F. Swaiter for calling their attention to Tomei's work. They are indebted to Prof. C. Tomei for several interesting discussions on the subject when most of the mathematical methods of the paper were learnt by the authors.

● RIASSUNTO (*)

In questo articolo si presentano le proprietà analitiche delle funzioni di Jost associate ad equazioni di quarto ordine mediante un esempio familiare (potenziale della funzione δ in una dimensione). Si mostra come costruire le funzioni di Jost e le due matrici di discontinuità associate alla linea di singolarità. Queste ultime dividono il piano k complesso in otto regioni di analiticità. Una di queste matrici è in relazione con il comportamento asintotico dello stato di scattering. Non lo è l'altra. Entrambe sono necessarie per risolvere il problema inverso. Oltre ai poli consueti collegati agli stati legati ce ne sono altri associati alla riflessione totale.

(*) *Traduzione a cura della Redazione.*

Аналитичность в волноых уравнениях четвертого порядка.

Резюме (*). — В этой работе, используя обычный пример (δ-функциональный потенциал в одном измерении), мы исследуем аналитические свойства функций Йоста, связанных с уравнениями четвертого порядка. Конструиружтся функции Йоста и две разрывные матрицы, связанные с линией сингулярностей. Комплексная k-плотность делится на восемь областей аналитичности. Одна из этих матриц связана с асимптотическим поведением состояния рассеяния. Другая не связана. Обе матрицы необходимы для решения обратной проблемы. Помимо обычных полюсов, соответствующих связанным состояниям, имеются также другие полюса, связанные с полным отражением.

(*) *Переведено редакцией.*