Journal of Russian Laser Research, Volume 32, Number 4, July, 2011

EVALUATION OF ENTANGLEMENT MEASURES IN SPIN SYSTEMS WITH THE RANDOM PHASE APPROXIMATION

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Abstract

We discuss a general formalism based on the mean field plus random phase approximation (RPA) for the evaluation of entanglement measures in the ground state of spin systems. The method provides a tractable scheme for determining the entanglement entropy as well as the negativity of finite subsystems, which becomes analytic in the case of systems with translational invariance, in one or D dimensions. The approach improves as the spin increases, and also as the interaction range or connectivity increases. Illustrative results for different types of entanglement entropies (single site, block and comb) in the ground state of a small spin lattice with ferromagnetic type XY couplings in a transverse field are shown and compared with the exact numerical result. Effects arising from symmetry breaking at the mean field level are also discussed.

Keywords: quantum entanglement, random phase approximation, bosonization.

1. Introduction

The entanglement properties of many-body systems are of great interest in quantum information theory [1] and condensed matter physics [2–4]. On the one hand, they allow one to determine the potential of a given many-body system for various quantum-information-processing tasks that require entanglement as an essential resource [1, 5–10]. On the other hand, they provide a deep understanding of quantum correlations and their relation with criticality [2–4, 11, 12]. Nonetheless, the evaluation of entanglement in strongly interacting many-body systems remains in general a difficult task. Although important advances have been made in recent years, with numerical treatments based on the quantum Monte Carlo method [13], density-matrix renormalization group (DMRG) [14, 15], or matrix products states (MPS) [16,17] providing accurate results in some systems with short-range couplings, systems with long-range interactions, high connectivity, or large dimensionality remain in general a challenge.

The aim of this paper is to discuss a general tractable scheme for approximating the ground state of an interacting spin system based on the mean field plus random phase approximation (RPA) [18, 19].

Manuscript submitted by the authors in English on August 7, 2011.

 1071-2836/11/3204-0322

Previously we have shown that such an approach was able to describe the main features of the pairwise entanglement, i.e., that between two spins, in both the ground state and the thermal state of finite spin systems with XY and XYZ couplings of different ranges in an applied magnetic field [20–22]. The RPA scheme was, in fact, able to predict correctly nontrivial results such as full-range pairwise entanglement in the vicinity of the factorizing field [22]. It was also shown that it improves for high connectivity or increasing coupling range, providing exact results for strong fields. Moreover, the approach can be rigorously derived from the path-integral representation of the partition function based on the Hubbard–Stratonovich transform [20–22]. While RPA assumes, in principle, short amplitude static and quantum fluctuations, the path-integral formalism also allows one to extend easily the approach, so to include large-amplitude static fluctuations (static path + RPA treatment [23–26]), relevant in critical domains of finite systems.

In this paper, we discuss the applicability of the RPA scheme to the problem of evaluating global entanglement measures such as the entanglement entropy [27] in the ground state of finite spin systems [18, 19]. We also show how it can be applied to evaluate approximately, by means of the negativity [28], the entanglement of arbitrary bipartitions of a subsystem [18]. We show that, in fact, the RPA approach can provide, through a bosonic treatment, a tractable scheme for evaluating these quantities, which is able to capture their main basic features and which becomes fully analytic in the case of systems with translational invariance [18, 19].

The main elements of the RPA formalism are discussed in Sec. 2, while Sec. 3 describes the evaluation of the entanglement entropy and negativity by means of the RPA-based bosonic treatment. The application to translationally invariant systems is considered in Sec. 4, while illustrative results for a small XY spin lattice are provided in Sec. 5. This section also describes the basic features of the RPA application to spin systems with XY couplings in an applied transverse field, in both the standard and spin-parity breaking phases. Comparison between exact and RPA results for the single site, block and "comb" entanglement entropies is made. Conclusions are finally drawn in Sec. 6.

2. Spin RPA Formalism

We start by considering a finite system of interacting spins s_i in an external magnetic field described by a general Hamiltonian of the form

$$H = \sum_{i,\mu} B_i^\mu s_{i\mu} - \tfrac{1}{2} \sum_{i\neq j,\,\mu,\nu} J_{ij}^{\mu\nu} s_{i\mu} s_{j\nu}, \tag{1}$$

where $s_{i\mu} = S_{i\mu}/\hbar$, with $\mu = x, y, z$ denoting the dimensionless spin at site i. We note that x, y, z may, in principle, refer to different local intrinsic axes at each site. For D-dimensional arrays, labels i, j stand for D-dimensional vectors.

In the RPA approach [18], the system is essentially mapped to a bosonic system. The spin RPA formalism can be summarized in the following steps: First, the mean-field ground state $|0\rangle = \otimes_{i=1}^n |0_i\rangle$, which is the separable state with the lowest energy $\langle H\rangle_0 = \langle 0|H|0\rangle$, is determined. This state can be obtained self-consistently as the lowest eigenstate of the noninteracting mean field Hamiltonian

$$h = \sum_{i,\mu} \lambda_i^\mu s_{i\mu}, \qquad \lambda_i^\mu = B_i^\mu - \sum_{j,\nu} J_{ij}^{\mu\nu} \langle s_{j\nu}\rangle_0, \tag{2}$$

where $\langle s_{j\nu}\rangle_0 = -s_j \lambda_j^\nu/\lambda_j$ with $\lambda_j = |\vec{\lambda}_j|$.

Choosing now the local z axis in the direction of the local mean field $\vec{\lambda}_i$, such that $\langle s_{j\nu}\rangle_0 = -s_j\delta_{\nu z}$ and $\lambda_i^\mu = \delta^{\mu z}\lambda_i^z$, with $\lambda_i^z = B_i^z + \sum_j J_{ij}^{zz} s_j > 0$, the next step is the approximate bosonization:

$$s_{i+} \to \sqrt{2s_i}\, b_i^\dagger, \qquad s_{i-} \to \sqrt{2s_i}\, b_i, \qquad s_{iz} \to b_i^\dagger b_i - s_i, \tag{3}$$

where $s_{i\pm} = s_{ix} \pm i s_{iy}$ and b_i and $b_i^\dagger$ are boson operators ($[b_i, b_j^\dagger] = \delta_{ij}$, $[b_i, b_j] = 0$). This bosonization coincides with the lowest order of the Holstein–Primakoff bosonization [29–32] and is in agreement with that implied by the path-integral formalism [22] in the $T \to 0$ limit. Neglecting cubic and quartic terms, replacement of Eq. (3) in (1) leads to the quadratic boson Hamiltonian,

$$H^b = \langle H\rangle_0 + \sum_{i,j}\left[(\lambda_i\delta_{ij} - \Delta_{ij}^+) b_i^\dagger b_j - \tfrac{1}{2}(\Delta_{ij}^- b_i^\dagger b_j^\dagger + \text{h.c.})\right], \tag{4}$$

or

$$H^b = \langle H\rangle_0 + \sum_\alpha \left[\omega_\alpha b'^\dagger_\alpha b'_\alpha + \tfrac{1}{2}(\omega_\alpha - \lambda_\alpha)\right], \tag{5}$$

where $\Delta_{ij}^\pm = \frac{1}{2}\sqrt{s_i s_j}[J_{ij}^{xx} \pm J_{ij}^{yy} - i(J_{ij}^{yx} \mp J_{ij}^{xy})]$ and the absence of linear terms in b_i, $b_i^\dagger$ in Eq. (4) is ensured by the mean-field choice of the local z axis. The diagonal form (5) is always feasible provided H^b corresponds to a stable boson system, i.e., provided $|0\rangle$ is a stable mean-field minimum, in which case the ω_α are all real and positive. They are the positive symplectic eigenvalues of the $2n \times 2n$ matrix $\mathcal{H}^b$ representing the bilinear form H^b [18], with b'_α and $b'^\dagger_\alpha$ collective normal boson operators related to the original ones through a Bogoliubov transform [29],

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \mathcal{W}\begin{pmatrix} b' \\ b'^\dagger \end{pmatrix}, \qquad \mathcal{W} = \begin{pmatrix} U & V \\ \bar{V} & \bar{U} \end{pmatrix}, \tag{6}$$

where $UU^\dagger - VV^\dagger = I$ and $UV^{\rm t} + VU^{\rm t} = 0$.

The bosonic RPA ground state is the vacuum $|0'\rangle$ of the operators b'_α, and is given explicitly by [29]

$$|0'_b\rangle = C_b \exp\left[\tfrac{1}{2}\sum_{i,j} Z_{ij} b_i^\dagger b_j^\dagger\right]|0_b\rangle, \qquad Z = V\bar{U}^{-1}, \tag{7}$$

where $C_b = \text{Det}[\bar{U}]^{-1/2}$. We can define the corresponding RPA spin state $|0_{\rm RPA}\rangle$ in a similar way, replacing $|0_b\rangle$ in (7) by the mean-field state $|0\rangle$, and $b_i^\dagger$ by $s_{i+}/\sqrt{2s_i}$ (with $C_b \to C_s$). In contrast to the original mean-field state $|0\rangle$, the RPA ground state is obviously entangled. We will see that it can, in fact, capture the essential aspects of the ground-state entanglement in such systems, at least when there is a well-defined mean-field minimum.

3. Bosonic Evaluation of Entanglement Measures

In the bosonic vacuum (7), the entanglement properties of the ground state can be easily evaluated through the general Gaussian-state formalism [33–38], which we here recast in terms of the nonnegative contraction matrix [18, 29],

$$\mathcal{D} = \left\langle \begin{pmatrix} b \\ b^\dagger \end{pmatrix}\begin{pmatrix} b^\dagger & b \end{pmatrix}\right\rangle_{0'} - \mathcal{M} = \begin{pmatrix} F^+ & F^- \\ \bar{F}^- & I + \bar{F}^+ \end{pmatrix}, \tag{8}$$

$$F_{ij}^+ = \langle b_j^\dagger b_i\rangle_{0'}, \qquad F_{ij}^- = \langle b_j b_i\rangle_{0'} = \langle b_i^\dagger b_j^\dagger\rangle_{0'}^*, \tag{9}$$

where $\mathcal{M}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Since in the vacuum $|0'\rangle$, $\langle b'^{\dagger}_{\alpha}b'_{\alpha'}\rangle_{0'}=\langle b'_{\alpha}b'_{\alpha'}\rangle_{0'}=0$, Eq. (6) leads to

$$F^{+}_{ij}=(VV^{\dagger})_{ij},\qquad F^{-}_{ij}=(VU^{t})_{ij}. \tag{10}$$

The elements of the Hermitian matrix (8) determine, through application of Wick's theorem [29], the average of any many-body operator. In particular, the reduced state $\rho_{\mathcal{A}}=\mathrm{Tr}_{\bar{\mathcal{A}}}|0'\rangle\langle 0'|$ of a subsystem $\mathcal{A}$ of $n_{\mathcal{A}}$ modes ($\bar{\mathcal{A}}$ denoting the complementary subsystem and $\mathrm{Tr}_{\bar{\mathcal{A}}}$ the partial trace) is fully determined by the corresponding submatrix $\mathcal{D}_{\mathcal{A}}$ [Eqs. (8)–(9) with $i,j\in\mathcal{A}$]. It is a thermal-like state of suitable $n_{\mathcal{A}}$ independent boson modes [18]. The entanglement entropy of the $(\mathcal{A},\bar{\mathcal{A}})$ partition,

$$E_{\mathcal{A}\bar{\mathcal{A}}}=S(\rho_{\mathcal{A}})=S(\rho_{\bar{\mathcal{A}}}), \tag{11}$$

is then the bosonic entropy determined by the positive symplectic eigenvalues $f^{\mathcal{A}}_{\alpha}$ of $\mathcal{D}_{\mathcal{A}}$, i.e., the standard eigenvalues of the matrix $\mathcal{D}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ associated to eigenvectors with positive norm in the symplectic metric $\mathcal{M}_{\mathcal{A}}$ (the eigenvalues of $\mathcal{D}_{\mathcal{A}}\mathcal{M}_{A}$ come in pairs $f^{\mathcal{A}}_{\alpha}$ and $-1-f^{\mathcal{A}}_{\alpha}$), which represent local boson average occupation numbers. Equation (11) is then given by

$$S(\rho_{\mathcal{A}})=-\mathrm{Tr}\,\rho_{\mathcal{A}}\log\rho_{\mathcal{A}}=-\sum_{\alpha=1}^{n_{\mathcal{A}}}[f^{\mathcal{A}}_{\alpha}\log f^{\mathcal{A}}_{\alpha}-(1+f^{\mathcal{A}}_{\alpha})\log(1+f^{\mathcal{A}}_{\alpha})]. \tag{12}$$

On the other hand, the entanglement between any two subsystems $\mathcal{A}$ and $\mathcal{B}$ (where the complement $\bar{\mathcal{A}\mathcal{B}}$ plays the role of an environment) can no longer be measured through the entropy of the reduced state $\rho_{\mathcal{A}}$ or $\rho_{\mathcal{B}}$, as $\rho_{\mathcal{A}\mathcal{B}}$ will be, in general, a mixed state. This entanglement can instead be approximately measured through the corresponding negativity [28], an entanglement monotone for mixed states, which is minus the sum of the negative eigenvalues of the partial transpose $\rho^{t_{\mathcal{B}}}_{\mathcal{A}\mathcal{B}}$ of the reduced state $\rho_{\mathcal{A}\mathcal{B}}$. Partial transposition implies corresponding replacements (i.e., $F^{+}_{ij}\leftrightarrow F^{-}_{ij}$ for $i\in\mathcal{A}$, $j\in\mathcal{B}$, etc.) in the contraction matrix for $\mathcal{A}\mathcal{B}$, leading to a matrix $\tilde{\mathcal{D}}_{\mathcal{A}\mathcal{B}}$ with symplectic eigenvalues $\tilde{f}^{\mathcal{A}\mathcal{B}}_{\alpha}$ and $-1-\tilde{f}^{\mathcal{A}\mathcal{B}}_{\alpha}$, where $\tilde{f}^{\mathcal{A}\mathcal{B}}_{\alpha}$ can now be negative (but above $-1/2$ [18]). The final result for the negativity can be expressed in terms of these negative symplectic eigenvalues $\tilde{f}^{\alpha}_{\mathcal{A}\mathcal{B}}$ of $\tilde{\mathcal{D}}_{\mathcal{A}\mathcal{B}}$ as follows [18]:

$$N_{\mathcal{A}\mathcal{B}}=\frac{1}{2}(\mathrm{Tr}\,|\rho^{t_B}_{\mathcal{A}\mathcal{B}}|-1),\qquad \mathrm{Tr}\,|\rho^{t_B}_{\mathcal{A}\mathcal{B}}|=\prod_{\tilde{f}^{\mathcal{A}\mathcal{B}}_{\alpha}<0}\frac{1}{1+2\tilde{f}^{\mathcal{A}\mathcal{B}}_{\alpha}}. \tag{13}$$

It should be remarked, however, that in the case of a degenerate symmetry-breaking mean field, symmetry restoration should be implemented on the RPA-spin state for a proper description of the entanglement entropy and negativity in the ground state, which will have a definite symmetry if nondegenerate. Nevertheless, in the case of a discrete broken symmetry and when the reduced states associated with different mean fields can be regarded as approximately orthogonal, the corrections derived from symmetry restoration can be easily evaluated [18]. They lead, for instance, just to an additive term in the entanglement entropy (12), accounting for the different mean fields (see Sec. 5).

4. Translationally Invariant Systems

These systems are characterized by a uniform magnetic field $B^{\mu}_{i}=B^{\mu}$ and separation-dependent couplings $J^{\mu\nu}_{ij}=J^{\mu\nu}(i-j)$, with $J^{\mu\nu}(-l)=J^{\mu\nu}(n-l)$ for a finite array. Through a discrete Fourier transform, the RPA treatment becomes then fully analytic in any dimension D, providing close expressions for elements (10) of the covariance matrix [18]. The ensuing RPA frequencies become [18]

$$\omega_k = \sqrt{(\lambda - \Delta_k^+)^2 - |\Delta_k^-|^2}\,, \tag{14}$$

where $\Delta_k^\pm = \sum_{l=0}^{n-1} e^{i2\pi kl/n}\Delta_\pm(l)$ are the discrete Fourier transforms of the original couplings $\Delta_{ij}^\pm = \Delta^\pm(i-j)$, and we have assumed, for simplicity, $\Delta^\pm(l) = \Delta^\pm(-l)$ [19]. The elements of the covariance matrix can then be obtained as

$$F_{ij}^\pm = \frac{1}{n}\sum_k e^{-i2\pi k(i-j)/n} f_k^\pm, \tag{15}$$

$$f_k^+ = \langle b_k^\dagger b_k\rangle_{0'} = \frac{\lambda - \Delta_k^+}{2\omega_k} - \frac{1}{2}\,, \qquad f_k^- = \langle b_k b_{-k}\rangle_{0'} = \frac{\Delta_k^-}{2\omega_k}\,, \tag{16}$$

where $b_k = \frac{1}{\sqrt{n}}\sum_j e^{i2\pi kj/n} b_j$ are the discrete Fourier transforms of the local bosons.

For instance, the single-site entropy, determining the entanglement of a single site i with the rest of the chain, is just

$$S(\rho_i) = -f\log f + (1+f)\log(1+f), \tag{17}$$

$$f = \sqrt{(\tfrac{1}{2} + F_{00}^+)^2 - |F_{00}^-|^2} - \tfrac{1}{2}\,, \tag{18}$$

where $F_{00}^\pm = \frac{1}{n}\sum_k f_k^\pm$ [Eq. (15)]. The local boson-occupation number f is the positive symplectic eigenvalue of the single site 2×2 covariance matrix, and represents, of course, the deviation from minimum uncertainty of the local mode: $(\frac{1}{2}+F_{00}^+)^2 - |F_{00}^-|^2 = \langle q_i^2\rangle_{0'}\langle p_i^2\rangle_{0'} - \langle q_i p_i + p_i q_i\rangle_{0'}^2/4$, where $q_i = (b_i + b_i^\dagger)/\sqrt{2}$ and $p_i = (b_i - b_i^\dagger)/\sqrt{2}i$ are the coordinate and momentum associated with the local boson operators b_i and $b_i^\dagger$, respectively.

5. Applications

As an illustration, we consider a ferromagnetic-type XY-spin s array in a uniform transverse field B, such that

$$H = B\sum_i s_{iz} - \frac{1}{2s}\sum_{i\neq j,\mu=x,y} J^\mu(i-j)s_{i\mu}s_{j\mu}, \tag{19}$$

with $|J_y(l)| \leq J_x(l)$ and $J^\mu(-l) = J^\mu(n-l)$. The s^{-1} scaling chosen here ensures a spin-independent mean field and RPA Hamiltonian, leading to a finite limit for high spin. This Hamiltonian commutes with the spin parity $P_z = e^{i\pi\sum_j(s_{jz}+s)}$, a symmetry which will be broken at the mean-field level for low fields $|B| < B_c$. Accordingly, two RPA phases arise, namely, normal RPA and parity-breaking RPA.

5.1. Normal RPA

For sufficiently strong fields B, the lowest mean-field state will be the aligned state $|0\rangle = |0_1 \ldots 0_n\rangle$, where $|0_i\rangle$ denotes the local state with maximum spin along the $-z$ axis ($s_{iz}|0_i\rangle = -s|0_i\rangle$). It is easy to show that, in the present situation, such a state will be the lowest separable state for

$$|B| > B_c = J_0^x = \sum_l J^x(l)\,. \tag{20}$$

At this phase, the RPA frequencies become

$$\omega_k = \sqrt{(\lambda - J_k^x)(\lambda - J_k^y)}, \tag{21}$$

where $\lambda = B$ and $J_k^\mu = \sum_l e^{i2\pi lk/n} J^\mu(l)$, with Eq. (20) ensuring ω_k real $\forall\, k$. It can be shown [18] that, at this phase, the RPA is exact for strong fields $|B| \gg B_c$, with the RPA-spin state coinciding with the exact first-order perturbative expansion of the ground state.

5.2. Parity Breaking RPA

For $|B| < B_c$, the normal state becomes unstable, and the lowest mean field corresponds to a parity breaking state $|\Theta\rangle = |\theta_1 \dots \theta_n\rangle$, with $|\theta_i\rangle = \exp[-i\theta s_{iy}]|0_i\rangle$ and

$$\cos\theta = |B|/B_c, \tag{22}$$

where all spins are aligned along the axis forming an angle θ with the z axis in the xz plane. Such mean-field state is obviously degenerate, as $|-\Theta\rangle = P_z|\Theta\rangle$ is also a separable state with the same energy.

At this phase, we may, in principle, apply the same previous expressions with the substitutions $\lambda \to J_0^x$ and $J_k^x \to J_k^x \cos^2\theta$ [18]. Nonetheless, important corrections due to parity breaking do arise in finite systems, which are to be taken into account for reproducing the entanglement properties of the exact definite-parity ground states. Notice that, at this phase, the ground state of a finite chain becomes almost degenerate but not exactly degenerate, preserving a definite parity P_z. It actually develops parity transitions [21], the last one at the factorizing field $B_s = J_0^x\sqrt{\chi}$ [39,40] when there is a common anisotropy $\chi = J^y(l)/J^x(l)\ \forall\, l$. Let us mention that at this field, the system exhibits a degenerate separable ground state, which is a combination of entangled definite parity states [39]. The latter provide the actual side limits at B_s and imply, therefore, nonzero-side limits of the entanglement entropy at B_s [39, 41].

The definite-parity spin-RPA states $(|0_{\rm RPA}(\theta)\rangle \pm |0_{\rm RPA}(-\theta)\rangle)/\sqrt{2}$ lead essentially to reduced states $\rho_{\mathcal{A}} \approx \frac{1}{2}(\rho_{\mathcal{A}}(\theta) + \rho_{\mathcal{A}}(-\theta))$, if the complementary overlap can be neglected. If the overlap between $\rho_{\mathcal{A}}(\theta)$ and $\rho_{\mathcal{A}}(-\theta)$ is also negligible (which is valid for a not too small subsystem [18]), the final effect in the entanglement entropy is just a constant shift:

$$S(\rho_{\mathcal{A}}) \approx S(\rho_{\mathcal{A}}(\theta)) + \delta, \tag{23}$$

where $S(\rho_{\mathcal{A}}(\theta))$ denotes the bosonic RPA entropy (12) for the parity-breaking RPA state and $\delta = \log 2$. The negativity changes as well to $N_{\mathcal{AB}} \approx 2N_{\mathcal{AB}}(\theta) + 1/2$. If the overlap cannot be neglected, the final effect is essentially a lower and θ-dependent value of the shift δ. Let us also note that, in the immediate vicinity of the factorizing field B_s, $S(\rho_{\mathcal{A}}(\theta))$ vanishes, and just the shift δ remains [18].

The RPA predictions for the entanglement measures become accurate in the high-spin limit as well as for systems with long-range interactions, such as a fully connected array [18]. Nonetheless, even in the opposite case of low spin and short range couplings, it provides a reasonable basic estimation of the main entanglement properties [19]. As illustration, we show in Figs. 1 and 2 results for the first neighbor XY couplings in small two-dimensional arrays, where the exact entanglement entropies were computed through full diagonalization. Notice that for D-dimensional arrays all previous expressions remain valid replacing i, k, n with vectors $\mathbf{i} = (i_1, \dots, i_D)$ and $\mathbf{k}$ and $\mathbf{n}$, with $e^{i2\pi kl/n} \to e^{i2\pi\sum_{j=1}^D k_j l_j/n_j}$. In Figs. 1 and 2, we have employed spatially isotropic first neighbor cyclic couplings, such that $J^\mu(\mathbf{l}) = J^\mu \delta_{|\mathbf{l}|,1}$.

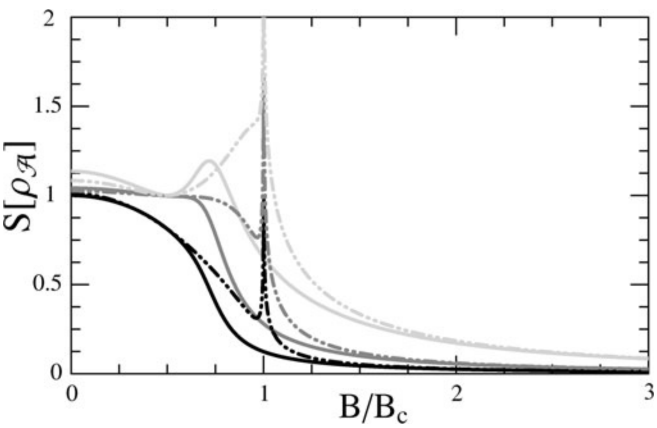

Fig. 1. Entanglement entropies of different partitions in a 4×4 spin-1/2 lattice as a function of the transverse applied field, for first neighbor XY couplings with anisotropy $J^y/J^x = 1/4$. The solid lines correspond to the exact results, whereas the dotted lines, to the RPA estimation. The black line denotes the entanglement of a single site with the rest of the chain, the dark gray line the entanglement of a 2×2 block with the rest, and light gray line shows the entanglement between all even sites ($(-1)^{i_x+i_y} = +1$) with all remaining odd sites ($(-1)^{i_x+i_y} = -1$), which is the maximum entangled bipartition. The RPA results become exact for strong fields $|B| \gg B_c$, as well as in the vicinity of the factorizing field $B_s = B_c/2$ (see text).

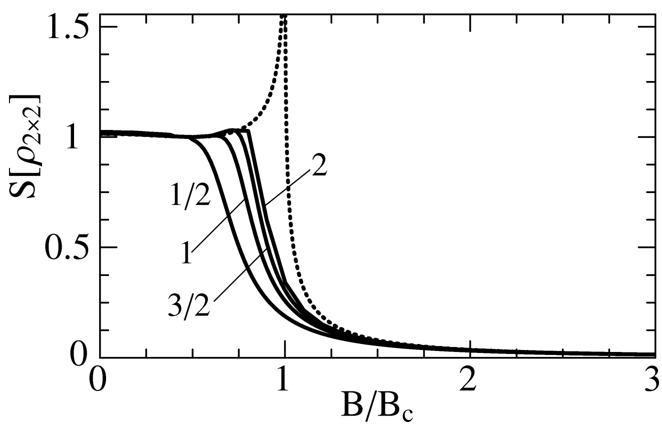

Fig. 2. Exact entanglement entropy for a block of 2×2 spins in a 4×2 lattice of spin $s = 1/2, 1, 3/2$ and 2 (shown by figures near the curves) as a function of the transverse applied field, together with the RPA result (shown by dotted line), for the same anisotropy $J^y/J^x = 1/4$. The exact entropies approach the RPA result as the spin increases.

As seen in Fig. 1, the RPA predictions for a small spin-1/2 lattice turn out to be quite satisfactory, except for the vicinity of the critical region, where the bosonic RPA results diverge while the exact results for the finite lattice remain obviously finite. The RPA results for a small system in a critical region can actually be much improved, if the RPA-spin state (rather than the bosonic treatment) is directly employed for evaluating the entanglement entropy [18]. We show in Fig. 1 three distinct types of entanglement entropies: (i) that of a single site (measuring the entanglement between the site and the rest of the chain); (ii) that of a 2×2 contiguous block; (iii) that of the noncontiguous even half ("comb" entropy [19, 42]), where the even subsystem is here defined by $(-1)^{i_x+i_y} = +1$, labeling the sites as (i_x, i_y) [19]. This is the maximum-entangled bipartition for the present first neighbor coupling, since it will break all coupling links. The RPA results for these three entropies are seen to be quite in agreement with the exact results, away from the critical domain. The bosonic RPA scheme becomes exact for strong fields $|B| \gg B_c$ for all these entropies, for any size n, spin s, geometry, or interaction range. Moreover, the RPA results improve as the spin s increases, as shown in Fig. 2 for the entropy of a 2×2 block, representing the high-spin limit.

We have used in Figs. 1 and 2 base 2 logarithm, such that the additive correction δ, Eq. (23), arising from parity restoration for $|B| < B_c$ is essentially +1 for the block and comb entropies. As seen in Figs. 1 and 2, such shift is in full agreement with the corresponding exact results for the definite-parity ground state, obtained from direct diagonalization. For the single-site entropy, the overlap between $\rho_i(\theta)$ and $\rho_i(-\theta)$ can no longer be neglected, which leads essentially to a lower θ-dependent value of δ as stated above. The ensuing RPA results remain again quite accurate below the critical domain. It should also be remarked that the RPA leads to the correct size dependence of the block and comb entropies, leading to correct area laws [4] for these quantities in the strong-field limit (see [19] for more details). It leads as well to correct results for the negativity of arbitrary subsystems [18].

Let us finally mention that, in the case of fully and uniformly connected XYZ spin arrays of arbitrary size and spin in a transverse field, the present bosonic RPA formalism is able to provide analytic final expressions for the entanglement entropy, as well as for the negativity of any partition of any subsystem [18], which becomes exact in the large spin limit at fixed size and also in the large size limit at fixed spin.

6. Conclusions

We have discussed a general mean field plus RPA formalism for approximating the ground state of many-body systems. We have shown that by means of an approximate bosonization, a straightforward estimation of the entanglement measures like the entanglement entropy and the negativity in the ground state of finite-spin systems is allowed. Moreover, the approach becomes fully analytic in systems with translational invariance. The formalism is able to capture the main features of these quantities in the exact ground state of finite ferromagnetic-type XY-spin arrays away from the critical domain, if basic symmetry-restoration effects, required for $|B| < B_c$, are taken into account. In small-finite systems, the predictions in the critical domain can, in fact, be also improved within the RPA scheme by extracting quantities directly from the RPA-spin state. When the system exhibits a well-defined minimum at the mean-field level, the approach improves for high spin, large connectivity, or long-range interactions, making it a useful technique that can complement other numerical methods aimed at short-range couplings or more specific problems. Several applications of this method, as well as its extension to finite temperatures, are currently under investigation.

Acknowledgments

N.C. and J.M.M. acknowledge support of Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) of Argentina and R.R. of the Comisión de Investigaciones Científicas (CIC) of Argentina.

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