ON THE UNMODELED PERTURBATIONS IN THE MOTION OF URANUS

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Abstract. In this paper we apply a numerical method to determine unmodeled perturbations in an attempt to explain the observed discrepancies in the motion of Uranus. We find that the estimated perturbation shows some significant periods that could be attributed to insufficient knowledge of the perturbations from some of the known planets. On the assumption that the gravitational attraction of an unknown planet is the origin of the deviations, the best planar solution of the inverse problem is a planet of 0.6 Earth masses, with true longitude of 133° (1990.5), semi major axis $a = 44$ AU and eccentricity $e = 0.007$.

Key words: Uranus, planetary perturbations.

1. Introduction

The source of the observed systematic deviations between the theory of motion of Uranus and its observations remains an unsolved problem in the outer Solar System.

In two recent works by Gomes and Ferraz-Mello (1987 and 1988a) the orbital parameters of Uranus and Neptune have been improved in order to eliminate (or, at least, to reduce) the systematic trend of the residuals. The authors have compared the observations with Bretagnon's VSOP82 theory (Bretagnon, 1982) and, after the improvement was made, some significant periods, close to the synodic period of Uranus and Neptune, were still present in the residuals. The authors conclude that these deviations from pure white noise could be attributed either to a poor determination of perturbations from some known planet or to a presence of a still unknown perturbing planet.

This last hypothesis is also supported by the presence of some problems in the orbits of a family of six periodic comets which have had more than one apparition near the Sun (Guliev, 1986). The orbits of these comets seem to be perturbed by an unmodeled force perpendicular to their orbital plane. A few years ago, Raup and Sepkosky (1984) suggested that biological extinction events on the Earth could be a consequence of cometary showers from the inner Oort Cloud of comets and, later, Matese and Withmire (1986) pointed out that these cometary showers could be produced by a trans-Neptunian Tenth planet circulating in an unusual, highly inclined orbit.

Although a variety of factors can contribute to the presence of such irregularities, a planet beyond Pluto has been the object of much attention in recent years. In fact, some astronomers have tried to find it, without success (Croswell, 1988).

Full details of the observational facts that support the possible existence of *planet X* can be read in Seidelmann and Harrington (1988) and Seidelmann and Williams (1988).

There are different ways to predict a hypothetical tenth planet from its perturbations on Uranus and Neptune. Nevertheless, some *a priori* assumptions must be made in order to reduce the free parameters involved in the problem (e.g. small eccentricity and inclination). Indeed, an exhaustive search of the optimum solution, leaving free all the parameters, would be impracticable from the computational point of view. Moreover, it is difficult simultaneously to make an accurate prediction of position and motion (Seidelmann and Harrington, 1988).

We mention here the early work by Rawlins and Hammerton (1973), and more recently Harrington (1988), Powell (1988), Gomes and Ferraz-Mello (1988b) and Gomes (1989). The best solution seems to be a planet with about one-half of the Earth's mass, at 45 AU from the Sun, presently in Cancer. However, there is a great number of possible solutions that improves the residuals of Uranus and Neptune (Gomes and Ferraz-Mello 1988b).

It is interesting to note that, in the case of the discovery of Neptune, although the elements computed by Le Verrier were accurate enough to accomplish their purpose, when the true orbital elements were avaiable, it was found that the predicted orbit was not correct at all (Tisserand, 1888; Brookes, 1970). Neptune was discovered within 55 arcminutes of the predicted position because the error in the adopted semimajor axis was partially compensated by errors in the eccentricity and mass, and the arc of orbit covered by the planet the years before the discovery was not too affected. In fact, the position of Neptune could have been predicted using any value of a between 30 AU and 40 AU, at any time after 1830 (Brookes, 1970).

To predict Pluto, an exactly similar procedure was followed (Lowell, 1915). Again, the predicted orbit and mass of Pluto were wrong, although it was detected by Clyde Tombaugh, within 6 degrees of one of the positions predicted by Lowell.

At this point, it might be interesting to see whether a method to estimate unmodeled perturbations can be used to investigate the nature of the observed residuals on Uranus and Neptune. This is of interest since the only *a priori* assumption in this case is that the residuals features have a dynamical origin. So, if we can compute numerical values of the perturbation, it will be possible to conceive some hypotheses about its nature. The final step is to establish an *inverse problem*, and try to determine the physical parameters in order to give a complete characterization of this unknown perturbation.

The main goal of the present paper, then, is to analyze whether this methodology can contribute to the explanation of the nature of the observed discrepancies in the outer Solar System. We will center our attention in the residuals of Uranus because this planet is a good detector of possible sources of perturbation (Gomes 1989). The effects on Neptune are very difficult to differentiate from errors in its orbital constants.

In Section 2 we summarize the method used to estimate the unmodeled perturbations. In Section 3, using 260 post-discovery observations of Uranus, we will describe the numerical results of the application of the method to the 'discovery of Neptune' where the solution is known; thus, our results may be tested. Finally, in Section 4, we will apply the same procedure to determine unmodeled perturbations on Uranus, covering the period 1781–1980.

The final section is devoted to some general remarks.

2. Estimation Algorithm

Recently, a very simple numerical method to estimate unmodeled forces was presented by Zadunaisky (1983) and was successfully used in problems related with planetary motion and with other dynamical problems (Zadunaisky and Sanchez Peña, 1988). An improved version of this method was later presented by Rodriguez and Zadunaisky (1986). In Zadunaisky's method, the only assumption is that the forces involved in the problem (including the unmodeled perturbations) can be developed in Taylor series for a short time span. In addition, Rodriguez (1988) developed a complete error analysis of the method. However, some improvements to Zadunaisky's methods must be made in order to apply it to real observations. This is the purpose of this section.

Although this method is applicable to any system of ordinary differential equations, for the sake of simplicity we will consider a one-dimensional initial-value problem (IVP) of the form:

$$y'(x) = f(x, y(x)) + P(x) \ldots y(x_0) = y_0 \,, \tag{2.1}$$

where $P(x)$ is an unknown perturbation depending generically on the independent variable x. We wish to compute this perturbation numerically in order to obtain a solution $y(x)$ of the IVP (2.1) which approximates a given data set.

If we consider that the available data are measurements y_n of the solution values $y(x_n)$, $(x_n = x_0 + n\,h \;\; n = 0, 1, \ldots, N)$ affected by small random errors ε_n, i.e.

$$y_n = y(x_n) + \varepsilon_n \,, \tag{2.2}$$

then Zadunaisky's method can be applied (see Rodriguez, 1988). However, this is not the real situation in practice, since, as we have pointed out, the measures are commonly nonlinear functions of the y-values. In order to show how we can apply the method in this case, let us consider indirect measures of the form

$$\alpha_n = \alpha(y(x_n)) + \delta_n \tag{2.3}$$

where δ_n are measurement errors.

Let us define, in the same way as Zadunaisky, a '*reference problem*' as:

$$z'(x) = f(x, z(x)) + P_0 \,, \tag{2.4}$$

obtained from Equation (2.1) by replacing the unknown perturbation $P(x)$ by a constant value P_0.

To obtain estimates of $P(x)$ at the point x_0, we must adjust an initial value $z(x_0)$ and a constant P_0 so that the solution of other point $x_j(x_j = x_0 + jh)$. In Section 2.1 we will show that the P_0 found in this way is an estimate of $P(x_0)$ and that this estimation may be combined with estimates from other points to obtain improved values of $P(x_0)$.

To carry out the indicated procedure numerically, we have used an adaptation of the so-called '*simple shooting method*' (Isaakson and Keller, 1966), as follows:

Let us consider the IVP

$$z'(x) = f(x, z(x)) + P_0 \dots z(x_0) = z_0 \ . \tag{2.5}$$

The solution of this IVP is a function of the numerical values of z_0 and P_0. Let us then write this solution in the form:

$$z(x) = z(z_0, P_0, x) \ . \tag{2.6}$$

The difference between measured values and computed ones may be written as some function of z_0 and P_0. Let us write R for this function, i.e.,

$$\begin{aligned} a_0 - \alpha(z(x_0)) &= R_0(z_0, P_0) \\ a_j - \alpha(z(x_j)) &= R_j(z_0, P_0) \ . \end{aligned} \tag{2.7}$$

Thus, the problem is reduced to the determination of the roots of (2.7), which can be found by applying some iterative procedure. Of course, in each step of such iteration scheme, at least one evaluation of $z(x_j)$ is required for each numerical value of z_0 and P_0. This may be done only approximately by numerical integration of (2.5).

Let us consider now the more general case, in which we have a system of N ODE, M of them perturbed $(M \leq N)$. In addition, let us suppose that we have, at each time, K independent measurements $(K \geq M + N)$; then, to estimate $P(x_0)$, we must solve a nonlinear system of the same form as system (2.7) with at least $M + N$ equations. If we form more than $(M + N)$ equations, we can solve the nonlinear system by the least- squares method. In as much as we have estimated the perturbation in one set of measures, we are able to repeat the process for another set.

2.1. Error Bounds

The conditions that must be fulfilled by the functions of the problem, in order to prove the existence and uniqueness of the solution of the system (2.7), as well as a complete numerical analysis of the estimation, lie outside the scope of the present paper. However, to complete this presentation, we wish to show the main results concerning the error bounds of the estimation. Here, it is sufficient to assume that P and f are regular functions; we particularly suppose that the function α has at least continuous first derivatives.

Whenever the reference problem is integrated numerically, computational errors arise from the determination of $z(x)$. However, these errors may be made small enough to be negligible just by solving the problem with a suitable numerical method.

In this situation, it may easily be proved that there are some values of z_0 and P_0 which satisfy the nonlinear system (2.7).

Under the imposed conditions, the solution of Equation (2.1) may be developed in Taylor series:

$$y(x_j) = y(x_0) + [f_{y_0} + P(x_0)]\, jh +$$
$$+ \Big[\frac{\partial f_{y_0}}{\partial y}\,(f_{y_0} + P(x_0)) + \frac{\partial f_{y_0}}{\partial x} + P'(x_0)\Big]\,(jh)^2/2 + \ldots \tag{2.8}$$

where $jh = (x_j - x_0)$, and the subscript y_0 means that the functions are evaluated at $y(x_0)$.

The same development can be performed for the solution of the '*reference problem*', leading to:

$$z(x_j) = z(x_0) + [f_{z_0} + P_0]\, jh + \Big[\frac{\partial f_{z_0}}{\partial z}\,(f_{z_0} + P_0) + \frac{\partial f_{z_0}}{\partial x}\Big]\,(jh)^2/2 + \ldots\,. \tag{2.9}$$

Subtracting (2.9) from (2.8), we may write:

$$y_j - z_j = y_0 - z_0 + (\Delta f_0 + \Delta P_0)\, h_j + \Delta\left(f\,\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\right)_0 (jh)^2/2 +$$
$$+ P'(x_0)\,(jh)^2/2 + \left(\frac{\partial f_{y_0}}{\partial y}\,P(x_0) - \frac{\partial f_{z_0}}{\partial y}\,P_0\right)\,(jh)^2/2 + \ldots \tag{2.10}$$

where Δ means the difference of the functions evaluated at $y(x_0)$ and $z(x_0)$, and $\Delta P_0 = P_0(x) - P_0$. In the last bracket, we may write:

$$\frac{\partial f_{y_0}}{\partial y} = \frac{\partial f_{z_0}}{\partial y} + \frac{\partial^2 f_{z_0}}{\partial y^2}\,(y_0 - z_0) + \ldots\,. \tag{2.11}$$

With the aid of similar developments for the terms involving differences of f and its derivatives, we may write:

$$y_j - z_j = (y_0 - z_0)\left(1 + L\,hj + (M + N\,P(x_0))\,(hj)^2/2\right) +$$
$$+ P'(x_0)\,(jh)^2/2 + \Delta P_0(hj)\,(1 + K\,hj/2) + \ldots \tag{2.12}$$

where

$$L = \frac{\partial f_{y0}}{\partial y} \; ; \; M = \frac{\partial}{\partial y}\left(f\,\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\right)_{y0} \; ; \; N = \frac{\partial^2 f_{z0}}{\partial y^2} \text{ and } K = \frac{\partial f_{z0}}{\partial y} \, .$$

Considering the relations (2.3), we may write, to first order:

$$\alpha(z_j) = \alpha(y_j) + \frac{\partial \alpha_j}{\partial y}\,(z_j - y_j) \tag{2.13}$$

and by virtue of (2.3), we have:

$$y_j - z_j = -\delta_j\,\frac{1}{\partial \alpha_j/\partial y} \, . \tag{2.14}$$

In this way, we have proved that the principal term in the asymptotic error expansion is:

$$P(x_0) - P_0 = -P'(x_0)\,jh/2 - (\Delta_j - \Delta_0)/jh \, . \tag{2.15}$$

where

$$\Delta_j = \delta_j\,\frac{1}{\partial \alpha_j/\partial y} \, .$$

In addition, we can compute the optimum step size, for which the error bound reaches the minimum, as:

$$(hj)_{op} = [2\,|\Delta_j - \Delta_0| \,/\, |P_0'|]^{1/2} \, . \tag{2.16}$$

Equation (2.15) shows that the error in the estimation of $P(x_0)$ has two principal components. However, the computational evidence shows that the principal source of error is not the truncation error coming from the representation of $P(x)$ by a constant value over the whole interval, but the measurement error, which is magnified by a factor $1/h$.

In order to minimize the effects of the measurement errors, let us suppose that we have two independent estimates of $P(x_0)$ (i.e. for $j = 1$ and $j = -1$);

$$\bullet\; x_{j=-1} \longleftarrow \bullet\; x_{j=0} \longrightarrow \bullet\; x_{j=1} \, .$$

The average $\hat{P}_0$ of both estimations has an error

$$\hat{P}_0 - P(x_0) = O(h^2) + (\Delta_j - \Delta_{-j})/2h \, . \tag{2.17}$$

This scheme of estimation (hereafter called '*three point scheme*') has a measurement error roughly one half of that of a single estimate, as is usual in this class of centered numerical schemes.

It is interesting to note that, by means of suitable linear combinations of estimate from different points, improvements in the order of magnitude of the truncation error can be obtained (Rodriguez, 1988). This shows that it is unnecessary to consider higher–order representations for the perturbation in the '*reference problem*'.

TABLE I

Accuracy in the estimation of $P(t)$.

year	$\lvert P_{true} \rvert$	$\lvert P_{est} \rvert$
1786	0.83×10^{-11}	0.69×10^{-11}
1822	10.62 "	10.67 "

3. An Application: The Discovery of Neptune

In this section we will present an application of the method to an old and classical problem of celestial mechanics whose solution is known: the search for Neptune from the disturbances that it causes on Uranus.

To do this we have used a collection of 260 observed apparent geocentric coordinates of Uranus, in the J2000 system, between the years 1781 and 1845. Pre-discovery observations were discarded.

At each instant, nine unknowns must be determined: six initial conditions and three components of the perturbation. So, as we have two independent measures at each instant (right ascension and declination), to apply our proposed three-point scheme we must use, for each estimation, five forward instants and five backward ones (we need at least nine independent measures for each forward estimation and another nine observations for each backward one). We have preferred to perform the estimation in the least-square sense, computing one averaged observation with all the observations of each opposition, instead of computing $P(t)$ and initial conditions fitting individual observations. In this way we may compute a *backward estimation and a forward one* for every mean date of opposition, and with both estimations we may apply the *three-point scheme* to finally obtain one average perturbation.

Previous simulations suggest that the best step size lies between 500 and 2000 days. So, for our estimation, we have used a step size equal to (or greater than) three consecutive oppositions of Uranus. It is worth noting that even when observations have unequal weights, this numerical method can still work.

To carry out the iterative process, one must determine starting conditions that guarantee convergence. As perturbations are small quantities, we have adopted zero as the initial value. Initial conditions for Uranus were taken from VSOP82. Starting with these values we have obtained convergence in all the points. The results are summarized in Table I, where we give the best estimation and the worst one on the whole interval.

These results look rather good. In all cases we have estimated at least one significant digit of the perturbation. This probably is due to the fact that the relative motion of the planets is slow, so the truncation error, which depends on $P'(t)$, is small in comparison to the component arising from the measurement errors.

The next step will be to make some hypothesis about the origin of the pertur-

TABLE II
Coordinates of Neptune.

	r_{calc}	r_{obs}	α_{calc}	α_{obs}	δ_{calc}	δ_{obs}
1786	36.27	30.32	200°.5	197°.4	−6°.7	−5°.6
1822	33.00	30.22	275°.8	275°.9	−22°.1	−22°.4

TABLE III
Orbital elements for the year 1800.

	α	e	i	ω	Ω
true	30.03	0.01	22°.3	45°.25	3°.47
estimated	33.19	0.04	22°.7	336°.24	3°.71
Le Verrier	36.15	0.11	—	284°.10*	—

* Not in J2000 system

bation. In this case, from the mathematical model of the three body problem, we may write an inverse problem of the form:

$$P_u = -G\,m_N \left[\frac{r_u - r_N}{\|r_u - r_n\|^3} - \frac{r_N}{\|r_n\|^3}\right] . \tag{3.1}$$

At each instant we may write a nonlinear system of three equations for the position vector and the mass of the unknown planet. A numerical value for the mass of the unknown planet must be adopted in order to make the system determinate. Following the same criterion of Le Verrier, we have adopted for Neptune a mass which is double the true one.

It is interesting to note that Equation (2.15) allows us to see that the direct problem of determining the perturbations by our proposed numerical method is not an ill-posed problem, in the sense that small perturbations in the data cannot introduce changes of great amplitude in the estimated perturbations. However, the practical inversion of (3.1) is ill-posed, and we have found that the iterative process sometimes oscillated around a local minimum far from the solution of the nonlinear system.

The results are summarized in Table II, where we give the best and the worst computed positions of Neptune.

Finally, we have calculated orbital elements from these coordinates. In Figure 1 we can see our computed orbit, the true one and the orbit predicted by the Le Verrier.

In Table III we compare the orbital elements of these three orbits.

The computed values show a good agreement with the true ones. The relatively small differences show that the position of the planet is predicted with the proposed procedure.

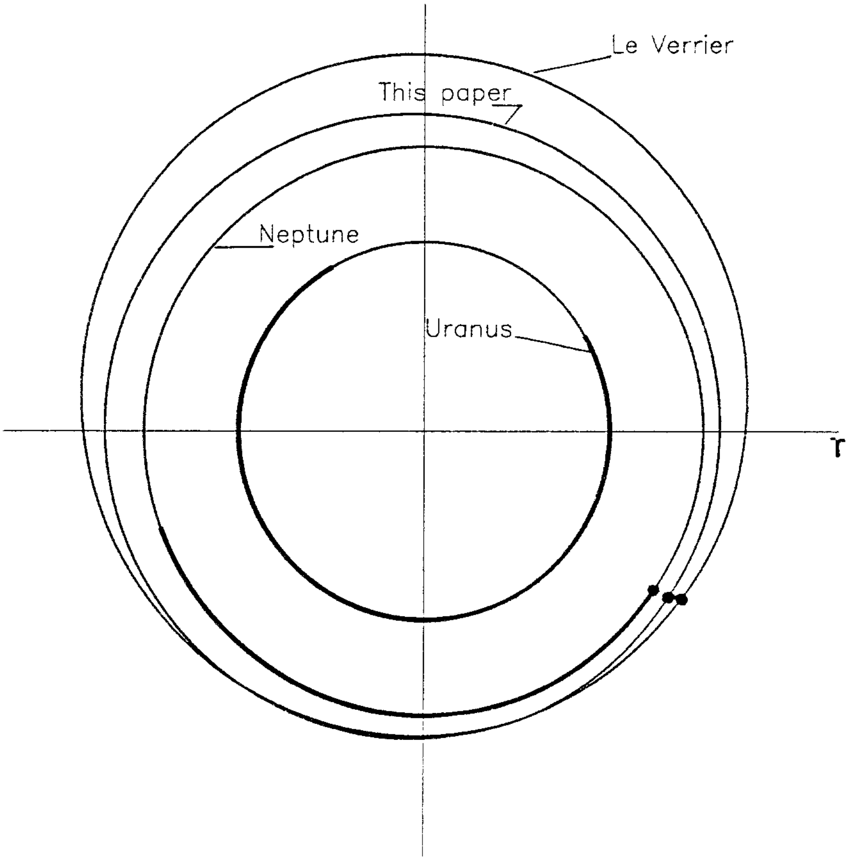

Fig. 1. Representation of the predicted orbit of Neptune by Le Verrier, the computed orbit in the present paper and the true one. The dots represent the location of the planets at the moment of the discovery. The thick lines represent the arcs covered by the planets Uranus and Neptune in the period 1781–1846.

4. Application to the Present Anomalies in the Residuals of Uranus

In this section we consider the observed discrepancies in the motion of Uranus as having a dynamic origin, using the same methodology used in the preceding section.

The data used are a collection of apparent geocentric longitudes of Uranus in the J2000 system, covering the period from the discovery up to 1980, which were provided by the U.S. Naval Observatory. The observations made before the discovery were not included in this analysis because of their relatively low precision.

The situation is certainly more complex here than in the case of the discovery of Neptune: the present observed residuals of Uranus are nearly 10 times smaller

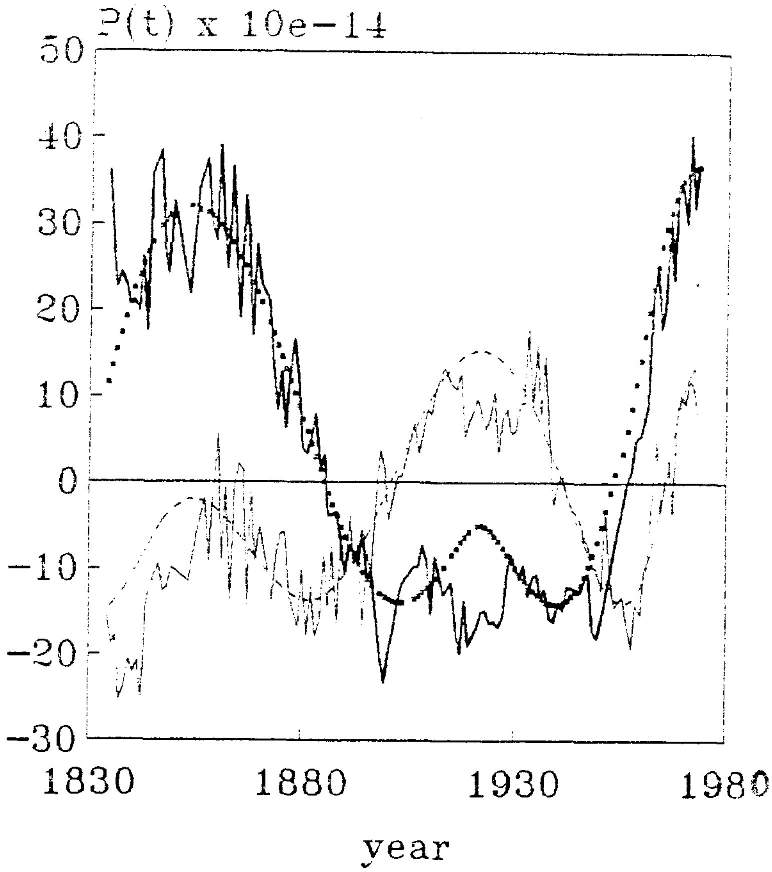

Fig. 2. Computed perturbations on Uranus in AU days^{-2} Radial (thin) and Tangential (thick) compared with the perturbation introduced by a planet of 0.6 Earth masses at 44 AU (dotted and dashed lines).

than those of Uranus before the discovery of Neptune. So the choice of the step size of estimation deserves some care. We have carried out several estimations using different step sizes. For step sizes lower than 3 and greater than 5 consecutive oppositions, the estimated perturbations were not at all distinguishable from white noise. This is due to the relative magnitude of observational errors and truncation ones (see Equation (2.13)). Using a step size of 4 consecutive oppositions the estimated perturbations have shown a clear systematic behavior, particularly in the period 1835–1975. We have plotted the tangential and radial components of the estimated perturbation in Figure 2 (solid lines). The estimated values before 1834 are rather wrong, due perhaps to the low precision of the observations of this period, as well as their low density (there is a significant gap of good observations prior to this year). Both the radial and tangential components show some significant periods. We have not found any peculiar feature in the *out-of-plane component.*

As the identification of the main periods can help our knowledge about the nature of the perturbation, we have performed a least-squares fit of a sinusoidal wave on the estimated values (Ferraz-Mello, 1980) and we have found two main associated periods: one of 152 y (reliability $\geq$ 98%) not very far from the Uranus–Neptune synodic period, and another period of 51 y (reliability $\geq$ 87%) that is close to the synodic period between Saturn and Uranus. The presence of these periods in the perturbations suggests that the best explanation of the observational facts could be a poor determination of the perturbations from the known planets, as was suggested by Gomes and Ferraz-Mello (1988).

The advantage of this indirect methodology in this class of problems is clear: several alternative models are actually proposed to explain the features in the residuals of Uranus. After the estimation process, we can select the physical model which best represents the computed perturbations.

If we assume the *planet X* hypothesis as the source of the perturbation, an *inverse problem*, such as those of Section 3 may be written. It is worth remarking that it is an ill-posed problem, but, in fact, this is a common characteristic of problems of inversion in general (Craig and Brown, 1986).

As before, we have one solution for each adopted numerical value of the mass of the unknown planet. It is clear, that if we adopt a wrong value of the mass of the unknown planet it will have a greater effect on the predicted distance than the direction, so, still adopting a wrong numerical value for the mass, we can expect good information about the region where the planet could be.

We have solved the *inverse problem* for 20 different numerical values of the *planet X's* mass, between 0.1 to 2.0 Earth masses. We have limited our search to this range of numerical values because the out-of-plane component of the perturbation must be small. So our hypothetical planet would be near the orbital plane of Uranus; a region of the sky well examined by C. Tombaugh. A planet greater than 2.0 Earth's masses would have been detected by him.

In Table IV we show some solutions of the inverse problem, as well as some numbers related with the goodness of fit: m = adopted mass value, a = semimajor axis, e = eccentricity, $\bar{\omega}$ = argument of the perihelion, ϕ = true longitude for 1990.5, σ = standard deviation of the residuals, κ = condition number of the nonlinear system, and N = a number related to the trend of the residuals (see explanation below).

It is interesting to note that all the solutions almost equally fit the computed perturbations. In fact, the slow variation of the standard deviation σ means that the prediction of an orbit from a relatively short arc is an ill-posed problem. As the variance-covariance matrix of the estimate gives an intuitive feeling of the degree to which the various parameters are well determined, we show its eigenvalues (for the case $m = 0.6$) which define the axis of the ellipsoid of uncertainty:

TABLE IV

m	a	e	$\tilde{\omega}$	ϕ	$\sigma(10^{-14})$	$\kappa(10^7)$	N
0.2	15.6	0.22	90	110	8.20	9.2	0.81
0.4	48.1	0.1	76	130	7.02	5.9	0.76
0.6	44.0	0.007	180	133	6.65	4.9	0.94
0.8	52.3	0.03	10	100	7.18	6.9	0.81
1.0	55	0.00	0	98	9.81	10.1	0.73
2.0	76	0.25	90	81	12.61	107	0.61

a	e	$\tilde{\omega}$	$\phi(1990.5)$
0.3	25.2	500.0	0.007

The most ill-determined parameter is the true longitude ϕ for 1990.5. This reflects the fact that there is a wide range of longitudes where the unknown planet could be with almost the same probability. We recall that this ill-determination of the parameters is inherent to this prediction problem, and not merely the result of poor data or inadequate methodology. This fact is also reflected in the condition number κ of the nonlinear least-squares system (computed in the L1 norm). The prediction of a whole orbit is very sensitive to errors in the determination of the perturbations, and the fact that the condition number increases with planet X's mass, arises from the relation between the arc of orbit covered by the observations, and the whole planet X's orbit, which decreases with the mass of the perturbing planet.

The presence of systematic trends in the residuals is another important question to examine. To detect systematic features, we have also tested the number of 'runs' (i.e. the sequence of residuals of equal sign). If n_1 and n_2 are the number of positive and negative residuals, respectively, then the more probable number of runs is (Acton, 1959):

$$N_p = 2n_1n_2 \,/\, (n_1 + n_2) + 1 \tag{4.1}$$

and, as a measure of the degree of deviation from pure white noise, we have computed:

$$N = N_o/N_p \tag{4.2}$$

where N_o is the number of runs of the residuals in both components of the perturbation, obviously $N = 0$ corresponds to the worst case and $N = 1$ to the best one. N is also shown in Table IV.

All our tests indicate that the solution of the *inverse problem* which best fits both components of the perturbation, in the least-square sense, is a planet of 0.6 Earth masses, with true longitude 133°

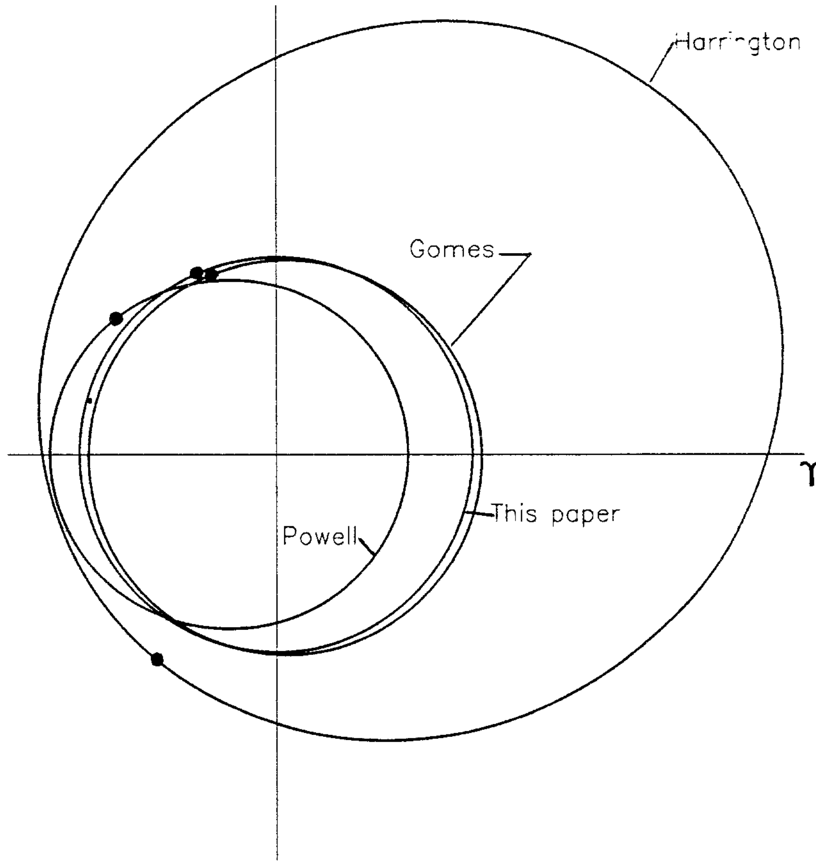

Fig. 3. Orbits of some actual predictions of planet X projected on the Ecliptic. The dots represent the location of the hypothetical planets for 1990.5.

(1990.5) at 44 AU from the Sun and with eccentricity $e = .007$, not far from the planet X proposed by Gomes (1989). Its perturbations on Uranus are shown in Figure 2 (dotted and dashed lines), where we can see a reasonable good fit with the computed ones.

It is important to bear in mind that our search for the planet from the best fit of the computed unmodeled perturbations was not exhaustive, but based on an iterative algorithm to solve the *inverse problem.* We have started the algorithm with $e = 0$ also assuming $i = 0$, because of the bad estimation of the *out-of-plane* component of the perturbation. This last is a strong condition for the planet X's mass and the solution of the inverse problem and, perhaps, explains the difference from the solution proposed by Harrington (1988), and the good accordance with Gomes (1989) as well as with Powell (1988).

In Figure 3 we show some of the actual predicted orbits of *planet X.*

5. Concluding Remarks

We must recall that the perturbations obtained in this way are affected by errors in the dynamical parameters involved in the problem, such as in the initial conditions of the known planets (Jupiter, Saturn and Neptune) as well as in the masses, which are usually computed from mutual planetary perturbations, and are thus affected by the unknown perturbation. This last problem could be partially solved by adopting determinations of the masses from the theory of motion of the best observed natural satellite of each planet (Rawlins and Hammerton, 1973). However, in spite of an obvious circularity, we have adopted masses from VSOP82 (Bretagnon, 1982), because there are serious discrepancies between different determinations of the masses from satellite observations. This is certainly a point that deserves more detailed study.

The numerical results of Section 3 indicate that the numerical scheme can be considered as a proposal to be used in the problem of the perturbations in the motion of Uranus, contributing to a better understanding of the dynamical nature of the problem. The estimated perturbations on Uranus present some significant periods, which, in some sense, means that the perturbation has been successfully estimated.

These periods indicate that errors in the masses of some of the known planets may be a possible source of the observed systematic residuals.

On the other hand, a relatively good fit of the computed perturbations can be made on the hypothesis that these are due to an unknown body beyond Pluto. However, a good fit does not prove that the model is correct.

It is interesting to note that, in 1930, our tentative planet X (that is in fact the same planet proposed by Gomes (1989)), and Lowell's planet X were within a few degrees of Pluto in the celestial sphere. Assuming for the planet a chemical composition similar to that of the bodies of the outer Solar System, a planet of 0.6 Earth masses at 44 AU from the Sun would be brighter than Pluto, and the real puzzle is that Clyde Tombaugh would have found it. If it was really there, it could be rocky rather than icy, with matter darker than normal.

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