General Relativity and Gravitation, Vol. 28, No. 3, 1996

The Newtonian Limit in a Family of Metric Affine Theories of Gravitation

D. E. Barraco,[1] R. Guibert,[1] V. H. Hamity[1] and H. Vucetich[2]

Received July 13, 1995. Rev. version October 5, 1995

A brief review of a first order theory with a quadratic Lagrangian $L = R + \omega_0 R^2$ is presented. It is shown that a test particle follows a geodesic of the metric connection. The theory behaves in the Newtonian limit as the Newtonian theory with a correction which is proportional to the matter density at the field point. This behavior can be produced by a Yukawa potential with an atomic scale characteristic range λ and a coupling constant α proportional to $1/\lambda^2$. This type of potential is not excluded by the present experimental data.

1. INTRODUCTION

In the limit of small energies the superstring theories give an action for spacetime in the form of the Einstein–Hilbert action plus terms which are quadratic in the scalar curvature and the Ricci tensor. On the other hand it is well known [1] that this action leads to a possible renormalizable quantum theory of gravity in the second order formalism; i.e., when we assume a Riemannian geometry and consider variations of the metric and its first derivatives equal to zero on the boundary of a space time region $\mathcal{U}$.

However, it is possible to modify the Einstein–Hilbert action by adding a boundary term such that, when the variation of this term is taken into account, it cancels the unwanted term which appears when we only impose

[1] Fa.M.A.F., Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina

[2] Observatorio Astronómico, Universidad Nacional de La Plata, La Plata 2000, Argentina

0001-7701/96/0300-0339$09.50/0

a null variation of the metric on the boundary and leave its derivatives unrestricted [2,3].

Alternatively, the Palatini approach or first order formalism can be applied to obtain the field equations in general relativity assuming the metric and the connection as independent variables. This formalism has also been applied to more general Lagrangian densities, with quadratic terms or a general function of the scalar curvature [4,5], to study other geometrical theories of gravitation. More recently [6], the latter theories have been extended by including a scalar field in the Lagrangian. One apparent conceptual advantage of these theories is that quantum fluctuations of the metric and the connection are independent of each other; i.e., the geometry of spacetime is less restrictive.

In general, the first and second order formalisms give different field equations although they may be related to each other using the theory of Ferraris et al. [7,8]. This theory associates an affine metric with a pure metric theory in a non vacuum region.

For $L = R + \omega_0 R^2$ it is easy to prove, by inspection, that the field equations of the first order formalism are different from those equations of the second order formalism. However, if we take the first order limit (linear in the parameter ω_0) both sets of equations are equivalent [9]. It is well known that the field equations in the second order formalism are of fourth order and lead to inflationary cosmological models [10]. Probably, the first order theory shares this kind of solution.

In this particular case ($L = R + \omega_0 R^2$) it was found that the cosmological solutions corresponding to dust and an equation of state $p = (\gamma - 1)\rho$ were free of the initial singularity, due to the presence of an "antigravity" force [11].

All the theories obtained from a Lagrangian density $L = f(R)$, in the first order formalism, share with the fourth-order equations corresponding to a Lagrangian density $L = R + \omega_0 R^2$ the vacuum solutions of general relativity. This may suggest that the classical tests of general relativity are automatically satisfied through the Schwarzschild solution [12,13]. However, the empty space solutions are to be matched to interior solutions and it may well occur that the matching conditions are not satisfied [14]. We need a further study of all these alternative theories, and in particular we need their weak field limit and a comparison with experimental data. Finally, we mention that many of the properties of the theories for a general Lagrangian density $L = f(R)$ are present in the simple case $L = R + \omega_0 R^2$ [4].

2. THE GENERAL STRUCTURE OF THE THEORY

Let us consider a Lagrangian density $L = R + \omega_0 R^2 + L_{\mathrm{M}}$, where the matter Lagrangian does not depend on the connection. Then the field equations, if we vary with respect to the metric, are

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + 2\omega_0 R (R_{\mu\nu} - \tfrac{1}{4} g_{\mu\nu} R) = 8\pi G T_{\mu\nu} \,. \tag{1}$$

The variation with respect to the connection gives

$$\nabla_\alpha g_{\mu\nu} = b_\alpha \, g_{\mu\nu} \,, \tag{2}$$

$$b_\alpha = [\ln(1 + 2\omega_0 R)\,]_{,\alpha} \,. \tag{3}$$

From (1) we obtain

$$R = -8\pi G T, \tag{4}$$

which shows that b_α is determined by T and its derivative. The connection is

$$\Gamma^\alpha_{\mu\nu} = C^\alpha_{\mu\nu} - \tfrac{1}{2} (\delta^\alpha_\mu b_\nu + \delta^\alpha_\nu b_\mu - g_{\nu\mu} b^\alpha), \tag{5}$$

where $C^\alpha_{\mu\nu}$ are the components of the metric connection. The Riemann tensor is defined as usual by

$$R^\lambda_{\mu\nu\tau} = \partial_\nu \Gamma^\lambda_{\mu\tau} - \partial_\tau \Gamma^\lambda_{\mu\nu} + \Gamma^\sigma_{\mu\tau} \Gamma^\lambda_{\nu\sigma} - \Gamma^\sigma_{\mu\nu} \Gamma^\lambda_{\tau\sigma} \,. \tag{6}$$

The Ricci tensor and scalar curvature are

$$R_{\mu\nu} = R^0_{\mu\nu} + \tfrac{3}{2} D_\mu b_\nu - \tfrac{1}{2} D_\nu b_\mu + \tfrac{1}{2} g_{\mu\nu} D \cdot b - \tfrac{1}{2} b_\mu b_\nu + \tfrac{1}{2} g_{\mu\nu} b^2 \tag{7}$$

$$R = R^0 + 3 D \cdot b + \tfrac{3}{2} b^2, \tag{8}$$

where $R^0_{\mu\nu}$, R^0 and D_α are the Ricci tensor, scalar curvature and covariant derivative defined from the metric connection.

The Bianchi identity gives [5,11,4]

$$\nabla_\mu \left[\frac{G^\mu_\nu}{1 - 2\omega_0 8\pi G T} \right] = 0 \,. \tag{9}$$

Using the definition of Einstein tensor and the field equations (1) we get

$$\nabla_\mu \left[\frac{T^\mu_\nu - (1/2)\omega_0 8\pi G g^\mu_\nu T^2}{[1 - 2\omega_0 \pi G T]^2} \right] = 0 \,. \tag{10}$$

It is apparent from (10) that a dust particle does not follow a geodesic of the affine connection.

However, we shall prove now that T^μ_ν is covariantly conserved if we use the metric connection D_μ as the derivative operator. Equation (10) can be put into the form

$$\nabla_\mu\left[\frac{T^\mu_\nu}{[1-2\omega_0\pi GT]^2}\right] = \frac{1}{4}T\left[\frac{1}{[1-2\omega_0\pi GT]^2}\right]_{,\nu} . \tag{11}$$

This last expression for the conservation law (9) can also be obtained as a generalized Bianchi identity, from the matter action, in the case that the matter Lagrangian does not depend on the connection.

Multiplying (11) by $[1-2\omega_0\pi GT]^2$ and using (3) we obtain

$$\nabla_\mu T^\mu_\nu + 2T^\mu_\nu b_\mu = \tfrac{1}{2}T\, b_\nu \,. \tag{12}$$

From (5) we can express ∇ in terms of D. Then,

$$\nabla_\mu T^\mu_\nu = D_\mu T^\mu_\nu - 2b_\mu T^\mu_\nu + \tfrac{1}{2}b_\nu T. \tag{13}$$

Finally, from this last equation and (12) we get

$$D_\mu T^\mu_\nu = 0\,. \tag{14}$$

We may conclude then that a test particle will follow the geodesics of the metric connection.

3. THE NEWTONIAN LIMIT

To solve the system of equations (1)–(2) we trace the metric field equations to replace R by $-8\pi GT$ and use (3) and (7) to obtain

$$\begin{aligned} R^{(0)}_{\mu\nu} &= \frac{1}{F}\left[8\pi GT_{\mu\nu} - 4\pi GTg_{\mu\nu}(1-8\pi GT)\,\right] \\ &\quad + \frac{F_{,\mu;\nu}}{F} - \frac{3}{2}\frac{F_{,\mu}F_{,\nu}}{F^2} + \frac{1}{2F}g_{\mu\nu}\,\Box F \end{aligned} \tag{15}$$

where $F = 1 - 16\pi\omega_0 GT$. The factor F in front warrants a modification of Newton's gravitational constant. Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the linearized Ricci tensor is found to be

$$R^{(0)}_{\mu\nu} = \partial^\alpha\partial_{(\nu}h_{\mu)\alpha} - \tfrac{1}{2}\partial^\alpha\partial_\alpha h_{\mu\nu} - \tfrac{1}{2}\partial_\mu\partial_\nu h.$$

The linearized field equation obtained from (15) is

$$\Box(h_{\mu\nu} - 16\pi G\omega_0\eta_{\mu\nu}T) = -16\pi G\Big(T_{\mu\nu} - \frac{T}{2}\,\eta_{\mu\nu}\Big), \tag{16}$$

where we have used a gauge similar to the harmonic one:

$$\partial^\mu h_{\mu\nu} - \tfrac{1}{2}\partial_\nu h + 2\omega_0(8\pi G)T_{,\nu} = 0\,. \tag{17}$$

When gravity is weak and the velocities are small the source has a Newtonian behavior,

$$T_{ab} \propto \rho t_a t_b\,, \tag{18}$$

where $t^\alpha = (\partial/\partial x^0)^\alpha$ is the time direction of our global inertial coordinate system. With these assumptions eq. (16) becomes

$$\begin{aligned} \nabla^2(h_{00} - 2\omega_0(8\pi G)\rho) &= -8\pi G\rho, \\ \nabla^2 h_{0i} &= 0, \\ \nabla^2(h_{ij} + \delta_{ij}2\omega_0(8\pi G)\rho) &= -8\pi G\rho\delta_{ij}\,. \end{aligned} \tag{19}$$

Then, in the Newtonian limit the equation of motion of a test particle is given by (14) with $u^\alpha \simeq \delta^\alpha_0$, and the proper time of the particle may be approximated by the coordinate time t. Thus we find

$$\frac{d^2x^i}{dx^2} = \frac{1}{2}\,\frac{\partial h_{00}}{\partial x^i}\,,$$

$$\tfrac{1}{2}h_{00} = -V_{\mathrm{N}} + \omega_0(8\pi G)\rho. \tag{20}$$

The integration constant in (20) is chosen to satisfy the condition that the metric be locally flat. We have a departure of the acceleration of a test body from the Newtonian theory value which is proportional to the gradient of the mass density.

This departure from the Newtonian value has to be measured when the body is moving "through" a matter filled region. Of course this statement can only be considered in the statistical sense. On the other hand, the experimental search for an extra force in the gravitational potential, between two mass points, has been put into the form of a Yukawa potential which is added to the Newtonian interaction through a coupling constant α and force range λ. The presence of a term proportional to the mass density, in the gravitational potential, would be precisely manifested as a Yukawa potential in the limit of a very short range λ ($\lambda \simeq$ atomic scale), i.e., mediated by a very massive particle, and a coupling constant $\alpha \equiv \omega_0/\lambda^2$. This can be proved by considering the expression

$$\lim_{\lambda\to 0}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{\omega_0}{4\pi\lambda^2}\frac{e^{-|\vec{x}-\vec{x}_0|/\lambda}}{|\vec{x}-\vec{x}_0|}\rho(\vec{x}_0)\,|\vec{x}-\vec{x}_0|^2 d|\vec{x}-\vec{x}_0|\,d\Omega$$
$$= \omega_0\,\rho(\vec{x}) \tag{21}$$

where we have used that

$$\lim_{\lambda\to 0}\int_0^\infty \frac{1}{\lambda^2}\,e^{-r/\lambda} r\,dr = 1. \tag{22}$$

On the other hand, it is interesting to notice that in the corresponding limit of the fourth-order theory, the gravitational potential is [1]

$$V = -G\int\left(\frac{1+(1/3)e^{-r/\sqrt{6\omega_0}}}{r}\right)\rho\,d^3x. \tag{23}$$

Thus, in the linear order in ω_0 we obtain

$$V \to V_{\rm N} - \frac{G}{3}\,6\omega_0\lim_{\omega_0\to 0}\int\frac{e^{-r/\sqrt{6\omega_0}}}{6\omega_0 r}\,\rho\,d^3x = V_{\rm N} - 2\omega_0 G4\pi\rho, \tag{24}$$

where we have again used (22). Thus the fourth-order theory, which is different from the theory obtained by Palatini's method, shares with the latter the same behaviour when we take the weak field limit and consider up to the linear term in the parameter ω_0.

4. EXPERIMENTAL CHECK

It would be interesting to obtain some experimental bound for the constant ω_0 in the present theory. However, this does not seem an easy task: all the departures from Newtonian behaviour are both very small and masked by other effects.

Bounds on a Yukawa tail of the gravitational interaction have been set in recent years, motivated by theoretical results similar to those in the present work. A review of them may be seen in [15,16]. The most accurate existing experimental data [17,18] do not strongly exclude the possibility that Newtonian gravity is violated for distances smaller than 1000 km, nor that there might exist some, as yet undiscovered, ultra-weak interactions of macroscopic range, coupling approximately to mass. On the other hand, the experimental data [19] would allow α to be very large if λ is very small. Then, the possibility that the term $\omega_0(8\pi G)\rho$ be the limit of a Yukawa potential, as explained above, would not be excluded. We have then a good example of a composition dependent force of a very short range.

Finally, let us mention that the range λ would be small, let us say smaller than the electron radius, but larger than the Planck length so we are not bound to use quantum gravity.

5. CONCLUSIONS

We have seen that the theories obtained by the Palatini formalism, when it is applied to the quadratic Lagrangian, behave in the Newtonian limit as the Newtonian theory with a correction which is proportional to the matter density at the field point. This behavior can be produced by a Yukawa potential with an atomic scale characteristic range λ and a coupling constant α proportional to $1/\lambda^2$. It is important to note that this type of potential is not excluded by the present experimental data.

ACKNOWLEDGEMENTS

We are grateful to the Consejo Nacional de Investigaciones Científicas y Técnicas and to CONICOR for financial support.

REFERENCES

1. Stelle, K. S. (1978). *Gen. Rel. Grav.* **9**, 353.
2. Wald, R. M. (1984). *General Relativity* (University of Chicago Press, Chicago).
3. Schmidt, H.-J. (1994). *Phys. Rev. D***49**, 6354.
4. Barraco, D., and Hamity, V. H. (1994). *Gen. Rel. Grav.* **25**, 461.
5. Shahid-Saless, B. (1987). *Phys. Rev. D***35**, 467.
6. Berthias, J. P. and Shahid-Saless, B. (1993). *Class. Quant. Grav.* **10**, 1039.
7. Ferraris, M., Francaviglia, M., Reina, C. (1982). *Gen. Rel. Grav.* **14**, 243.
8. Ferraris, M., and Kijowski, J. (1982). *Gen. Rel. Grav.* **14**, 37.
9. Shahid-Saless, B. (1991). *J. Math. Phys.* **32**, 694.
10. Mijic, M. B., Morris, M. S., and Wai-Mo Suen (1989). *Phys. Rev. D***39**, 1496.
11. Shahid-Saless, B. (1990). *J. Math. Phys.* **31**, 2429.
12. Weyl, H. (1919). *Ann. Phys. (Leipzig)* **59**, 101.
13. Eddington, A. (1924). *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge).
14. Pechlaner, E., and Sexl, R. (1966). *Commun. Math. Phys.* **2**, 165; Deser, S., van Nieuwenhuizen, P., and Tsao, H. S. (1974). *Phys. Rev. D***10**, 3337.
15. Will, C. M. (1992). *Int. J. Mod. Phys. D***1**, 92.
16. Vucetich, H. (1995). *Bol. Acad. Nac. de Ciencias (Córdoba)* **61**, to appear.
17. Zumberge, M. A., et al. (1991). *Phys. Rev. Lett.* **67**, 3051.
18. Moody, M. V., and Paik, H. J. (1993). *Phys. Rev. Lett.* **70**, 1195.
19. Moore, M. W., et al. (1994). *Class. Quant. Grav.* **11**, A97.