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On some Classes of Heyting Algebras with Successor that have the Amalgamation Property

Abstract. In this paper we shall prove that certain subvarieties of the variety of S-algebras (Heyting algebras with successor) has amalgamation. This result together with an appropriate version of Theorem 1 of [L. L. Maksimova, *Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-boolean algebras*, Algebra i Logika, **16(6)**:643-681, 1977] allows us to show interpolation in the calculus $IPC_S(n)$, associated with these varieties.

We use that every algebra in any of the varieties of S-algebras studied in this work has a canonical extension, to show completeness of the calculus $IPC_S(n)$ with respect to appropriate Kripke models.

Keywords: Amalgamation property, Craig's interpolation theorem, Heyting algebras with operators, Extensions of intuitionistic propositional calculus.

Introduction

In [16], Kuznetsov introduced an operation on Heyting algebras as an attempt to build an intuitionistic version of the provability logic of Gödel-Löb, which formalizes the concept of provability in Peano Arithmetic. This unary operation, which we shall call *successor* [6], was also studied by Caicedo and Cignoli in [6] and by Esakia in [12]. In particular, Caicedo and Cignoli considered it as an example of an implicit compatible operation on Heyting algebras.

A set $E(f)$ of equations in the signature of Heyting algebras augmented with a unary function symbol f is said to define an implicit operation if for any Heyting algebra H there is at most one function $f_H : H \to H$. The function f is an implicit compatible operation provided all f_H are compatible.

The successor, S, can be defined on the variety of Heyting algebras by the following set of equations:

(S1) $x \leq S(x)$,

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(S2) $S(x) \leq y \vee (y \rightarrow x)$,

(S3) $S(x) \rightarrow x = x$.

We shall call *S-algebra* to a Heyting algebra endowed with its successor function, when it exists.

Recall that Esakia duality (see for example [11]) establishes a dual equivalence between the category $\mathcal{H}$ of Heyting algebras and the category $\mathcal{E}$ of Esakia spaces,

$$\mathbf{X} : \mathcal{H} \leftrightarrows \mathcal{E}^{op} : \mathbf{D}$$

Here, $\mathbf{X}(H)$ is the set of prime filters of the Heyting algebra H and $\mathbf{D}(X)$ is the set of clopen upsets of the Esakia space X. The morphisms of $\mathcal{E}$ are called Esakia morphisms. The unit and counit of the adjuntion are $\varphi_H(x) = \{P \in \mathbf{X}(H) : x \in P\}$ and $\epsilon_X(x) = \{U \in \mathbf{D}(X) : x \in U\}$, respectively.

Write $\mathcal{SH}$ for the category whose objets are S-algebras and whose morphisms are the Heyting algebra morphisms that commute with the successor. If X is a poset and $V \subseteq X$, we write V_M for the set of maximal elements in V. An Esakia space X is a S-space if for every $U \in \mathbf{D}(X)$ the set $U \cup (U^c)_M$ is clopen. Observe that X is a S-space if and only if it is an Esakia space such that for every clopen downset V the set V_M is clopen. Let X and Y be S-spaces. A morphism of Esakia spaces $g : X \rightarrow Y$ is a S-morphism if for every V clopen downset in X it holds that $g^{-1}(V_M) = [g^{-1}(V)]_M$. We shall write $\mathcal{SE}$ for the category whose objects are S-spaces and whose morphisms are S-morphisms. It was proved in [8] that there exists a dual categorical equivalence between the category $\mathcal{SH}$ and the category $\mathcal{SE}$ and that if X is a S-space then in $\mathbf{D}(X)$ the successor function takes the form

$$S(U) = U \cup (U^c)_M.$$

In this paper we shall prove that certain subvarieties of the variety of S-algebras has amalgamation (Theorem 3.4). This result together with an appropriate version of Theorem 1 of [17] allows us to show interpolation in the calculus $IPC_S(n)$, associated with these varieties (Corollary 4.4).

Interpolation is gaining importance in the applications of Logic to Computer Science. Besides its applicability in software design, where it is connected to the modularization property, the importance of interpolation has been recently recognized in Model Checking, where non-classical logics are used to specify and verify properties of software and reactive systems [10].

We use that every algebra in any of the varieties of S-algebras studied in this work has a canonical extension (Proposition 2.3) to show completeness of the calculus $IPC_S(n)$ with respect to appropriate Kripke models (Proposition 5.4).

1. The height in posets, topological spaces and algebras

Let X be a poset and let $\mathbb{N}$ be the set of natural numbers greater than 1. For every $n \in \mathbb{N}$ we define an increasing sequence of sets $\{X_n\}_{n\geq 1}$ as follows:

$$\begin{aligned} X_1 \quad &= X_M, \\ X_{n+1} &= X_n \cup (X_n^c)_M. \end{aligned}$$

Then for $n \geq 2$ we define the sets $\hat{X}_n$ by

$$\begin{aligned} \hat{X}_1 &= X_1\ , \\ \hat{X}_n &= (X_{n-1}^c)_M = X_n \backslash X_{n-1}. \end{aligned}$$

We say that the poset X has height n if $X = X_n$ and n is the minimum natural number with this property.

Remark 1.1. (i) For every $n \in \mathbb{N}$, X_n is an upset.

(ii) For every $n \in \mathbb{N}$,

$$X_n = \bigcup_{i=1}^{n} \hat{X}_i.$$

In fact,

$$X_n = \hat{X}_n \cup X_{n-1} = \hat{X}_n \cup \hat{X}_{n-1} \cup X_{n-2} = \cdots = \bigcup_{i=1}^{n} \hat{X}_i,$$

so we can equivalently say that the poset X has height less or equal to n iff $X = \bigcup_{i=1}^{n} \hat{X}_i$.

We define the height of a S-space as the height of its underlying poset. We write $\mathcal{SE}_n$ for the full subcategory of $\mathcal{SE}$ whose objects are S-spaces of height less or equal to n.

Remark 1.2. It is customary to define the height of an element x of a poset X as the length of a maximal upward chain starting at x (when this length exists) and, when each element of X has a length, the height of the poset X, as the maximum of the heights of its elements. Note that in case that X is of height n, $\hat{X}_i = \{x \in X \mid \text{height}(x) = i\}$, for $i = 1, \ldots, n$, and $X_j = \bigcup_{i=1}^{j} \hat{X}_i$ $(j = 1, \ldots, n)$. In particular, $X = X_n = \bigcup_{i=1}^{n} \hat{X}_i$.

If X is a poset of height $n \in \mathbb{N}$, then the family $F = \{\hat{X}_i\}_{i=1}^n$ is completely characterized by the following properties (see Lemma 2.2 bellow):
1. F is a partition of X with n elements,
2. if $x \leq y$ and $x \in F_i$, then there exists $j < i$ such that $y \in F_j$, and
3. for $i \geq 2$, if $x \in F_i$ then there exists $y \in F_{i-1}$ such that $x < y$.

We say that an S-algebra H has height n if $S^{(n)}(0) = 1$ and n is the minimum natural number with this property. We write $\mathcal{SH}_n$ for the class of S-algebras of height less or equal to n. This class is a variety with defining equations of S-algebras and the additional equation

$$S^{(n)}(0) = 1,$$

or, equivalently, the equation $S^{(n)}(x) = 1$. Note that $\mathcal{SH}_1$ is just the variety of Boolean algebras, and that we have that

$$\mathcal{SH}_1 \subseteq \mathcal{SH}_2 \subseteq \ldots \mathcal{SH}_n \subseteq \ldots$$

We also write $\mathcal{SH}_n$ for the category of S-algebras of height less or equal to n.

The following is an immediate consequence of theorems 4.4 and 4.10 of [8]:

THEOREM 1.3. There is a categorical dual equivalence between $\mathcal{SH}_n$ and $\mathcal{SE}_n$.

The next lemma and proposition will be used to give a characterization of the morphisms in $\mathcal{SE}_n$.

LEMMA 1.4. Let $f : (X, \leq) \to (Y, \leq)$ be a morphism of S-spaces. Then the following conditions hold:

(a) $f^{-1}(Y_i) = X_i$.

(b) $f^{-1}(\hat{Y}_i) = \hat{X}_i$.

(c) If X and Y are S-spaces of height k and l respectively then $k \leq l$.

(d) If f is surjective in (c) then $k = l$.

PROOF. (a) We will do the proof by induction. Case $i = 1$: we have that $f^{-1}(Y_1) = f^{-1}(Y_M) = X_M = X_1$. Suppose that the assertion holds for some $i \in \mathbb{N}$. Then

$$f^{-1}(Y_{i+1}) = f^{-1}(S(Y_i)) = S(f^{-1}(Y_i)) = S(X_i) = X_{i+1}.$$

(b) It follows from item (a). The case $i = 1$ is immediate. For $i > 1$ we have that $f^{-1}(Y_i) = X_i$, so $f^{-1}(Y_{i-1}) \cup f^{-1}(\hat{Y}_i) = X_{i-1} \cup \hat{X}_i$. Taking the intersection with respect to $\hat{X}_i$ in both members (and using (a)) we conclude that $\hat{X}_i \subseteq f^{-1}(\hat{Y}_i)$. Taking the intersection with respect to $f^{-1}(\hat{Y}_i)$ in both members (and using (a) again) we conclude that $f^{-1}(\hat{Y}_i) \subseteq \hat{X}_i$.

(c) By (a) we have that $X = f^{-1}(Y) = f^{-1}(Y_l) = X_l$. Thus $k \leq l$.

(d) By (c) we only prove that $l \leq k$. Suppose that $Y_k^c \neq \emptyset$. By (a) and surjectivity of f, we conclude that $X_k^c \neq \emptyset$. It is a contradiction because $X_k = X$. ■

Let $f : X \to Y$ be a morphism in $\mathcal{E}$ and X, Y be S-spaces. Consider the following condition:

(E) For every $x, y \in X$, $x < y$ implies that $f(x) < f(y)$.

Proposition 1.5. If f satisfies condition (E) then f is a morphism in $\mathcal{SE}$. Conversely, if $f : X \to Y \in \mathcal{SE}_n$ for some $n \in \mathbb{N}$ then f satisfies condition (E). In particular, $f \in \mathcal{SE}_n$ iff f satisfies condition (E).

Proof. Suppose that f satisfies condition (E) and let $U \in \mathbf{D}(X)$. We know that $(f^{-1}(U^c))_M \subseteq f^{-1}(U_M^c)$. Conversely, let $x \in f^{-1}(U_M^c)$ and let $x \leq z$ with $z \in f^{-1}(U^c)$. If $x < z$ then $f(x) < f(z)$ (by condition (E)), a contradiction. Then f is a morphism in $\mathcal{SE}$.

Conversely, suppose that $f : X \to Y \in \mathcal{SE}_n$ and let $x < y$, so $f(x) \leq f(y)$. Suppose that $f(x) = f(y)$. Thus there is i a natural number such that $f(x) \in \hat{Y}_i$, so by item (b) of Lemma 1.4 we have that $x, y \in \hat{X}_i$. Using that $x \leq y$ we conclude that $x = y$, a contradiction. ■

2. Canonical extension of S-algebras

Canonical extensions of distributive lattices with operators were introduced by Gehrke and Jónsson [3] as a natural generalization of canonical extensions of Boolean algebras with operators. They were further generalized to lattices with operators by Gehrke and Harding [2].

Definition 2.1. (Def. 2.1 of [5]) Suppose that L is a lattice. A pair (C, e) is a completion of L if C is a complete lattice and $e : L \to C$ is a lattice embedding. A completion (C, e) of L is dense if each $c \in C$ is a join of meets and a meet of joins of elements of $e(L)$. A completion (C, e) of L is compact if for each pair of sets A, B of L with $\bigwedge e(A) \leq \bigvee e(B)$, there are finite subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $\bigwedge e(A_0) \leq \bigvee e(B_0)$. A canonical extension of L is a dense and compact completion of L.

Throughout the paper we will slightly abuse notation and call a complete lattice C a completion of a lattice L. Every lattice admits a unique, up to isomorphism, dense and compact completion (Theorem 2.2 of [5]).

If H is a Heyting algebra, a canonical extension of H is a complete Heyting algebra H' such that there is a Heyting algebras embedding $e: H \to H'$ for which, as a lattice, H' is a canonical extension of H.

If X is a poset, we write X^+ for the Heyting algebra of upsets of X. Then for a Heyting algebra H we have that $\mathbf{X}(H)^+$ is a canonical extension of H. Similarly, we shall define a canonical extension of an S-algebra.

In this section we show that the canonical extension of an algebra in $\mathcal{SH}_n$ is in $\mathcal{SH}_n$.

LEMMA 2.2. Let X be a poset. Then

(a) If $i \neq j$ then $\hat{X}_i \cap \hat{X}_j = \emptyset$.

(b) If $x \leq y$, $x \in \hat{X}_i$ and $y \in \hat{X}_j$ then $j \leq i$.

(c) If $i \geq 2$ and $x \in \hat{X}_i$ then there is $y \in \hat{X}_{i-1}$ such that $x < y$.

PROOF. (a) Suppose that $\hat{X}_i \cap \hat{X}_j \neq \emptyset$ and $i < j$. Then there exists $x \in X$ such that $x \in \hat{X}_i \cap \hat{X}_j$ and $i \leq j-1$. In particular $x \notin X_{j-1}$, so $x \notin X_i$. Thus $x \notin \hat{X}_i$, a contradiction.

(b) Let $x \leq y$, $x \in \hat{X}_i$ and $y \in \hat{X}_j$. We have that $x \in X_i$ and hence, $y \in X_i$. Then there is $k \leq i$ such that $y \in \hat{X}_k$. By (a), we have that $j = k \leq i$.

(c) Let us consider two cases:

Case $i = 2$. Suppose that $x \in \hat{X}_2$, then $x \notin X_M$. Thus there is $y \in X$ such that $x < y$. Using that $x \in X_2$ we have that $y \in X_2$. Hence, since $x < y$ we conclude that $y \in \hat{X}_1$.

Case $i > 2$. Let $x \in \hat{X}_i$, so by (a) we have that $x \notin \hat{X}_{i-1}$. Thus there is $y \notin X_{i-2}$ such that $x < y$. Besides there is $j = 1, \ldots, i$ such that $y \in \hat{X}_j$. If $j = i$, using that $x, y \in \hat{X}_i$ and $x < y$, we have a contradiction. Suppose that $j \leq i-2$. Using the fact that $y \notin X_{i-2}$, we have that $y \notin \hat{X}_k$ for every $k = 1, \ldots, i-2$. Thus $y \notin \hat{X}_j$, a contradiction. Therefore $j = i-1$. ■

If X is a poset and $V \subseteq X$ we write

$$[V) = \{x \in X : x \geq v \text{ for some } v \in V\},$$

$$(V] = \{x \in X : x \leq v \text{ for some } v \in V\}.$$

If $x \in X$ we write $[x)$ in place of $(\{x\}]$.

Proposition 2.3. Let X be a poset. If X has height $n \in \mathbb{N}$ then X^+ is a S-algebra of height n, where for every $U \in X^+$ we have that

$$S(U) = U \cup (U^c)_M.$$

Proof. First, we will prove that function $S : X^+ \to X^+$ is the successor function. Equations ($S1$) and ($S2$) follows from an easy computation (see page 352 of [12] for an alternative approach). To prove equation ($S3$) we only need to prove that for every downset V in X it holds that $V = (V_M]$. Take $x \in V$. We have that there is $i \in \{1, \ldots, n\}$ such that $x \in \hat{X}_i$. Thus $[x) \cap V \cap \hat{X}_i \neq \emptyset$. We consider the set

$$M = \{k = 1, \ldots, n : [x) \cap V \cap \hat{X}_i \neq \emptyset\}.$$

Since $i \in M$, we conclude that $M \neq \emptyset$ and take $j = min\ M$. Then $[x) \cap V \cap \hat{X}_j \neq \emptyset$, so there is $v \in [x) \cap V \cap \hat{X}_j$. Thus $x \leq v$ and $v \in V$. Suppose that $v \leq w$ for some $w \in V$. There exists $k \in \{1, \ldots, n\}$ such that $w \in \hat{X}_k$, so $w \in [x) \cap V \cap \hat{X}_k$. Thus $[x) \cap V \cap \hat{X}_k \neq \emptyset$, and $k \in M$. Then $j \leq k$.

On the other hand by (b) of Lemma 2.2 we have that $k \leq j$, so $k = j$. Using that $v \leq w$ with $v, w \in \hat{X}_k$ we conclude that $v = w$, and hence $v \in V_M$. Thus $x \in (V_M]$. Therefore equation ($S3$) holds. Finally by hypothesis we have that $S^{(n)}(\emptyset) = X_n = X$. If $S^{(k)}(\emptyset) = X_k = X$ then $n \leq k$. ■

Note that not every S-algebra have an extension of the form of previous Proposition. For instance, if $\mathbb{N}^0$ is the set of natural numbers with its inverse order, and $\oplus$ is the ordinal sum of posets (see [1], p. 39), then $\mathbb{N} \oplus \mathbb{N}^0$ is a S-algebra. However, $(X(\mathbb{N} \oplus \mathbb{N}^0))^+$ is not a S-algebra.

3. Amalgamation property in $\mathcal{SH}_n$

The amalgamation property was first considered by Schrwier in [20], where it was investigated for groups. In a general form, the amalgamation property was first formulated by Fraïssé ([14]) in connection with certain embedding properties.

Definition 3.1. Let K be a class of algebras. We say that K has the *amalgamation property* if for $H_0, H_1, H_2 \in K$, and embeddings $i_1 : H_0 \to H_1$ and $i_2 : H_0 \to H_2$, there exists some $H \in K$ and embeddings $\epsilon_1 : H_1 \to H$ and $\epsilon_2 : H_2 \to H$ such that $\epsilon_1 i_1 = \epsilon_2 i_2$. That is, if the following diagram commutes:

$$\begin{array}{ccc} H_0 & \xrightarrow{i_1} & H_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle \epsilon_1} \\ H_2 & \xrightarrow[\epsilon_2]{} & H \end{array}$$

In this section we will prove that for every $n \in \mathbb{N}$, $\mathcal{SH}_n$ has the amalgamation property. In order to prove it, we will use the dual categorical equivalence given in Theorem 1.3, and Proposition 1.5.

For $j = 1, 2$, let $i_j : H_0 \to H_j$ be embeddings in $\mathcal{SH}_n$. If we take $\alpha : \mathbf{X}(H_1) \to \mathbf{X}(H_0)$, given by $\alpha = \mathbf{X}(i_1)$ and $\beta : \mathbf{X}(H_2) \to \mathbf{X}(H_0)$, given by $\beta = \mathbf{X}(i_2)$, we have that α and β are epimorphisms in $\mathcal{SE}_n$. Take $X = \mathbf{X}(H_1)$, $Y = \mathbf{X}(H_2)$, $Z = \mathbf{X}(H_0)$ and consider the set

$$W = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}.$$

By (d) of Lemma 1.4 we have that X, Y and Z are S-spaces of the same height $h \leq n$. Note that

$$W \subseteq \bigcup_{i=1}^{h} \hat{X}_i \times \hat{Y}_i. \tag{I}$$

In order to prove this, let $\alpha(x) = \beta(y)$. Then there is $i \in \{1, \ldots, h\}$ such that $x \in \hat{X}_i$. By (b) of Lemma 1.4 we have that $\beta(y) = \alpha(x) \in \hat{Z}_i$. Thus $y \in \hat{Y}_i$, and $(x, y) \in \hat{X}_i \times \hat{Y}_i$.

Let W be the poset with the order induced by the product order of $X \times Y$.

With the above notation, we have the following

Lemma 3.2. The poset W has height h.

Proof. We first prove by induction on the level index $i \in \mathbb{N}$ that

$$W \cap (\hat{X}_i \times \hat{Y}_i) \subseteq \hat{W}_i. \tag{II}$$

Let $(x, y) \in W$, $x \in X_M$ and $y \in Y_M$. Suppose that $(z, w) \in W$ and $(x, y) \leq (z, w)$. Using that $x \in X_M$ and $y \in Y_M$ we conclude that $(x, y) = (z, w)$, and $(x, y) \in W_M$. Then $W \cap (\hat{X}_1 \times \hat{Y}_1) \subseteq \hat{W}_1$.

Suppose that $W \cap (\hat{X}_i \times \hat{Y}_i) \subseteq \hat{W}_i$ for all $i \leq k$ ($k \in \mathbb{N}$). Take $(x, y) \in W \cap (\hat{X}_{k+1} \times \hat{Y}_{k+1})$. Thus $\alpha(x) = \beta(y)$, $x \in \hat{X}_{k+1}$ and $y \in \hat{Y}_{k+1}$. Suppose that $(x, y) \in W_k$, so there is $i \leq k$ such that $(x, y) \in \hat{W}_i$. By (b) of Lemma 2.2 there is $x_i \in \hat{X}_i$ such that $x < x_i$, so by Proposition 1.5 we have that $\alpha(x) < \alpha(x_i)$. On the other hand, as β is surjective we have that there is $z_i \in Y$ such that $\alpha(x_i) = \beta(z_i)$, so by (b) of Lemma 1.4 we have that

$z_i \in \hat{Y}_i$. Using that β is a p-morphism and the fact that $\beta(y) < \beta(z_i)$, we have that there is $y_i \in Y$ such that $y \leq y_i$ and $\beta(y_i) = \beta(z_i)$, so by (b) of Lemma 1.4 again we have that $y_i \in \hat{Y}_i$. Thus $(x, y) \in \hat{W}_i$ and $(x, y) < (x_i, y_i) \in W \cap (\hat{X}_i \times \hat{Y}_i) \subseteq \hat{W}_i$, a contradiction. Therefore $(x, y) \in (W_k)^c$. Let $(x, y) \leq (z, w)$ with $(z, w) \in (W_k)^c$. Suppose that $x < z$, so by (b) of Lemma 2.2 we have that $z \in \hat{X}_i$ for some $i \leq k$. Using Proposition 1.5 we have that $\alpha(x) < \alpha(z)$, so $\beta(y) < \beta(w)$. Using that $(z, w) \in W$, (I), the fact that $z \in \hat{X}_i$ and (a) of Lemma 2.2, we conclude that $w \in \hat{Y}_i$ and thus $(z, w) \in W \cap (\hat{X}_i \times \hat{Y}_i) \subseteq \hat{W}_i$, a contradiction because $(z, w) \in W_k^c$. Thus $(x, y) \in \hat{W}_{k+1}$, and $W \cap (\hat{X}_{k+1} \times \hat{Y}_{k+1}) \subseteq \hat{W}_{k+1}$. Then condition (II) holds for every $i \in \mathbb{N}$. By (I) and (a) of Lemma 2.2, we conclude that

$$W \cap (\hat{X}_i \times \hat{Y}_i) = \hat{W}_i$$

for every $i \in \mathbb{N}$. Therefore by (I) we conclude that

$$W = \bigcup_{i=1}^{h} (W \cap (\hat{X}_i \times \hat{Y}_i)) = \bigcup_{i=1}^{h} \hat{W}_i.$$

In particular, $W_h = W$. In order to prove that h is the minimum natural number with the previous property, let m be a natural number such that $W_m = W$ and suppose that h is the height of $(X, \leq)$. We will prove that $X = X_h = X_m$. Let $x \in X_h$, so there is $i \in \{1, \ldots, h\}$ such that $x \in \hat{X}_i$. By (b) of Lemma 1.4 we have that $\alpha(x) \in \hat{Z}_i$. As β is surjective, there is $y \in Y$ such that $\alpha(x) = \beta(y)$ (note that by (b) of Lemma 1.4 we have that $y \in \hat{Y}_i$). Then $(x, y) \in W = W_m$. Thus there is $j \in \{1, \ldots, m\}$ such that $(x, y) \in \hat{W}_j$, so $x \in \hat{X}_j$ and then $x \in X_m$. Hence $X = X_h \subseteq X_m$, i.e , $X = X_h = X_m$. As the height of X is h we have that $h \leq m$. Therefore the poset W has height h. ■

By lemmas 3.2, 2.3 and with the notation of above, we have the following

COROLLARY 3.3. W^+ is a S-algebra of height h.

For $i = 1, 2$ we define maps $f_i : H_i \to W^+$ by

$$f_i(b) = \{(P_1, P_2) \in W : b \in P_i\}.$$

These maps are embeddings of Heyting algebras such that $f_1 i_1 = f_2 i_2$ (see [15], section 2.5). Using that $\varphi_{H_i}(S(b)) = \varphi_{H_i}(b) \cup (\varphi_{H_i}(b))^c_M$ we can prove that $f_i(S(b)) = S(f_i(b))$ (for $i = 1, 2$). Therefore we have the following

THEOREM 3.4. $\mathcal{SH}_n$ has the amalgamation property.

4. Craig's interpolation theorem

Let L be the language of formulas of the intuitionistic propositional calculus (IPC), built in the usual way from the connective symbols $\to$, $\wedge$, $\vee$, $\neg$, corresponding to implication, conjunction, disjunction and negation, respectively, and the propositional variables π_i, $i = 0, 1, \ldots$ We write $\varphi \leftrightarrow \psi$ as a short hand for $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

Let ∇ be a distinct connective symbol (of arbitrary arity), and $L(\nabla)$ will denote the propositional language obtained by allowing ∇ in the formation rules of formulas. To each set of formulas $A(\nabla) \subseteq L(\nabla)$, associate the axiomatic system having $A(\nabla) \cup Int$ for axiom schemas, where Int is a complete system of schemas for the intuitionistic propositional calculus (as given for example in [18] and [19]), and substitution in axiom schemas and Modus Ponens as only rules. Only this kind of systems will be considered. Given $\Gamma \cup \{\varphi\} \subseteq L(\nabla)$, the notation $\Gamma \vdash_{A(\nabla)} \varphi$ will indicate that φ is deducible from Γ in this calculus. We write $\vdash_{A(\nabla)} \varphi$ if $\Gamma = \emptyset$. It is immediate that the deduction theorem is satisfied:

$$\Gamma \cup \{\alpha\} \vdash_{A(\nabla)} \varphi \text{ implies } \Gamma \vdash_{A(\nabla)} \alpha \to \varphi.$$

Each formula $\varphi \in L(\nabla)$ may be seen as a term in the variables π_i, in the type $\tau \cup \nabla$ of Heyting algebras enlarged with the operation symbol ∇. Therefore, to each extension $A(\nabla)$ of the intuitionistic calculus we may associate the system of equations $E(\nabla) = \{\varphi = 1 : \varphi \in A(\nabla) \cup Int\}$, and the corresponding variety of Heyting algebras

$$V(A(\nabla)) = V(E(\nabla)).$$

Definition 4.1. (Def. 4.2 of [6]) A set of formulas $A(\nabla)$ will be said to define axiomatically a connective ∇ provided that

$$\vdash_{A(\nabla) \cup A(\hat{\nabla})} \nabla(\pi_1, \ldots, \pi_n) \leftrightarrow \hat{\nabla}(\pi_1, \ldots, \pi_n),$$

where $\hat{\nabla}$ is a new n-ary connective and $A(\hat{\nabla}) = \{\varphi(\nabla/\hat{\nabla}) : \varphi \in A(\nabla)\}$.

In the following we will assume that $A(\nabla)$ defines axiomatically a connective. We write IPC_{∇} for the propositional intuitionistic calculus with the additional axioms given by the formulas of $A(\nabla)$. For the logic IPC_{∇} we consider the variety $V(A(\nabla))$.

Definition 4.2. By the *Craig's interpolation theorem* (CIT) in the logic IPC_{∇} we mean the following proposition: for any formulas α, $\beta \in L(\nabla)$, if $\vdash_{A(\nabla)} \alpha \to \beta$, there is a formula γ in $L(\nabla)$ such that $\vdash_{A(\nabla)} \alpha \to \gamma$, $\vdash_{A(\nabla)} \beta \to \gamma$ and γ contains only those variables which occur simultaneously in both α and β.

Suppose that K is an arbitrary class of algebras partially ordered. If x is a set of k variables and $p(x)$ is an equation in K, we write $\models_K p(x)$ in case that for any $H \in K$ and for any $a_1, \dots, a_n \in H$ we have that equality given by $p(a_1, \dots, a_k)$ is true in H. Besides, we define $\models_K t(x) \leq u(x)$ (where t and u are terms) in case that for any $H \in K$ and for any $a_1, \dots, a_k \in H$ we have that inequality given by $t(a_1, \dots, a_k) \leq u(a_1, \dots, a_k)$ is true in H.

By the *interpolation principle for identities* (IPE) in the class K we mean the following proposition: for any pairwise disjoint sets of variables x, y, z and identities $p_1(x, y)$, ...,$p_n(x, y)$, $q(x, z)$, if

$$\models_K \bigwedge_{i=1}^{n} p_i(x, y) \Rightarrow q(x, z)$$

then there exist m and identities $\tau_1(x), \dots, \tau_m(x)$ such that

$$\models_K \textstyle\bigwedge_{i=1}^{n} p_i(x, y) \Rightarrow \bigwedge_{j=1}^{m} \tau_j(x) \text{ and } \models_K \bigwedge_{j=1}^{m} \tau_j(x) \Rightarrow q(x, z).$$

We also define the *interpolation principle for inequalities* (IPI): for any terms $t(x, y)$ and $u(x, z)$, if $\models_K t(x, y) \leq u(x, z)$, then there is a term $v(x)$ such that $\models_K t(x, y) \leq v(x) \leq u(x, z)$.

A class K is called *strongly amalgamable* if for any $H_0, H_1 H_2 \in K$ Definition 3.1 is satisfied and $\epsilon_1(H_1) \cap \epsilon_2(H_2) = \epsilon_1 i_1(H_0)$. A class K of partially ordered algebras is called *superamalgamable* if Definition 3.1 is satisfied for $H_0, H_1, H_2 \in K$, and if for $j, k \in \{0, 1\}$, if $\epsilon_j(x) \leq \epsilon_j(y)$ then there is $z \in H_0$ such that $x \leq_j i_j(z)$ and $i_k(z) \leq_k y$ (where $\leq_i$ is the order in H_i for $i = 1, 2$).

Let K be a class of algebras such that in each $H \in K$ we can define the supreme of two elements. We say that H is *completely connected* if for all $x, y \in H$, if $x \vee y = 1$ then $x = 1$ or $y = 1$.

The following theorem is a reformulation of Theorem 1 given in [17], using also Theorem 4.1, Theorem 4.2 and Corollary 4.4 of [6].

THEOREM 4.3. We suppose that the set of formulas $A(\nabla)$ defines axiomatically a connective ∇. Then the following conditions are equivalent:

1) CIT is true in IPC_∇;

2) The variety $V(A(\nabla))$ satisfies the IPI;

3) The variety $V(A(\nabla))$ satisfies the IPE;

4) $V(A(\nabla))$ is superamalgamable;

5) $V(A(\nabla))$ is strongly amalgamable;

6) $V(A(\nabla))$ has the amalgamation property;

7) Definition 3.1 is satisfied for any completely connected $H_0, H_1, H_2 \in V(A(\nabla))$.

Let $n \in \mathbb{N}$. The following set $A(S)_n$ (see Example 5.2 of [6]) of schemas defines an implicit connective of intuitionistic calculus:

(Sn1) $\alpha \to S(\alpha)$,

(Sn2) $S(\alpha) \to (\beta \vee (\beta \to \alpha))$,

(Sn3) $(S(\alpha) \to \alpha) \to \alpha$,

(Sn4) $S^{(n)}(\alpha)$.

In particular, we have that $V(A(S)_n) = \mathcal{SH}_n$. We write $IPC_S(n)$ for the propositional intuitionistic calculus with the additional axioms $(Sn1)$-$(Sn4)$, with Modus Ponens and substitution as the only rules. Then by theorems 3.4 and 4.3 we have the following

Corollary 4.4. $IPC_S(n)$ satisfies the CIT.

5. Kripke models

Associating algebraic models to propositional logics is often achieved by an easy transcription of syntactic specifications of such logics. As a consequence, semantic modeling by such algebras is often not far removed from the syntactic treatment of the logics.

In [7] X. Caicedo proposes a Kripke semantics for IPC_S. In this section we apply a variation on this semantics to the study of completeness results for $IPC_S(n)$.

Let $n \in \mathbb{N}$. Set $\mathcal{M} = (X, K)$, where X is a poset of height less or equal to n and $K : L(S) \to X^+$ is a function. For α in $L(S)$ and $p \in X$, write

$$\models_p \alpha \text{ if and only if } p \in K(\alpha)$$

If the relation $\models_p$ satisfies the following conditions, we say that $\mathcal{M}$ is a *Kripke model* [13] of $IPC_S(n)$:

(1)	$\models_p \neg\alpha$	iff	$q \geq p$ implies $\not\models_q \alpha$,
(2)	$\models_p \alpha \vee \beta$	iff	$\models_p \alpha$ or $\models_p \beta$,
(3)	$\models_p \alpha \wedge \beta$	iff	$\models_p \alpha$ and $\models_p \beta$,
(4)	$\models_p \alpha \to \beta$	iff	for $q \geq p$: if $\models_q \alpha$ then $\models_q \beta$,
(5)	$\models_p S(\alpha)$	iff	for every $q > p$ we have that $\models_q \alpha$.

It can be proved that condition (5) is equivalent to

$$K(S(\alpha)) = K(\alpha) \cup (K(\alpha)^c)_M.$$

Notice that, although the displayed equation does not hold for any poset, it holds in this case because any subset of X has maximal elements.

Let T be a theory of $IPC_S(n)$. We define the following equivalence relation in $L(S)$:

$$\alpha \equiv_T \beta \text{ iff } T \vdash_{A(S)_n} \alpha \leftrightarrow \beta.$$

Write $[\alpha]_T$ for the class of equivalence of α. Note that the binary relation in $L(S)$ given by $\alpha \leq \beta$ iff $T \vdash_{A(S)_n} (\alpha \to \beta)$ is a pre-order (i.e reflexive and transitive). Then we have the following order in the quotient: $[\alpha]_T \leq [\beta]_T$ iff $\alpha \leq \beta$.

Lemma 5.1. Let $\alpha \in L(S)$ and let T be a theory of $IPC_S(n)$. Then

(a) $[\alpha]_T = 1$ iff $T \vdash_{A(S)_n} \alpha$.

(b) $L(S)/\equiv_T$ is a S-algebra of height less or equal to n. In this case we define

$$S([\alpha]_T) = [S(\alpha)]_T.$$

Proof. The first assertion is proved in the same way as for the intuitionistic case. The second one follows from the fact that S is an implicit connective (see Theorem 4.1, Theorem 4.2 and Corollary 4.4 of [6]). ■

Definition 5.2. Let $\alpha \in L(S)$ and let T be a theory of $IPC_S(n)$. We say that α holds in a Kripke model $\mathcal{M} = (X, K)$ of T iff for each $p \in X$, we have that if for all $\gamma \in T$, $p \in K(\gamma)$ then $p \in K(\alpha)$.

Any function $v : \{\pi_1, \pi_2, \dots\} \to Domain(H)$ with $H \in V(A(S)_n)$ (called H-valuation) may be extend to a unique homomorphism $\overline{v} : L(S) \to H$. Here we see $L(S)$ as the algebra of terms in the signature of Heyting algebras with S. Then we define for any set $\Gamma \cup \{\varphi\} \subseteq L(S)$ an algebraic consequence relation as follows

Definition 5.3. (Def. 4.1 of [6])

$\Gamma \Vdash_{A(S)_n} \varphi$ iff for any $H \in V(A(S)_n)$ and H-valuation v: $\overline{v}(\gamma) = 1$ for all $\gamma \in \Gamma$ implies $\overline{v}(\varphi) = 1$.

It is easy to check, by induction on the length of proofs, that $\vdash_{A(S)_n}$ is sound with respect to this semantics. That is,

$$\Gamma \vdash_{A(S)_n} \varphi \ \textit{implies}\ \Gamma \Vdash_{A(S)_n} \varphi. \tag{III}$$

PROPOSITION 5.4. Let $\alpha \in L(S)$ and let T be a theory of $IPC_S(n)$. Then

$$T \vdash_{A(S)_n} \alpha \text{ iff } \alpha \text{ holds in every Kripke model of } T.$$

PROOF. Let $\mathcal{M} = (X, K)$ be a Kripke model of T and suppose that $T \vdash_{A(S)_n} \alpha$. By (III) we have that $T \Vdash_{A(S)_n} \alpha$. Using that X is a poset of height less or equal to n and Proposition 2.3, we have that $X^+ \in \mathcal{SH}_n$, where $S(U) = U \cup (U^c)_M$ for every $U \in X^+$. Then $K(S(\alpha)) = S(K(\alpha))$. Thus $K : L(S) \to X^+$ is an homomorphism, so if $K(\gamma) = X$ for all $\gamma \in T$ then $K(\alpha) = X$. Therefore α holds in the Kripke model $\mathcal{M}$.

For the converse we shall use Theorem 1.3 and Proposition 2.3. Let $\rho : L(S) \to L(S)/\equiv_T$ given by $\rho(\alpha) = [\alpha]_T$. By (b) of Lemma 5.1 we have that $L(S)/\equiv_T \in \mathcal{SH}_n$, so in particular $\mathbf{X}(L(S)/\equiv_T)$ is a poset of height less or equal to n. Let $i : \mathbf{D}(\mathbf{X}(L(S)/\equiv_T)) \to (\mathbf{X}(L(S)/\equiv_T))^+$ be the inclusion morphism, and define $K : L(S) \to (\mathbf{X}(L(S)/\equiv_T))^+$ as $K = i\varphi\rho$, with $\varphi = \varphi_{L(S)/\equiv_T}$. By (b) of Lemma 5.1 we have that $K(S(\alpha)) = i\varphi\rho(S(\alpha)) = i\varphi(S(\rho(\alpha))) = i(\varphi(\rho(\alpha)) \cup [\varphi(\rho(\alpha))]^c_M) = \varphi(\rho(\alpha)) \cup [\varphi(\rho(\alpha))]^c_M$. On the other hand, by Proposition 2.3 we have that $S(K(\alpha)) = K(\alpha) \cup K(\alpha)^c_M = i(\varphi(\rho(\alpha)) \cup [i(\varphi(\rho(\alpha))]^c_M = \varphi(\rho(\alpha)) \cup [\varphi(\rho(\alpha))]^c_M$. Hence, it follows that $\mathcal{M} = (\mathbf{X}(L(S)/\equiv_T), K)$ is a Kripke model of T.

Suppose that $T \vdash_{A(S)_n} \gamma$ for all $\gamma \in T$. By (a) of Lemma 5.1 we have that $[\gamma]_T = 1$ for every $\gamma \in T$. In particular we have that $K(\gamma) = \mathbf{X}(L(S)/\equiv_T)$ for every $\gamma \in T$, and hence $K(\alpha) = \mathbf{X}(L(S)/\equiv_T)$. Using that $i\varphi$ is injective we have that $[\alpha]_T = 1$. Using (a) of Lemma 5.1 again, we conclude that $T \vdash_{A(S)_n} \alpha$. ■

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