# The Behaviour of Evanescent Waves under Lorentz Transformations.

J. L. AGUDÍN (*) and A. M. PLATZECK (*)

*Observatorio Astronómico, Universidad Nacional de La Plata - La Plata, Argentina*

J. R. ALBANO (**)

*ASTRA, Compañía Argentina de Petróleo - Buenos Aires, Argentina*

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In this work we shall show that the transversal magnetic (TM) or transversal electric (TE) electromagnetic evanescent waves have not independent existence: under Lorentz transformations an original TM *or* TE mode is transformed in a mixture of TM *and* TE modes out of phase in $\pi/2$. Through the well-known TORALDO DI FRANCIA expansion of the field of an uniformly moving charge in evanescent waves (1), we are able to give a physical reason for this result. An interesting feature of these transformations is that there exist two linear combinations of TM and TE modes (the left- and right-handed circularly polarized modes) which remain the same in all systems. These modes, which propagates with a mean velocity of energy transport equal to $c$ (2), have an interesting property concerning their helicities which has been recently discussed in the literature (3).

Let us begin by discussing polarization in the case of evanescent waves. Besides the direction of phase propagation (= real part of the wave vector $\boldsymbol{k}$), evanescent waves have another distinctive direction which is the direction of attenuation. These both directions define a plane which enables us to classify the evanescent solutions of Maxwell's equations in vacuum into two types according their polarization, the general case of arbitrary polarization being obtained by an appropriate linear combination of these two modes. Taking $\boldsymbol{k} = (k_x, k_y, iK) = (|\mathrm{Re}\,\boldsymbol{k}|\cos\theta, |\mathrm{Re}\,\boldsymbol{k}|\sin\theta, iK)$, with $k_x^2 + k_y^2 - K^2 = \omega^2/c^2$, we have

(*) Fellow of the Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.

(**) Postal address: ASTRA, Gcia. de Exp., Dto. de Inv., Leandro N. Alem 651, Buenos Aires, Argentina.

(1) G. TORALDO DI FRANCIA: *Nuovo Cimento*, **16**, 61 (1960).

(2) The mean velocity of energy transport (MVET), defined as the ratio of the time average of the Poynting vector to that of the energy density, has been introduced by SOMMERFELD and BRILLOUIN in their famous work on electromagnetic wave propagation in the region of anomalous dispersion (see for example W. K. H. PANOFSKY and M. PHILLIPS: *Classical Electricity and Magnetism*, Second Edition (Reading, Mass., 1962), p. 413). The fact that for a circularly polarized evanescent wave the MVET reaches $c$, has been given independently by J. RICARD (*Compt. Rend.*, **270**, 381 B (1970)) and by J. L. AGUDÍN (ICTP, Trieste, Int. Rep. IC/70/64 and *Phys. Lett.*, **35** A, 107 (1971)).

(3) J. L. AGUDÍN, A. M. PLATZECK and J. R. ALBANO: *Phys. Lett.*, **65** A, 77 (1978).

i) the transversal magnetic (TM) solution, with the magnetic field polarized normal to this plane

$$(1)\quad \begin{cases} E_x = i[k_x K/(k_x^2+k_y^2)] \ \ A_{\text{TM}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \\ E_y = i[k_y K/(k_x^2+k_y^2)] \ \ A_{\text{TM}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \\ E_z = -A_{\text{TM}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \\ H_x = -[\omega k_y/c(k_x^2+k_y^2)] A_{\text{TM}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \\ H_y = \quad [\omega k_x/c(k_x^2+k_y^2)] A_{\text{TM}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \\ H_z = 0 \,. \end{cases}$$

ii) The transversal electric (TE) solution, with the electric field polarized normal to this plane

$$(2)\quad \begin{cases} E_x = -[\omega k_y/c(k_x^2+k_y^2)] \ \ A_{\text{TE}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \\ E_y = \quad [\omega k_x/c(k_x^2+k_y^2)] \ \ A_{\text{TE}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \\ E_z = 0 \,, \\ H_x = -i[k_x K/(k_x^2+k_y^2)] A_{\text{TE}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \\ H_y = -i[k_y K/(k_x^2+k_y^2)] A_{\text{TE}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \\ H_z = A_{\text{TE}} \exp[-Kz] \exp[i(k_x x + k_y y - \omega t)] \,, \end{cases}$$

where $A_{\text{TM}}$ and $A_{\text{TE}}$ are arbitrary complex constants.

We shall now consider in one co-ordinate frame $S$ a TM solution as given by eq. (1). Let us imagine another system $S'$ which moves relative to $S$ with velocity $\boldsymbol{v} = (v, 0, 0)$. The transformed wave vector and frequency are

$$(3)\quad \boldsymbol{k}' = [k'_x, k'_y, k'_z] = [\gamma(k_x - \beta\omega/c), k_y, iK] \,, \qquad \omega' = \gamma(\omega - \beta c k_x) \,,$$

where $\gamma = (1-\beta^2)^{-\frac{1}{2}}$ and $\beta = v/c$.

The transformed fields are

$$(4)\quad \begin{cases} E'_x = i[k_x K/(k_x^2+k_y^2)] \, A_{\text{TM}} \exp[-Kz'] \exp[i(k'_x x' + k_y y' - \omega' t')] \,, \\ E'_y = i\gamma[k_y K/(k_x^2+k_y^2)] \, A_{\text{TM}} \exp[-Kz'] \exp[i(k'_x x' + k_y y' - \omega' t')] \,, \\ E'_z = \gamma[-1 + \{\beta\omega k_x/c(k_x^2+k_y^2)\}] A_{\text{TM}} \exp[-Kz'] \exp[i(k'_x x' + k_y y' - \omega' t')] \,, \\ H'_x = -[\omega k_y/c(k_x^2+k_y^2)] \, A_{\text{TM}} \exp[-Kz'] \exp[i(k'_x x' + k_y y' - \omega' t')] \,, \\ H'_y = \gamma[\{\omega k_x/c(k_x^2+k_y^2)\} - \beta] \, A_{\text{TM}} \exp[-Kz'] \exp[i(k'_x x' + k_y y' - \omega' t')] \,, \\ H'_z = -i\gamma\beta[k_y K/(k_x^2+k_y^2)] \, A_{\text{TM}} \exp[-Kz'] \exp[i(k'_x x' + k_y y' - \omega' t')] \,. \end{cases}$$

Since $H'_z \neq 0$, it follows at once that these fields do not belong to a pure TM evanescent wave in $S'$ (cf. eq. (1)). We shall show that these fields can be written as a linear combination of TM and TE evanescent waves of wave vector $\boldsymbol{k}'$.

The most general linear combination of TM and TE modes in $S'$ can be written as (cf. eqs. (1) and (2))

$$(5)\quad \begin{cases} E'_x = \{i[k'_x K/(k'^2_x + k^2_y)]A'_{\text{TM}} - [\omega' k_y/c(k'^2_x + k^2_y)]A'_{\text{TE}}\}\cdot \\ \qquad\qquad \cdot\exp[-Kz']\exp[i(k'_x x' + k_y y' - \omega' t')]\,, \\ E'_y = \{i[k_y K/(k'^2_x + k^2_y)]A'_{\text{TM}} + [\omega' k'_x/c(k'^2_x + k^2_y)]A'_{\text{TE}}\}\cdot \\ \qquad\qquad \cdot\exp[-Kz']\exp[i(k'_x x' + k_y y' - \omega' t')]\,, \\ E'_z = \{-A'_{\text{TM}}\}\cdot\exp[-Kz']\exp[i(k'_x x' + k_y y' - \omega' t')]\,, \\ H'_x = \{-[\omega' k_y/c(k'^2_x + k^2_y)]A'_{\text{TM}} - i[k'_x K/(k'^2_x + k^2_y)]A'_{\text{TE}}\}\cdot \\ \qquad\qquad \cdot\exp[-Kz']\exp[i(k'_x x' + k_y y' - \omega' t')]\,, \\ H'_y = \{[\omega' k'_x/c(k'^2_x + k^2_y)]A'_{\text{TM}} - i[k_y K/(k'^2_x + k^2_y)]A'_{\text{TE}}\}\cdot \\ \qquad\qquad \cdot\exp[-Kz']\exp[i(k'_x x' + k_y y' - \omega' t')]\,, \\ H'_z = \{A'_{\text{TE}}\}\cdot\exp[-Kz']\exp[i(k'_x x' + k_y y' - \omega' t')]\,, \end{cases}$$

where $A'_{\text{TM}}$ and $A'_{\text{TE}}$ are arbitrary complex constants. Thus what we must do now is to equate the fields given by eq. (4) to those given by eq. (5) and to determine $A'_{\text{TM}}$ and $A'_{\text{TE}}$. Equating the $z$-components of the electric field it results

$$(6)\qquad A'_{\text{TM}} = \gamma\{1 - [\beta\omega k_x/c(k^2_x + k^2_y)]\}A_{\text{TM}}\,.$$

Equating the $z$-component of the magnetic field it results

$$(7)\qquad A'_{\text{TE}} = -i\gamma\beta[k_y K/(k^2_x + k^2_y)]A_{\text{TM}}\,.$$

With a rather tedious algebra it can be shown that with these values of $A'_{\text{TM}}$ and $A'_{\text{TE}}$, the other field's components coincide.

To stress the fact that the amplitudes $A'_{\text{TM}}$ and $A'_{\text{TE}}$ in the $S'$ system stems from an original TM mode in $S$, it is convenient to write eqs. (6) and (7) as

$$(8)\qquad \begin{cases} A'_{\text{TM}} = C_{\text{TM,TM}}A_{\text{TM}}\,, \\ A'_{\text{TE}} = C_{\text{TE,TM}}A_{\text{TM}}\,, \end{cases}$$

where

$$(9)\qquad \begin{cases} C_{\text{TM,TM}} = \gamma\{1 - [\beta\omega k_x/c(k^2_x + k^2_y)]\} = \gamma\{1 - [\beta\omega/c|\text{Re}\,\boldsymbol{k}|]\cos\theta\}\,, \\ C_{\text{TE,TM}} = -i\beta\gamma k_y K/(k^2_x + k^2_y) = -i\beta\gamma[K/|\text{Re}\,\boldsymbol{k}|]\sin\theta\,. \end{cases}$$

In a similar way it can be shown that an original TE mode in $S$ as given by eq. (2), gives rise to a linear combination of TE and TM modes in $S'$, with amplitudes $A'_{\text{TE}}$ and $A'_{\text{TM}}$ given by

$$\text{(10)} \qquad \begin{cases} A'_{\text{TE}} = C_{\text{TE,TE}} A_{\text{TE}}\,, \\ A'_{\text{TM}} = C_{\text{TM,TE}} A_{\text{TE}}\,, \end{cases}$$

where

$$\text{(11)} \qquad \begin{cases} C_{\text{TE,TE}} = \gamma\{1 - [\beta\omega k_x/c(k_x^2 + k_y^2)]\} = \gamma\{1 - [\beta\omega/c|\text{Re}\,\boldsymbol{k}|]\cos\theta\}\,, \\ C_{\text{TM,TE}} = i\beta\gamma k_y K/(k_x^2 + k_y^2) = i\beta\gamma[K/|\text{Re}\,\boldsymbol{k}|]\sin\theta\,. \end{cases}$$

Expressions (8) and (9) (or (10) and (11)) gives us the way in which an original TM (or TE) evanescent mode in $S$ is transformed in a linear combination of TM and TE evanescent modes in $S'$.

Let us discuss these results. For $K = 0$ (plane waves), we have $C_{\text{TE,TM}} = C_{\text{TM,TE}} = 0$ and thus polarization is conserved (4). For $\theta = 0$ we also obtain $C_{\text{TE,TM}} = C_{\text{TM,TE}} = 0$, which shows that polarization is conserved when moving along the phase propagation direction. For any other value of $\theta$, the original TM (TE) evanescent wave in $S$ gives rise to a mixture of TM and TE evanescent waves in $S'$, out of phase in $\pi/2$.

This is a good place to discuss our work in relation to a previous one by G. TORALDO DI FRANCIA (1): this will give us some insight into the physical reasons by which a TM mode in $S$ gives rise to a mixture of TM and TE modes in $S'$. TORALDO DI FRANCIA have shown that the field generated by a charged particle moving freely in empty space with velocity $\boldsymbol{v} = (v, 0, 0)$ can be exactly expanded into a set of TM and TE evanescent waves, the representation being valid in a whole half-space having no points in common with the particle's track. Since this expansion is exact, it is possible to take the limit when the particle's velocity goes to zero. It can be seen that in this limit the contribution of each TE mode vanishes, and each TM mode degenerates into a TM evanescent wave of zero frequency (for which the magnetic field vanishes). Thus, what we obtain in the $S$ system by taking the limit $\beta \to 0$ in the TORALDO DI FRANCIA expansion is *a representation of the Coulomb electrostatic field in* TM *evanescent waves of zero frequency* (5). Let us now perform a Lorentz transformation to a system $S'$ which is moving with velocity $\boldsymbol{V} = (-v, 0, 0)$ with respect to the system $S$ in which the charge is now at rest. In this system $S'$ it is evident that the original TORALDO DI FRANCIA expansion in TM *and* TE evanescent waves remains valid. Thus, it is a physical requirement that each TM mode of the expansion of the Coulomb electrostatic field in $S$ must be transformed in a mixture of TM *and* TE modes in $S'$.

We shall give now what we think it is an interesting property of the transformations given by eqs. (8) to (11). Let us consider in the $S$ system a linear combination of TM and TE modes of equal wave vector $\boldsymbol{k} = (k_x, k_y, iK)$ and amplitudes $A_{\text{TM}}$ and $A_{\text{TE}} = \alpha A_{\text{TM}}$. This gives rise in the $S'$ system to another linear combination of TM and TE modes

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(4) For the case of plane waves ($K = 0$), the definition of polarization given early for evanescent waves losses its meaning. What we mean by « conservation of polarization » in this case is that by performing a Lorentz transformation in the plane determined by the direction of propagation and the magnetic (electric) field, the magnetic (electric) field remains parallel to this plane, and the electric (magnetic) field remains normal to it.

(5) J. L. AGUDÍN and A. M. PLATZECK: *On the limit for $\beta \to 0$ of the Toraldo di Francia expansion*, *Lett. Nuovo Cimento*, **22**, 13 (1978).

of amplitudes $A'_{\text{TM}}$ and $A'_{\text{TE}}$ given by

$$A'_{\text{TM}} = C_{\text{TM,TM}} A_{\text{TM}} + C_{\text{TM,TE}} A_{\text{TE}} = A_{\text{TM}}(C_{\text{TM,TM}} + \alpha C_{\text{TM,TE}}) ,$$
$$A'_{\text{TE}} = C_{\text{TE,TM}} A_{\text{TM}} + C_{\text{TE,TE}} A_{\text{TE}} = A_{\text{TM}}(C_{\text{TE,TM}} + \alpha C_{\text{TE,TE}}) .$$

Is there any linear combination in $S$ which remains the same in $S'$ under *any* Lorentz transformation? Clearly we must have $A'_{\text{TE}} = \alpha A'_{\text{TM}}$ which implies

$$C_{\text{TE,TM}} + \alpha C_{\text{TE,TE}} = \alpha \{C_{\text{TM,TM}} + \alpha C_{\text{TM,TE}}\} .$$

By taking into account eqs. (9) and (11), this condition reduces to

$$\beta K \sin\theta \, \alpha^2 = -\beta K \sin\theta . \tag{12}$$

In the most general case ($\beta \neq 0, \theta \neq 0, K \neq 0$), eq. (12) implies $\alpha = \pm i$. These are just the left- and right-handed circularly polarized modes which has been discussed in ref. ([3]). What has been proved there is that the projection of the angular momentum of a circularly polarized electromagnetic evanescent wave along the mean velocity of energy transport (= helicity) can be reversed by a Lorentz transformation, in spite of the fact that this velocity is $c$ ([6]).

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([6]) The fact that a circularly polarized electromagnetic *homogeneous plane* wave carries angular momentum (which results proportional to its energy), stems from the model discussed, among others, by R. FEYNMAN (*The Feynman Lectures on Physics*, Vol. **3**, Chap. 17 (Reading, Mass., 1965), p. 10) and by B. ROSSI (*Optics* (Reading, Mass., 1957), p. 411). In ref. ([3]) it has been tacitly assumed that this model remains valid for circularly polarized *evanescent* waves.