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WEIGHTED A PRIORI ESTIMATES FOR SOLUTION OF $(-\Delta)^m u = f$ WITH HOMOGENEOUS DIRICHLET CONDITIONS

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Abstract. Let u be a weak solution of $(-\Delta)^m u = f$ with Dirichlet boundary conditions in a smooth bounded domain $\Omega \subset \mathbf{R}^n$. Then, the main goal of this paper is to prove the following a priori estimate:

$$\|u\|_{W^{2m,p}_\omega(\Omega)} \le C\,\|f\|_{L^p_\omega(\Omega)},$$

where ω is a weight in the Muckenhoupt class A_p.

Key words: *Dirichlet problem, Green function, Calderón-Zygmund theory, weighted Sobolev space*

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1 Introduction

We will use the standard notation for Sobolev spaces and for derivatives, namely, if α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{Z}^n_+$ we denote $|\alpha| = \sum_{j=1}^{n} \alpha_j$, $D^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$ and

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), \quad \forall\, |\alpha| \le k\}.$$

For $u \in W^{k,p}(\Omega)$, its norm is given by

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}.$$

We consider the homogeneous problem

$$\begin{cases} (-\Delta)^m u = f, & \text{in } \Omega, \\ \\ \left(\frac{\partial}{\partial \nu}\right)^j u = 0, & \text{in } \partial\Omega, \qquad 0 \le j \le m-1, \end{cases} \tag{1.1}$$

where $\dfrac{\partial}{\partial \nu}$ is the normal derivative.

In the classic paper [1], the authors obtained a priori estimates for solutions of (1.1) for a smooth domain Ω given by

$$\|u\|_{W^{2m,p}(\Omega)} \le C \|f\|_{L^p(\Omega)}.$$

A key tool to prove those estimates was the Calderón-Zygmund theory for singular integral operators.

On the other hand, after the pioneering work of Muckenhoupt [7], a lot of work on continuity in weighted norms has been developed. In particular, weighted estimates for a wide class of singular integral operators have been obtained for weights in the class of Muckenhoupt A_p. Therefore, it is a natural question whether analogous weighted a priori estimates can be proved for the derivatives of solutions of elliptic equations.

For the Laplace equation ($m = 1$), it was proved in[5] that for a weight ω belonging to the Muckenhoupt class A_p

$$\|u\|_{W^{2,p}_\omega(\Omega)} \le C \|f\|_{L^p_\omega(\Omega)}$$

on a bounded domain Ω with $\partial\Omega \in C^2$.

The goal of this paper is to extend the results of [5] for powers of the Laplacian operator with homogeneous Dirichlet boundary conditions, i.e. it is to prove that

$$\|u\|_{W^{2m,p}_\omega(\Omega)} \le C \|f\|_{L^p_\omega(\Omega)}, \tag{1.2}$$

for $\omega \in A_p$, where the constant C depends on Ω, m, n and the weight ω.

The main ideas for the proof of these estimates are similar to those given in [5]. However, non trivial technical modifications are needed because, for $m \ge 2$, the Green function is not positive in general and therefore, we cannot apply the maximum principle.

2 Preliminaries

In what follows we consider the problem (1.1) in a bounded domain Ω with $\partial\Omega \in C^{6m+4}$ for $n = 2$ and $\partial\Omega \in C^{5m+2}$ for $n > 2$ (the regularity on the boundary is necessary in order to use the results of the Green function given in [6]).

The solution of (1.1) is given by

$$u(x)=\int_\Omega G_m(x,y)\,f(y)\,\mathrm{d}y, \tag{2.1}$$

where $G_m(x,y)$ is the Green function of the operator $(-\Delta)^m$ in Ω which can be written as

$$G_m(x,y)=\Gamma(x-y)+h(x,y), \tag{2.2}$$

where $\Gamma(x-y)$ is a fundamental solution and $h(x,y)$ satisfies

$$\begin{cases} (-\Delta_x)^m h(x,y)=0, & x\in\Omega,\\ \left(\frac{\partial}{\partial\nu}\right)^j h(x,y)=-\left(\frac{\partial}{\partial\nu}\right)^j\Gamma(x-y), & x\in\partial\Omega,\quad 0\le j\le m-1\end{cases}$$

for each fixed $y\in\Omega$.

Then

$$h(x,y)=-\sum_{j=0}^{m-1}\int_{\partial\Omega}K_j(y,P)\left(\frac{\partial}{\partial\nu}\right)^j\Gamma(P-x)\,\mathrm{d}S, \tag{2.3}$$

where $K_j(y,P)$ are the Poisson kernels and dS denotes the surface measure on $\partial\Omega$.

We recall that any fundamental solution associated to (1.1) is smooth away from the origin and it is homogeneous of degree $2m-n$ if n is odd or if $2m<n$ and the logarithmic function appears if n is even and $2m\ge n$. However, in both cases, under our assumption on the boundary domain, we have the known estimates of the Green function $G_m(x,y)$ and the Poisson kernels $K_j(x,y)$. In what follows the letter C will denote a generic constant not necessarily the same at each occurrence.

The following are known facts:

$$|D_x^\alpha G_m(x,y)|\le C\quad \text{for } |\alpha|<2m-n, \tag{2.4}$$

$$|D_x^\alpha G_m(x,y)|\le C\log\left(\frac{2\,diam(\Omega)}{|x-y|}\right)\quad \text{for } |\alpha|=2m-n, \tag{2.5}$$

$$|D_x^\alpha G_m(x,y)|\le C\,|x-y|^{2m-n-|\alpha|}\quad \text{for } |\alpha|>2m-n, \tag{2.6}$$

$$|D_x^\alpha G_m(x,y)|\le C\,\frac{1}{|x-y|^n}\,\min\left\{1,\,\frac{d(y)}{|x-y|}\right\}^m\quad \text{for}\quad |\alpha|=2m, \tag{2.7}$$

$$|K_j(x,y)|\le C\,\frac{d(x)^m}{|x-y|^{n-j+m-1}}\quad \text{for } 0\le j\le m-1, \tag{2.8}$$

where $d(x):=\mathrm{dist}(x,\partial\Omega)$ (see [6] for (2.4), (2.5) and (2.6) and [4] for (2.7) and (2.8)).

3 The Estimates for the Derivatives of u

In this section we state pointwise estimates for the first $2m-1$ derivatives of the function u and a weak estimate for the $2m$ derivative. These estimates are needed for proving the main result of this work.

Lemma 3.1. *Let $u(x)$ be the solution of (1.1). Then, for $|\alpha| \leq 2m-1$ we have*

$$|D_x^\alpha u(x)| \leq C M f(x),$$

where $Mf(x)$ is the usual Hardy- Littlewood maximal function of f.

Proof.

$$\begin{aligned} |D_x^\alpha u(x)| &\leq \int_\Omega |D^\alpha G_m(x,y)|\,|f(y)|\,\mathrm{d}y \\ &\leq C \int_\Omega \frac{|f(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \leq C M f(x), \end{aligned}$$

by (2.4), if $2m-n+1 \leq |\alpha| \leq 2m-1$ and by (2.5) and (2.6), if $|\alpha| \leq 2m-n$.

Proposition 3.1. *Given two measurable functions f and g in Ω, for $|\alpha| = 2m$ we have that*

$$\int_D |D_x^\alpha G_m(x,y)\, f(y)\, g(x)|\,\mathrm{d}y\,\mathrm{d}x \leq C \left(\int_\Omega Mf(x)\,|g(x)|\,\mathrm{d}x + \int_\Omega Mg(y)\,|f(y)|\,\mathrm{d}y \right),$$

where $D := \{(x,y) \in \Omega \times \Omega : |x-y| > d(x)\}$.

Proof. We write $D = D_1 \cup D_2$, where

$$D_1 = \{(x,y) \in D : d(y) \leq 2d(x)\} \quad \text{and} \quad D_2 = \{(x,y) \in D : d(y) > 2d(x)\}.$$

Then, using (2.7) we have

$$\begin{aligned} \int_D |D_x^\alpha G_m(x,y)\, f(y)\, g(x)|\,\mathrm{d}y\,\mathrm{d}x &\leq \int_D \frac{\mathrm{d}(y)^m}{|x-y|^{n+m}}\,|f(y)|\,|g(x)|\,\mathrm{d}y\,\mathrm{d}x \\ &\leq 2^m \int_{D_1} \frac{\mathrm{d}(x)^m}{|x-y|^{n+m}}\,|f(y)|\,|g(x)|\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_{D_2} \frac{\mathrm{d}(y)^m}{|x-y|^{n+m}}\,|f(y)|\,|g(x)|\,\mathrm{d}y\,\mathrm{d}x = I + II. \end{aligned} \tag{3.1}$$

Calling $\Omega_k(x) = \{z \in \Omega : 2^k d(x) \leq |x-z| < 2^{k+1} d(x)\}$,

$$\begin{aligned} \int_{D_1} \frac{\mathrm{d}(x)^m}{|x-y|^{n+m}}\,|f(y)|\,|g(x)|\,\mathrm{d}y\,\mathrm{d}x &\leq \int_\Omega \sum_{k=1}^{\infty} \int_{\Omega_k(x)} \frac{\mathrm{d}(x)}{|x-y|^{n+1}}\,|f(y)|\,\mathrm{d}y\,|g(x)|\,\mathrm{d}x \\ &= \int_\Omega A(x)\,|g(x)|\,\mathrm{d}x \end{aligned}$$

with

$$A(x) \leq \sum_{k=1}^{\infty} \int_{\{|x-y|<2^{k+1}\mathrm{d}(x)\}} \frac{d(x)}{|x-y|^{n+1}} |f(y)| \mathrm{d}y \leq 2^n \sum_{k=1}^{\infty} \frac{1}{2^k} Mf(x) = 2^n Mf(x).$$

In order to estimate the term II in (3.1), we first observe that for $(x,y) \in D_2$, we have that $|x-y| \geq \frac{1}{2} d(y)$. Then

$$\begin{aligned} \int_{D_2} \frac{\mathrm{d}(y)^m}{|x-y|^{n+m}} |f(y)| |g(x)| \mathrm{d}y \mathrm{d}x &\leq C \int_{\Omega} \sum_{k=1}^{\infty} \int_{\Omega_{k-1}(y)} \frac{\mathrm{d}(y)}{|x-y|^{n+1}} |g(x)| \mathrm{d}x |f(y)| \mathrm{d}y \\ &= \int_{\Omega} B(y) |f(y)| \mathrm{d}y \end{aligned}$$

and therefore, by the same arguments used before we have that

$$B(y) \leq CMg(y)$$

and the Proposition is proved.

In order to see how to estimate in $\Omega \setminus D$, we consider separately the function h and Γ involved in G_m.

Proposition 3.2. *If* $|\alpha| \geq 2m-n+1$*, there exists a constant C such that*

$$|D^{\alpha} h(x,y)| \leq C d(x)^{2m-n-|\alpha|} \tag{3.2}$$

for $|x-y| \leq d(x)$.

Proof. In view of (2.3) we must find estimates for $D_x^{\alpha}(\frac{\partial}{\partial \nu})^j \Gamma(P-x)$ and $K_j(y,P)$.

From the general properties of the fundamental solution $\Gamma(x-y)$ we have that

$$\left| D_x^{\alpha}(\frac{\partial}{\partial \nu})^j \Gamma(P-x) \right| \leq C |P-x|^{2m-n-|\alpha|-j} \tag{3.3}$$

for $|\alpha|+j \geq 2m-n+1$, and for $0 \leq j \leq m-1$, by (2.8) we have that

$$|K_j(y,P)| \leq C \frac{\mathrm{d}(y)^m}{|y-P|^{n-j+m-1}} \tag{3.4}$$

for $y \in \Omega$ and $P \in \partial\Omega$.

Then by (3.3), (3.4) and the fact that if $|x-y| \leq \mathrm{d}(x)$ then $\mathrm{d}(y) < 2\,\mathrm{d}(x)$, we have for $|\alpha|+j \geq 2m-n+1$

$$\begin{aligned} |D_x^{\alpha} h(x,y)| &\leq C \sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{\mathrm{d}(y)^m}{|y-P|^{n-1+m-j}} |P-x|^{2m-n-|\alpha|-j} \mathrm{d}S \\ &\leq C \mathrm{d}(x)^{2m-n-|\alpha|} \sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{\mathrm{d}(y)^{m-j}}{|y-P|^{n-1+m-j}} \mathrm{d}S. \end{aligned}$$

In order to see that each integral is finite we write $\partial\Omega = F_1 \cup F_2$, with

$$F_1 = \{P \in \partial\Omega : |P_0 - P| > 2\,\mathrm{d}(y)\} \quad \textit{and} \quad F_2 = \{P \in \partial\Omega : |P_0 - P| \leq 2\,\mathrm{d}(y)\},$$

where $P_0 \in \partial\Omega$ is that $|y - P_0| = \mathrm{d}(y)$. And now, the convergence of these integrals follows in a standard way.

It follows from the previous Proposition that for each $x \in \Omega$ and $|\alpha| \geq 2m - n + 1$ we have that $D_x^\alpha h(x,y)$ is bounded uniformly in a neighborhood of x and so is

$$D_x^\alpha \int_\Omega h(x,y)\, f(y)\,\mathrm{d}y = \int_\Omega D_x^\alpha h(x,y)\, f(y)\,\mathrm{d}y. \tag{3.5}$$

On the other hand, although $D_x^\alpha \Gamma$ is a singular kernel for $|\alpha| = 2m$, taking β such that $|\beta| = 2m - 1$, we have

$$D_{x_i} \int_\Omega D_x^\beta \Gamma(x-y)\, f(y)\,\mathrm{d}y = K f(x) + c(x) f(x), \tag{3.6}$$

where c is a bounded function and K is a Calderón - Zygmund operator given by

$$K f(x) = \lim_{\varepsilon \to 0} K_\varepsilon f(x), \;\; \text{with} \;\; K_\varepsilon f(x) = \int_{|x-y|>\varepsilon} D_x^\alpha \Gamma(x-y)\, f(y)\,\mathrm{d}y. \tag{3.7}$$

We will also make use of the maximal operator $\tilde{K} f(x) = \sup_{\varepsilon>0} |K_\varepsilon f(x)|$. Here and in what follows we consider f defined in $\mathbf{R}^n$ extending the original f by zero.

Now we are in conditions to give the following estimate:

Theorem 3.3. *Given g a measurable function and* $|\alpha| = 2m$. *Then there exists a constant C depending only on n, m and* Ω *such that for any* $x \in \Omega$,

$$\begin{aligned} \int_\Omega |D_x^\alpha u(x)\, g(x)|\,\mathrm{d}x \;\; &\leq \;\; C \left(\int_\Omega \widetilde{K} f(x)\, |g(x)|\,\mathrm{d}x + \int_\Omega M f(x)\, |g(x)|\,\mathrm{d}x \right. \\ &+ \;\; \left. \int_\Omega M g(y)\, |f(y)|\,\mathrm{d}y + \int_\Omega |f(x)|\, |g(x)|\,\mathrm{d}x \right) \end{aligned}$$

Proof. Using the representation formula for u, by (3.5), (3.6) and (3.7) we have

$$D_x^\alpha u(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon < |x-y| \leq \mathrm{d}(x)} D_x^\alpha \Gamma(x-y)\, f(y)\,\mathrm{d}y + c(x) f(x)$$

$$+ \int_{|x-y| \leq d(x)} D_x^\alpha h(x,y)\, f(y)\,\mathrm{d}y + \int_{|x-y| > d(x)} D_x^\alpha G(x,y)\, f(y)\,\mathrm{d}y$$

$$=: I + II + III + IV. \tag{3.8}$$

By the results given above, for I, II and III we have pointwise estimates, and obtain (in the same way as in [5]) that

$$|I+II+III| \leq C\left(\widetilde{K}f(x)+|f(x)|+Mf(x)\right).$$

However, for IV we have just a weak estimate. Indeed, from Proposition **??** we have

$$\int_\Omega |IV|\,|g(x)|\,\mathrm{d}x \quad \leq \quad C\left(\int_\Omega Mf(x)\,|g(x)|\,\mathrm{d}x+\int_\Omega Mg(y)\,|f(y)|\,\mathrm{d}y\right)$$

and the Theorem is proved.

4 Main Result

We can now state and prove our main result. First we recall the definition of the A_p class for $1<p<\infty$. A non-negative locally integrable function ω belongs to A_p if there exists a constant C such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1} \leq C$$

for all cubes $Q\subset \mathbf{R}^n$.

For any weight ω, $L^p_\omega(\Omega)$ is the space of measurable functions f defined in Ω such that

$$\|f\|_{L^p_\omega(\Omega)}=\left(\int_\Omega |f(x)|^p\,\omega(x)\,\mathrm{d}x\right)^{1/p}<\infty$$

and $W^{k,p}_\omega(\Omega)$ is the space of functions such that

$$\|f\|_{W^{k,p}_\omega(\Omega)}=\left(\sum_{|\alpha|\leq k}\|D^\alpha f\|^p_{L^p_\omega(\Omega)}\right)^{1/p}<\infty.$$

Theorem 4.1. *Let $\Omega\subset\mathbf{R}^n$ be a bounded domain such that $\partial\Omega$ is of class C^{6m+4} for $n=2$ and $\partial\Omega$ is of class C^{5m+2} for $n\geq 2$. If $\omega\in A_p$, $f\in L^p_\omega(\Omega)$ and u a weak solution of* (1.1), *then there exists a constant C depending only on n, m, ω and Ω such that*

$$\|u\|_{W^{2m,p}_\omega(\Omega)}\leq C\,\|f\|_{L^p_\omega(\Omega)}.$$

Proof. Since M is a bounded operator in $L^p_\omega(\Omega)$, by Lemma 3.1 it follows that

$$\sum_{|\alpha|\leq 2m-1}\|D^\alpha_x u\|_{L^p_\omega(\Omega)}\leq C\,\|f\|_{L^p_\omega(\Omega)}.$$

Therefore, it only remains to estimate $\|D^\alpha_x u\|_{L^p_\omega(\Omega)}$ for $|\alpha|=2m$.

Let $\omega \in A_p$ and $g(x) := (D_x^\alpha u(x))^{p-1}\,\omega(x)$. By Theorem **??** we see that

$$\begin{aligned}\int_\Omega |D_x^\alpha u(x)|^p\,\omega(x)\,\mathrm{d}x &= \int_\Omega |D_x^\alpha u(x)|\,g(x)\,\mathrm{d}x\\ &\le C\left(\int_\Omega \widetilde{K}f(x)\,|g(x)|\,\mathrm{d}x + \int_\Omega Mf(x)\,|g(x)|\,\mathrm{d}x\right.\\ &+ \left.\int_\Omega Mg(y)\,|f(y)|\,\mathrm{d}y + \int_\Omega |f(x)|\,|g(x)|\,\mathrm{d}x\right). \qquad (4.1)\end{aligned}$$

Since $\tilde{K}$ and M are bounded operators in $L^p_\omega(\Omega)$, applying the Hölder inequality, it follows that

$$\begin{aligned}\int_\Omega \widetilde{K}f(x)\,|g(x)|\,\mathrm{d}x &= \int_\Omega \widetilde{K}f(x)\,|g(x)|\,\frac{1}{\omega(x)^{1/p}}\,\omega(x)^{1/p}\,\mathrm{d}x\\ &\le \left(\int_\Omega \widetilde{K}f(x)^p\,\omega(x)\,\mathrm{d}x\right)^{1/p}\left(\int_\Omega |g(x)|^q\,\frac{1}{\omega(x)^{q/p}}\,\mathrm{d}x\right)^{1/q}\\ &\le \|f\|_{L^p_\omega(\Omega)}\left(\int_\Omega |g(x)|^q\,\frac{1}{\omega(x)^{q/p}}\,\mathrm{d}x\right)^{1/q}, \qquad (4.2)\end{aligned}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$.

In the same way, we obtain that

$$\int_\Omega Mf(x)\,|g(x)|\,\mathrm{d}x \le \|f\|_{L^p_\omega(\Omega)}\left(\int_\Omega |g(x)|^q\,\frac{1}{\omega(x)^{q/p}}\,\mathrm{d}x\right)^{1/q} \qquad (4.3)$$

and

$$\int_\Omega |f(x)|\,|g(x)|\,\mathrm{d}x \le \|f\|_{L^p_\omega(\Omega)}\left(\int_\Omega |g(x)|^q\,\frac{1}{\omega(x)^{q/p}}\,\mathrm{d}x\right)^{1/q}. \qquad (4.4)$$

For the last term in (4.1), taking into account that $\omega^{-q/p} \in A_q$, we have

$$\begin{aligned}\int_\Omega Mg(y)\,|f(y)|\,\mathrm{d}y &\le \|f\|_{L^p_\omega(\Omega)}\left(\int_\Omega Mg(y)^q\,\frac{1}{\omega(y)^{q/p}}\,\mathrm{d}y\right)^{1/q} \qquad (4.5)\\ &\le \|f\|_{L^p_\omega(\Omega)}\left(\int_\Omega |g(x)|^q\,\frac{1}{\omega(x)^{q/p}}\,\mathrm{d}x\right)^{1/q}.\end{aligned}$$

Then, by (4.2), (4.3), (4.4) and (4.5)we have

$$\|D_x^\alpha u\|^p_{L^p_\omega(\Omega)} \le C\,\|f\|_{L^p_\omega(\Omega)}\left(\int_\Omega |g(x)|^q\,\frac{1}{\omega(x)^{q/p}}\,\mathrm{d}x\right)^{1/q}.$$

By the definition of $g(x)$,

$$\begin{aligned}\left(\int_\Omega |g(x)|^q \frac{1}{\omega(x)^{q/p}}\,\mathrm{d}x\right)^{1/q} &= \left(\int_\Omega |D_x^\alpha u|^{(p-1)q}\,\omega(x)^q \frac{1}{\omega(x)^{q/p}}\,\mathrm{d}x\right)^{1/q}\\ &= \left(\int_\Omega |D_x^\alpha u|^p\,\omega(x)\,\mathrm{d}x\right)^{1/q} = \|D_x^\alpha u\|_{L^p_\omega(\Omega)}^{p/q}.\end{aligned}$$

Then we obtain

$$\|D_x^\alpha u\|^p_{L^p_\omega(\Omega)} \le C\,\|f\|_{L^p_\omega(\Omega)}\,\|D_x^\alpha u\|^{p/q}_{L^p_\omega(\Omega)} \tag{4.6}$$

and the Theorem is proved for $u \in W_\omega^{2m,p}(\Omega)$.

Finally, we will show that the weak solutio u of (1.1) belongs to $W_\omega^{2m,p}(\Omega)$.

We have $(-\Delta)^m u = f$, with $f \in L^p_\omega(\Omega)$, then there exists a sequence $f_k \in C^\infty(\mathbf{R}^n)$ such that $\lim\limits_{k\to\infty} f_k = f$ in $L^p_\omega(\Omega)$ [3].

For each k, there exists $u_k \in C^\infty(\Omega)$ satisfying

$$\begin{cases} (-\Delta)^m u_k = f_k, & \text{in } \Omega,\\ \left(\frac{\partial}{\partial\nu}\right)^j u_k = 0, & \text{in } \partial\Omega \quad 0 \le j \le m-1.\end{cases}$$

It is easy to see, from Lemma 3.1 that $u_k \in W_\omega^{2m-1,p}(\Omega)$, and obviously $u_k \in W^{2m,p}_{\omega,loc}(\Omega)$. Moreover, for all compact sets $K \subset \Omega$, we have

$$\|u_k\|_{W_\omega^{2m,p}(K)} \le C(K),$$

where $C(K)$ is a constant depending on the measure of K. Indeed, taking $v_k = u_k\varphi$ with $\varphi \in C_0^\infty(K)$, it follows that $v_k \in W_\omega^{2m,p}(\Omega)$, satisfies (1.1) with $f = g_k \in L^p_\omega(\Omega)$, and we can use (4.6).

Then, it follows from the dominated convergence theorem that $u_k \in W_\omega^{2m,p}(\Omega)$ and applying (4.6), we have

$$\|u_k\|_{W_\omega^{2m,p}(\Omega)} \le C\,\|f_k\|_{L^p_\omega(\Omega)}.$$

Therefore, $\{u_k\}$ is a Cauchy sequence in $W_\omega^{2m,p}(\Omega)$ and there exists $v \in W_\omega^{2m,p}(\Omega)$ such that $\lim\limits_{k\to\infty} u_k = v$ in $W_\omega^{2m,p}(\Omega)$. Let us see now that v solves (1.1).

Obviously, $f = \lim\limits_{k\to\infty} f_k = \lim\limits_{k\to\infty}(-\Delta)^m u_k = (-\Delta)^m v$ in $L^p_\omega(\Omega)$ and by the classical trace theorems in Sobolev spaces and the definition of $\omega \in A_p$, it follows that v satisfies the homogeneous boundary conditions and by uniqueness of the solution, the Theorem is proved.

Remark 4.2. The result of Theorem 4.1 is also valid for u a weak solution of

$$\begin{cases} \mathcal{L}u = f, & \text{in } \Omega, \\ \\ \mathcal{B}_j u = 0, & \text{in } \partial\Omega \quad 0 \le j \le m-1 \end{cases}$$

when $\mathcal{L} := \sum_{|\alpha|\le 2m} a_\alpha D^\alpha$ is uniformly elliptic and $\mathcal{B}_j := \sum_{|\alpha|\le m_j} b_\alpha D^\alpha$, $0 \le j \le m-1$ are the boundary operators defined in [1].

Indeed, we define $l_1 > \max_j(2m - m_j)$ and $l_0 = \max_j(2m - m_j)$. If the coefficients $a_\alpha \in C^{l_1+1}(\overline{\Omega})$, $b_{j,\alpha} \in C^{l_1+1}(\partial\Omega)$ and $\partial\Omega \in C^{l_1+2m+1}$ we have that the Green function G_m and the Poisson kernels K_j for $0 \le j \le m-1$ exist whenever $l_1 > 2(l_0+1)$ for $n = 2$ and $l_1 > \dfrac{3}{2} l_0$ for $n \ge 3$.

Moreover, wherever they are defined, the Green function and the Poisson kernels of the operator $\mathcal{L}$ with these boundary conditions satisfy the estimates (2.4), (2.5), (2.6), (2.7) and (2.8) (see [4] and [6]).

Remark 4.3. Using the fact that $d(x)^\beta \in A_p$ for $-1 < \beta < p-1$ and generalizing the classical imbedding Theorems for Sobolev spaces to weighted Sobolev spaces (as we have done in [5], Theorem 3.4) we have as a consequence of the main result: Under the hypotheses of Theorem **??** with $\omega = d^\gamma$, where $\gamma = k\beta$, $k \in \mathbf{N}$ and $0 \le \beta \le 1$. If $0 \le \gamma < p-1$ and $1/p - 1/q \le 2m/(n+k)$ (with $q < \infty$ when $2mp = n+k$), then there exists a constant C depending only on γ , p, q, n and Ω such that

$$\|u\|_{L^q_{d^\gamma}(\Omega)} \le C\,\|f\|_{L^p_{d^\gamma}(\Omega)}. \tag{4.7}$$

Finally, as a particular case of (4.7) taking $\gamma = m$ we have

$$\|u\|_{L^q_{d^m}(\Omega)} \le C\,\|f\|_{L^p_{d^m}(\Omega)}$$

for $p > m+1$ and $1/p - 1/q \le 2m/(n+1)$ (with $q < \infty$ when $2mp = n+m$).

This result is proved in [4] using different arguments for the case $1/p - 1/q < 2m/(n+1)$. Our results show that, at least in the case $p > m+1$, the estimate remains valid when

$$\frac{1}{p} - \frac{1}{q} = \frac{2m}{n+m}.$$

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