IL NUOVO CIMENTO VOL. 103 A, N. 4 Aprile 1990

Quantization of a 4th-Order Wave Equation (*).

D. G. BARCI and M. C. ROCCA
Departamento de Fisica, Facultad de Ciencias Exactas
Universidad Nacional de la Plata, Argentina
Consejo Nacional de Investigaciones Cientificas y Tecnicas, Argentina

C. G. BOLLINI
Departamento de Fisica, Facultad de Ciencias Exactas
Universidad Nacional de La Plata, Argentina
Comision de Investigaciones Cientificas de la Provincia de Buenos Aires, Argentina

(ricevuto il 30 Maggio 1989)

Summary. — The quantization of a scalar field obeying a fourth-order wave equation is described. Faster-than-light particles appearing in the process are treated as ghosts, with the wrong relation between spin and statistics. In this way, positive energy spectrum, as well as positive metric in Fock space are obtained. Finally, some implications of an interaction described by a Lagrangian symmetrized in the scalar fields are briefly discussed.

PACS 03.70 – Theory of quantized fields.
PACS 14.80.Kx – Others and hypothetical particles (including photons and tachyons).
PACS 11.30.Pb – Supersymmetry.

1. – Introduction.

The use of higher-order differential equations in particle theory has been considered every now and then, but they have almost always been discarded due to the difficulties they present. For example, the energy spectrum can be

(*) To speed up publication, the authors of this paper have agreed to not receive the proofs for correction.

unbounded from below, the Hilbert space of states could have an indefinite metric, the S-matrix may turn out not to be unitary, etc. (see however ref. [1]). The case of only one significative variable has been studied by mathematicians [2]. They establish a one-to-one correspondence between an N-th-order differential operator and the so-called «scattering data», obtained as the discontinuities of the Jost-functions at certain rays of the complex energy plane. (See ref. [3] for an application to a 4th-order wave equation.)

One can observe that these data contain physical as well as unphysical jumps and poles of the Jost functions. The inverse problem, which is a generalization of the classical Gelfand-Levitan result [4] has also been solved (see ref. [2] and [5]).

If we then restrict the data to be physical, a definite (small) subset of differential equation emerges, which should be able to represent physical processes [6]. These equations have a well-defined relation between the coefficient functions of the different derivatives (the «potentials») so they cannot be arbitrary.

Recently, an extension of the Wess-Zumino supersymmetric model [7] has been proposed, which ties the order of the wave equation to the dimensionality of space-time [8]. Perhaps, the simplest case, within this proposal, is to take $d = 6$, for which a fourth-order wave equation is obeyed by the components of the chiral superfield.

We have studied this case and the corresponding quantization will be presented in a forthcoming paper. There are some hopes that supersymmetry may provide the necessary relation between the different «potentials» (couplings) to have a physical wave equation. Nevertheless, we think that some particularities we find are interesting enough in themselves to deserve an independent presentation. For this reason, we decided to extract the scalar field (or scalar component of the chiral superfield) from the model and impose directly on it the fourth-order wave equation (see eq. (1)). That equation has normal solutions of the Klein-Gordon type together with other solutions with imaginary mass corresponding to «tachyons» (see eq. (2)). Of course, when one deals with faster-than-light particles one cannot help naming the pioneers in the field, who have clarified much of the misunderstandings on the subject [9-12]. It was Feinberg, in ref. [11] who gave the name «tachyon» to particles with spacelike momentum.

In the complete model, the quantization is carried out in such a way that the spinor conserved charge acts as a generator of supersymmetry transformations. The Heisenberg equations of motion are then satisfied as a consequence of the supersymmetry algebra. For the simplified scalar model we are here considering, we want the operator P_μ acting as the infinitesimal generator of space-time displacements. We do not suppose that the tachyonic components are observable. So we are free to impose commutation or anticommutation relations for them. We choose to have non-negative energy, then it follows that we should take anticommutation rules. Tachyons have then ghostlike behaviour.

On the other hand, the normal part of the scalar field should obey the usual commutation relations. It is possible that those characteristics remain in other higher-order equations. The fields are then mixed objects having normal parts with the usual relations between spin and statistics [13] and also abnormal parts with unusual commutation relations. (See, however ref. [14].) Note that supersymmetry transformations relate normal components to normal components and tachyons to tachyons, so that the inversion of statistics for all tachyons do not impair the symmetry.

In sect. 2 we write the differential equation and the Lagrangian from which it follows. The energy momentum tensor is then evaluated. In sect. 3 we quantize the field. In sect. 4 we evaluate the propagator. In sect. 5 a discussion is given on the principal aspects of this paper.

2. – Lagrangian.

The complex scalar field satisfies the equation (see ref. [8])

$$(\Box^2 - m^4)\,\varphi(x) = J(x) \tag{1}$$

or

$$(\Box - m^2)(\Box + m^2)\,\varphi(x) = J(x)\,. \tag{2}$$

It is clear that $\varphi(x)$ has two components. One of them is related to the normal Klein-Gordon equation, and the other to the «tachyonic» Klein-Gordon equation ($m^2 < 0$).

When $\varphi(x)$ is a complex free field ($J = 0$), eq. (1) (or (2)) implies for the Fourier transform

$$\varphi(x) = \frac{1}{(2\pi)^3}\int \mathrm{d}^6 k\, \tilde{\varphi}(k) \exp[ik_\mu x^\mu] \tag{3}$$

that

$$\varphi(x) = \frac{1}{(2\pi)^3 m}\sqrt{\frac{\pi}{2}}\int \mathrm{d}^5 k \left[\frac{1}{\sqrt{\omega}}(\phi_1^\dagger(\boldsymbol{k})\exp[i\omega x^0] + \phi_2(\boldsymbol{k})\exp[-i\omega x^0]) + \right.$$
$$\left. + \frac{1}{\sqrt{\omega^1}}(\phi_3^\dagger(\boldsymbol{k})\exp[i\omega' x^0] + \phi_4(\boldsymbol{k})\exp[-i\omega' x^0])\right]\exp[i\boldsymbol{k}\cdot\boldsymbol{x}] \tag{4}$$

where

$$w = \sqrt{k^2 + m^2}\,, \qquad w' = \sqrt{k^2 - m^2}\,.$$

At this point, in ref. [11, 12] the sphere $k^2 < m^2$ is suppressed. We are not going to worry about the division of space into $k^2 < m^2$ and $k^2 > m^2$. We will essentially work as if the above-mentioned prescription is followed, but finally we will take an analytical continuation of the physical interesting quantities to $k^2 < m^2$. (See the evaluation of the propagator in sect. 4.)
The Lagrangian density by means of which we obtain eq. (1) is

$$\mathcal{L} = \varphi^\dagger(x)\,\Box^2\varphi(x) - m^4\varphi^\dagger(x)\,\varphi(x)\,. \tag{5}$$

We built the canonical energy momentum density following the prescription of ref. [15] (see also [16])

$$T^\nu_\mu = -\partial^\nu\Box\varphi^\dagger(x)\,\partial_\mu\varphi(x) + \Box\varphi^\dagger(x)\,\partial^\nu\partial_\mu\varphi(x) - \partial^\nu\varphi^\dagger(x)\,\Box\partial_\mu\varphi(x) + \varphi^\dagger(x)\,\partial^\nu\Box\partial_\mu\varphi(x) \tag{6}$$

which is divergenceless

$$\partial_\nu T^\nu_\mu = 0\,. \tag{7}$$

Equation (7) implies the conservation of P_ν given by

$$P_\nu = \int \mathrm{d}^5x\, T^0_\nu\,. \tag{8}$$

Using the Fourier components given in eq. (4) we obtain

$$P_\nu = \int \mathrm{d}^5k\,[k_\nu(\phi_1^\dagger(\bar{k})\,\phi_1(\bar{k}) + \phi_2^\dagger(\bar{k})\,\phi_2(\bar{k})) - k'_\nu(\phi_3^\dagger(\bar{k})\,\phi_3(\bar{k}) + \phi_4^\dagger(\boldsymbol{k})\,\phi_4(\boldsymbol{k}))]\,, \tag{9}$$

where

$$k_\nu = (w_1\,\boldsymbol{k})\,, \qquad k'_\nu = (w'_1\,\boldsymbol{k})\,.$$

Note that one can naturally divide (9) in two parts, one corresponding to normal particles and the other to tachyons

$$P_\nu = P^{(\mathrm{N})}_\nu + P^{(\mathrm{T})}_\nu\,. \tag{10}$$

3. – Quantization.

As we pointed out in the introduction, the complete model is the Wess-Zumino model in $d = 6$. Naturally $\varphi(x)$ is one of the components of a chiral superfield. The other components are two spinor fields, one vector field and one auxiliary scalar field [17]. (See ref. [18, 19] for another SUSY model in $d = 6$.)

Supersymmetry implies the existence of a conserved fermionic current J^{μ}_{α}. The fermionic charge

$$Q_\alpha = \int d^5x\, J^0_\alpha(x) \tag{11}$$

is the generator of the transformation of supersymmetry. We quantize this model in such a way that the operator Q_α acts as the infinitesimal generator of supersymmetry. (The complete procedure will be presented elsewhere.) In other words, if we make an infinitesimal supersymmetry transformation with parameters η^α we have

$$\delta\varphi(x) = i[\eta^\alpha Q_\alpha, \varphi(x)]\,. \tag{12}$$

From (12) it follows

$$[P_\nu, \varphi(x)] = -i\partial_\nu \varphi(x)\,, \tag{13}$$

when P_ν is defined by the relation

$$P^{\dot{\alpha}}_{\alpha} = \{Q_\alpha, \overline{Q}^{\dot{\alpha}}\}\,. \tag{14}$$

Keeping in mind the division of $\varphi(x)$ and P_ν in normal and tachyonic parts we have (cf. (13))

$$[P^{(\mathrm{N})}_\nu, \varphi_i(x)] = -i\partial_\nu \varphi_i(x) \qquad i = 1, 2\,, \tag{15a}$$

$$[P^{(\mathrm{T})}_\nu, \varphi_i(x)] = -i\partial_\nu \varphi_i(x) \qquad i = 3, 4\,. \tag{15b}$$

Equation (15*a*) is the usual relation implying normal quantization with the canonical procedure. But if in (15*b*) we follow the usual method we get negative-energy states or an indefinite metric of the Hilbert space of states. In fact, if we follow the usual spin statistics connection which is valid for any number of dimension (see ref. [13]) we should impose the commutation rules

$$[\phi_i(\boldsymbol{k}), \phi_i^\dagger(\boldsymbol{k}')] = \delta(\bar{k} - \boldsymbol{k}')\,, \qquad i = 1, 2\,, \tag{16a}$$

$$[\phi_i(\bar{k}), \phi_i^\dagger(\boldsymbol{k}')] = \delta(\bar{k} - \boldsymbol{k}')\,, \qquad i = 3, 4\,. \tag{16b}$$

The minus sign in (16*b*) has the consequence that the operators $\phi_i^\dagger(\boldsymbol{k})$ $(i = 1, 2)$

create negative-energy states or the metric of the Hilbert space is indefinite.

Since tachyons would not seem to be observable physical particles, we are not obliged to use commutation relations for them. Therefore, we will consider the tachyonic component as ghostlike objects, in the sense that they have a spin-statistics relation opposite to the normal ones.

It is true that the commutator of $\varphi(x)$ with $\varphi^\dagger(x)$ has no definite significance, but we will see that the vacuum expectation value of the time-ordered product is well defined and it has sense. So instead of (16b) we choose

$$\{\phi_i(\boldsymbol{k}), \phi_i^\dagger(\boldsymbol{k}')\} = \delta(\bar{k} - \bar{k}') \qquad i = 3, 4 . \tag{17}$$

We now have P_ν normally ordered

$$P_\nu = \int \mathrm{d}^5 k [k_\nu (\phi_1^\dagger(\bar{k})\, \phi_1(\bar{k}) + \phi_2^\dagger(\bar{k})\, \phi_2(\bar{k})) + k'_\nu (\phi_3^\dagger(\bar{k})\, \phi_3(\bar{k}) + \phi_4^\dagger(\bar{k})\, \phi_4(\bar{k}))] , \tag{18}$$

where the change of sign relative to eq. (9) is due to the change of statistics of the tachyonic component. In the complete supersymmetric model, this is the operator that we obtain defining $P_\alpha^{\dot{\alpha}} = \{Q_\alpha, \overline{Q}^{\dot{\alpha}}\}$ and imposing relations (16a) and (17).

The Fock space is built up applying the creation operators to the vacuum $|0\rangle$, and all the states have non-negative energy and positive metric too.

4. – Propagator.

In the interaction picture, the perturbative solution of the states evolution equation is given as a function of the time-ordered products of the interaction Hamiltonian. The difference between time-ordered and normal-ordered products defines the field propagator. In other words, the propagator is the vacuum expectation value of the time-ordered product

$$\Delta(x, y) = \langle 0|T[\varphi(x)\, \varphi^\dagger(y)]|0\rangle . \tag{19}$$

Keeping in mind the division into normal and tachyonic parts we have

$$\Delta(x, y) = \langle 0|T[\varphi^{(N)}(x)\, \varphi^{(N)\dagger}(y)]|0\rangle + \langle 0|T[\varphi^{(T)}(x)\, \varphi^{(T)\dagger}(y)]|0\rangle . \tag{20}$$

The first term of the right-hand side of (20) is the usual Feynman propagator of a scalar field that obeys the Klein-Gordon equation. The second term is the propagator of the ghost particle.

The Fourier transform of the sum of both terms is given by

$$(21)\qquad \begin{cases} \Delta(k) = -\dfrac{i}{2m}\left(\dfrac{1}{k^2 - i\varepsilon - m^2} + \dfrac{1}{k^2 - i\varepsilon + m^2}\right), \\ \Delta(k) = -\dfrac{i}{m^2}\,\dfrac{k^2}{(k^2 - i\varepsilon)^2 - m^4}\,. \end{cases}$$

The change in the statistic of the tachyonic part produces a change in the sign of the second term of (21). This is the reason behind the fact that expression (21) differs (by the k^2 factor) from the inverse of the differential operator acting on $\varphi(x)$.

When one now adds to the Lagrangian (5), the self-coupling of the scalar field, present in the complete model, we get an interaction of the form

$$(22)\qquad \mathscr{L}' = -gm^2\varphi^{\dagger 2}(x)\,\varphi(x) + \text{h.c.} - g^2\varphi^{\dagger 2}(x)\,\varphi^2(x)\,,$$

where all terms should be symmetrized in the fields φ and $\varphi^\dagger$. Due to this circumstance, no term of $\mathscr{L}'$ contains more than one tachyonic component, as they obey anticommutation relations. In other words, tachyonic internal lines are always disconnected and as a consequence no tachyonic closed loop can occur in a diagram.

5. – Discussion.

The scalar field $\varphi(x)$ satisfies the fourth-order eq. (1). This equation implies that one part of the field satisfies the normal Klein-Gordon equation, while the other part satisfies a tachyonic one.

By taking $\varphi(x)$ to be a mixed object whose normal part obeys the usual connection between spin and statistics and whose tachyonic part is subject to anticommutation rules, the energy of the field and the matrix of the Fock space turn out to be positive.

There is no conflict with supersymmetry, due to the fact that the transformation that maps the scalar field into the spinor field (and this into the scalar and vector field) maps normal components into normal components. In other words, the supersymmetry transformation does not mix normal and tachyonic parts. So the change of the statistics of all the tachyonic components allow us to keep supersymmetry invariance. The coupling of the scalar field cannot be arbitrary, because it is a component of a supersymmetric multiplet. In this form, apart from the self-coupling of the field, the coupling with the spinors and the vector fields must be present. Furthermore, due to the symmetrized form of the interaction Lagrangian and the remarks made at the end of sect. 4, at least some of the objections against tachyons pointed out in ref. [20] are absent here. Of course, in the complete supersymmetric model all coupling are related and the nonrenormalization theorem assures the compensation of all masses generated in self-energy processes.

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● RIASSUNTO (*)

Si descrive la quantizzazione di un campo scalare che obbedisce ad un'equazione d'onda di quarto ordine. Le particelle piú veloci della luce che appaiono nel processo sono trattate come fantasmi con la relazione errata tra spin e statistica. In questo modo si ottengono lo spettro d'energia positiva nonché la metrica positiva nello spazio di Fock. Infine si discutono brevemente alcune implicazioni di un'interazione descritte da una lagrangiana simmetrizzata nei campi scalari.

(*) *Traduzione a cura della Redazione.*

Квантование волнового уравнения четвертого порядка.

Резюме (*). — Описывается квантование скалярного поля, подчиняющегося волновому уравнению четвертого порядка. Частицы, движущиеся быстрее скорости света, которые появляются в процессе, трактуются как духи с неправильной связью между спином и статистикой. В этом подходе получаются спектр положительных энергий, а также положительная метрика в пространстве Фока. В заключение, вкратце обсуждаются некоторые применения взаимодействия, описываемого с помощью Лагранжиана, симметризованного в скалярных полях.

(*) *Переведено редакцией.*