quantum reports

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Statistical Quantifiers Resolve a Nuclear Theory Controversy

Diana Monteoliva [1], Angelo Plastino [2,*] and Angel Ricardo Plastino [3]

[1] UNLP-Comisión de Investigaciones Científicas, Provincia de Buenos Aires, La Plata 1900, Argentina; monteoli@fisica.unlp.edu.ar
[2] Instituto de Física La Plata-CCT-CONICET, Universidad Nacional de La Plata, C.C. 727, La Plata 1900, Argentina
[3] CeBio-Departamento de Ciencias Básicas Universidad Nacional del Noroeste de la Prov. de Buenos Aires (UNNOBA), CONICET, Junin 1988, Argentina; arplastino@unnoba.edu.ar
* Correspondence: plastino@fisica.unlp.edu.ar

Abstract: We deal here with an exactly solvable N-nucleon system that has been used to mimic typical features of quantum many-body systems. There is in the literature some controversy regarding the possible existence of a quantum phase transition in the model. We show here that an appeal to a suitable statistical quantifier called thermal efficiency puts an end to the controversy.

Keywords: quantum phase transitions; exactly solvable models; statistical quantifiers

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Citation: Monteoliva, D.; Plastino, A.; Plastino, A.R. Statistical Quantifiers Resolve a Nuclear Theory Controversy. *Quantum Rep.* **2022**, *4*, 127–134. https://doi.org/10.3390/quantum4010009

Academic Editor: Henry Chermette

Received: 7 January 2022
Accepted: 18 February 2022
Published: 22 February 2022

1. Introduction

Colloquially, quantum theorists often speak of "phase transitions" with reference to a particular situation in which a certain system's parameter slightly varies and, as a consequence, the ground state experiences a crossover. Such a situation is the subject of the present considerations.

The degree of understanding of finite, quantum many-body systems has been greatly augmented recently thanks to new statistical tools derived from information theory. We mean here information measures and complexity measures. They were fruitfully used to discuss variegated facets of the physics of atoms, molecules, and atomic nuclei. Now, nuclear systems rarely permit an exact analytical treatment. As a consequence, most concomitant theoretical research heavily rests on numerical solutions.

Obviously, exactly solvable models are most welcome to elucidate some theoretical difficulties. Applying in such instances information-theoretic tools to them yields often useful insights. Here, we used rather recent information techniques to elucidate the properties of exactly solvablemany-fermion models of the Lipkin kind, which play in nuclear physics a role similar to the Hubbard model for solid-state physics [1]. Information treatments of the many-body behavior at a finite temperature have been recently reported as very useful ones [2].

We appeal here for the rather novel concepts of disequilibrium D and statistical complexity C. Statistical-complexity-associated traits were experimentally observed long ago in microscopic systems (nuclear physics and metal clusters) (see [3–13] and the references therein).

Motivated by the above-cited results, we revisit here an exactly solvable (finite) fermion model of the Lipkin sort (see [14,15] and the references therein). The associated structural details are well described by the canonical ensemble methodology. At low enough temperatures T, we can also glance at the details of the ground state phenomenology.

1.1. Statistical Order

The concomitant (statistical) order–disorder transactions here involved were studied using Gibbs' canonical ensemble methodology [16]. In it, the pertinent probability distribution (PD) is proportional to $\exp(-\beta\hat{H})$, $\hat{H}$ standing of course for the Hamiltonian, while β

does so for the inverse temperature. Maximal statistical disorder is usually associated with a uniform distribution (UD), since for it, all micro-states are equi-probable. In the opposite fashion, the maximal order is associated with systems for which just a very small set of micro-states displays non-vanishing probabilities.

1.2. Disequilibrium

We call disequilibrium D the distance (in probability space) between the current probability distribution (PD) and the UD D. D is a statistical quantifier that grows as the degree of order increases. If S is the thermodynamic entropy, then the product DS is denominated the statistical complexity [17].

1.3. Exactly Solvable Lipkin-like Models

Lipkin-like models are arguably the simplest non-trivial finite many-fermion systems that can easily be exactly solved [14,15]. They are excellent testing grounds for devising, trying, and applying new nuclear physics' theoretical methodologies. Amongst these, *we focused our attention here on information-theoretic methods*. We concentrated our attention on a Lipkin-associated model called the Abecasis–Faessler–Plastino (AFP) model [18–21] The AFP model represents two-level nuclei, an extremely simplified nuclear model. Nuclear spectra display a complex discrete spectrum and also a continuous one. The AFP model retains only the two lowest lying levels and yields their exact energies and eigenvectors.

2. The AFP Model

The model [19] deals with $N = 2\,\Omega$ fermions distributed amongst $(2\,\Omega)$-fold degenerate single-particle (sp) levels. Our two AFP energy levels are separated by an sp energy gap ϵ. We have $4\,\Omega$ sp micro-states, but just two level-energies, which exhibit degeneration. Two quantum numbers (μ and p) are attributed to a general sp micro-state. The first adopts the values $\mu = -1$ (lower level) and $\mu = +1$ (upper level). The remaining quantum number p, called the quasi-spin or pseudo-spin, singles out a specific micro-state belonging to the N-fold degeneracy. The pair p, μ is seen as a "site" that is occupied by a fermion or empty. We have:

$$N = 2J, \tag{1}$$

where J stands for a kind of angular momentum. Following Lipkin et al. [14], were appeal to the quasi-spin operators:

$$\hat{J}_+ = \sum_p C^\dagger_{p,+} C_{p,-}, \tag{2}$$

$$\hat{J}_- = \sum_p C^\dagger_{p,-} C_{p,+}, \tag{3}$$

$$\hat{J}_z = \sum_{p,\mu} \mu\, C^\dagger_{p,\mu} C_{p,\mu}, \tag{4}$$

$$\hat{J}^2 = \hat{J}_z^2 + \frac{1}{2}\,(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+), \tag{5}$$

where the eigenvalues of $\hat{J}^2$ are of the form $J(J+1)$. It is useful to introduce also the operators [14,18]:

$$\hat{G}_{ij} = \sum_{p=1}^{2\Omega} C^\dagger_{p,i} C_{p,j}. \tag{6}$$

The AFP Hamiltonian was devised by AFP [18,19]. We call v the coupling constant for the pertinent two-body interaction. v is the control parameter of the system and plays a central role in our statistical considerations below.

The all-important AFP feature is that the AFP model displays a level-crossing in the ground-state energy (see [19]). The lowest lying of the Hamiltonian's eigenvalues is, of course, the ground state level (gsl). As v increases, one encounters, suddenly, the Hamiltonian's eigenvalue representing the gsl moves. The v-value at which the level-crossing takes place is called a "critical coupling constant" (CCC). There are several CCCs. The larger N, the larger the CCC number is [19]. These level-crossings are colloquially called phase transitions. The Hamiltonian reads:

$$\hat{H}_{AFP} = \epsilon \sum_{i}^{N} \hat{G}_{ii} + v(\hat{J}_x - \hat{J}_x^2), \tag{7}$$

where $\hat{J}_x$ is the well-known sum $[\hat{J}_+ + \hat{J}_-]/2$. $\hat{H}_{AFP}$ commutes with all the $\hat{J}$ operators. Whenever we state that v is large or small, this is always in relation to the ϵ-value.

3. Hamiltonian Matrices

For the AFP, one deals (see Equation (6) of [19]) with the Hamiltonian matrix:

$$\begin{aligned} \langle n'|H_{AFP}|n\rangle \quad = \quad & (n-J)\delta_{n',n} + \frac{1}{2}v\{2(2J^2 + J + n^2 - 2Jn)\delta_{n',n} \\ & +2\sqrt{(2J-n)(n+1)}\delta_{n',n+1} + 2\sqrt{(2J-n+1)n}\delta_{n',n-1} \\ & -\sqrt{(2J-n-1)(n+2)(2J-n)(n+1)}\delta_{n',n+2} \\ & -\sqrt{(2J-n+2)(n-1)(2J-n+1)n}\delta_{n',n-2}. \end{aligned} \tag{8}$$

For comparison, we sometimes use Lipkin results below. The pertinent Lipkin Hamiltonian matrix is [15]:

$$\begin{aligned} \langle n'|H_L|n\rangle \quad = \quad & \left\{\frac{N}{2} - n + 1 - \left(Nn - \frac{N}{2} - n^2 + 2n - 1\right)\omega\right\}\delta_{n',n} \\ & -\frac{v}{2}\sqrt{(N-n)(N-n+1)(n+1)n}\,\delta_{n',n+2} \\ & -\frac{v}{2}\sqrt{(N-n)(N-n+1)(n+1)n}\,\delta_{n',n-2}, \end{aligned} \tag{9}$$

with $n = 0, 1, \ldots, N$ for $N = 2, 4, 6, \ldots$ and $J = N/2$. After numerically diagonalizing the matrices, we found energy-eigenvalues $E_n(v, J)$ for both Hamiltonians. With them, we can perform statistical mechanics calculations in the canonical ensemble.

4. Thermal Quantifiers

The main thermal quantifiers were obtained from the partition function Z [16]. Z is built up using the probabilities associated with the concomitant microscopic states, which have energies E_i [16]. The most important quantifiers are the mean energy U, the entropy S, and the free energy F [16]. Z [16] and its associated quantifiers are obtained from the canonical probability distributions [16] $P_n(v, J\beta)$, where *beta* is the inverse temperature. The pertinent quantifier expressions are:

$$P_n(v, J, \beta) \quad = \quad \frac{1}{Z(v, J, \beta)} \mathrm{e}^{-\beta E_n(v,J)} \tag{10}$$

$$Z(v, J, \beta) \quad = \quad \sum_{n=0}^{N} \mathrm{e}^{-\beta E_n(v,J)} \tag{11}$$

$$
\begin{aligned}
U(v,J,\beta) &= \langle E\rangle = -\frac{\partial lnZ(v,J,\beta)}{\partial\beta} \\
&= \sum_{n=0}^{N} E_n(v,J)P_n(v,J,\beta) \\
&= \frac{1}{Z(v,J,\beta)}\sum_{n=0}^{N} E_n(v,J)\mathrm{e}^{-\beta E_n(v,J)}
\end{aligned} \tag{12}
$$

$$
S(v,J,\beta) = -\sum_{n=0}^{N} P_n(v,J,\beta)\ln[P_n(v,J,\beta)] \tag{13}
$$

$$
F(v,J,\beta) = U(v,J,\beta) - T\,S(v,J,\beta). \tag{14}
$$

4.1. Complexity-Associated Quantum Quantifiers

Other quantifiers were advanced some 25 y ago [17,22–26]. We discuss them in the context of them being specialized for the AFP model. We remark that they depend on the coupling constant of the concomitant two-body interaction. We refer to the disequilibrium D_{AFP} and the statistical complexity C_{AFP} (quantities with sub-index "L" refer to the Lipkin model). One deals with:

$$
D_{AFP}(v,J,\beta) = \sum_{n=0}^{N}\left(P_n^{AFP}(v,J,\beta) - P_n^u\right)^2 \tag{15}
$$

$$
D_L(v,J,\beta) = \sum_{n=0}^{N}\left(P_n^{L}(v,J,\beta) - P_n^u\right)^2 \tag{16}
$$

$$
C_{AFP}(v,J,\beta) = S(v,J,\beta)D(v,J,\beta), \tag{17}
$$

with an identical expression for $C_L(v,J,\beta)$. Remember that the J-multiplets contain $2J+1 = N+1$ possible micro-states. Thus,

$$
P_n^u = \frac{1}{N+1}\;\forall\, n = 0,1,\ldots N. \tag{18}
$$

The disequilibrium D measures the statistical order [17,22–27]. $D = 0$ entails total r (randomization) [17]. C vanishes both for the total order and the total disorder [17]. It is maximal if the system attains particular sorts of states, amongst them, those linked to "phase transitions" or crossing levels.

4.2. Thermal Efficiency

In addition, we employed here still another, new, thermal quantifier. It is called the thermal efficiency η [28,29] with reference to the control parameter v, our coupling constant (k_B is Boltzmann's constant):

$$
\eta = -\frac{1}{k_B}\frac{\partial S/\partial v}{\partial F/\partial v}, \tag{19}
$$

which represents the work one needs to perform to change the coupling constant value from v to $v + \mathrm{d}v$ for positive η and received from the system for a negative one [29].

5. The Controversy

It was affirmed in [21] that the AFP model exhibits a phase transition (PT) or crossover at a critical value for v that depends on N. However, this was contradicted in [18,20], who declared that this PT disappears if we replace v with v/N. This is the controversy that we resolve below with the help of the thermal efficiency $\eta(v)$.

We depict the behavior of some of our information-theoretic quantifiers (vertical axis) *vs. the interaction strength* v (horizontal axis), with $\beta = 5.0$ and several values for N. We deal in Figure 1 with η_{AFP} (red) and, for comparison, its Lipkin counterpart η_L (blue). Figure 2 is identical to Figure 1, except for the fact that it displays the disequilibrium D. D_{AFP} displays a minimum at the critical v value. The physical explanation is clear enough. In between two different micro-states, the ground state becomes disordered.

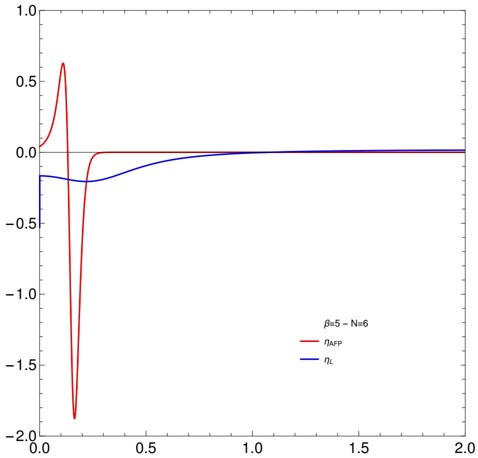

Figure 1. η_{AFP} (red) and Lipkin's η_L (blue) vs. v, for $N = 6$ and the low temperature $\beta = 5$. Note that η_{AFP} (red) clearly detects a phase transition. There is none for the Lipkin model.

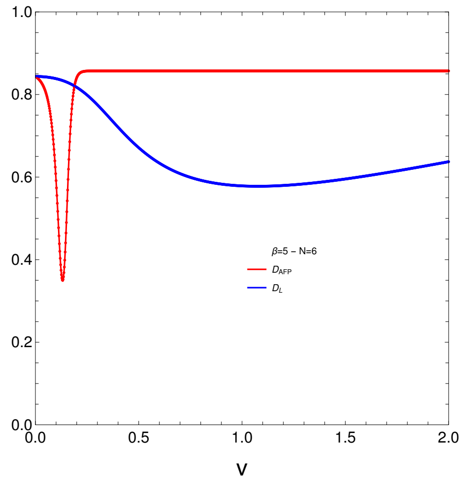

Figure 2. Vertical axis: D_{AFP} (red) and Lipkin's D_L (blue) versus v (horizontal axis). The remaining details are as in Figure 1.

Figure 3 depicts the N-dependence of the phase transition by displaying η_{AFP} vs. v at $\beta = 5$ for several N values. We see that the critical v values diminish with N. This is a typical quantum many-body effect. The interaction among fermions increases its effect the larger the N value is.

Renormalized Coupling v/N

We pass now to our central theme. It was argued in [18] that the above-described phase transitions are an artifice of not replacing v with v/N. We show below that this is not so. We start by reminding the reader of a well-known behavior of the statistical complexity C in the presence of a phase transition. It exhibits in that case a double peak with a small valley between the two peaks [30]. Such is that which one observes in Figures 4 and 5 for the AFP model. We plot in them, for a coupling constant of the form v/N, both the statistical complexity C (Figure 4) and the efficiency η (Figure 5) versus v for several values of N and $\beta = 5$.

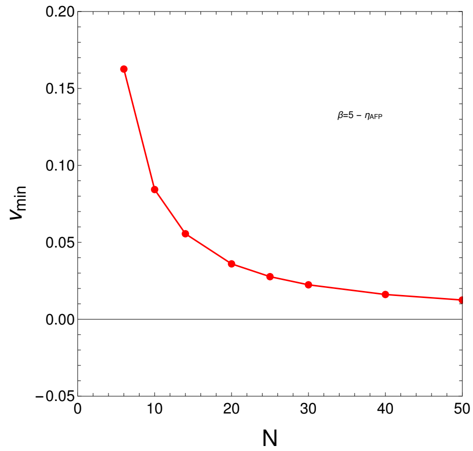

Figure 3. Critical v values versus N. Note that $N > 250$ does not make sense in a nuclear context.

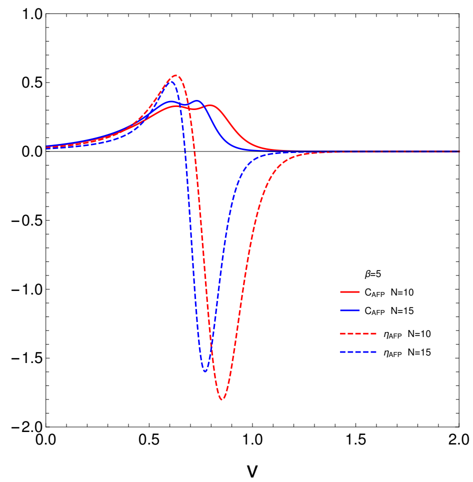

Figure 4. Complexity C (continuous line) and ηAFP (dashed) plotted versus v. The effective coupling constant is v/N. Two N values are used.

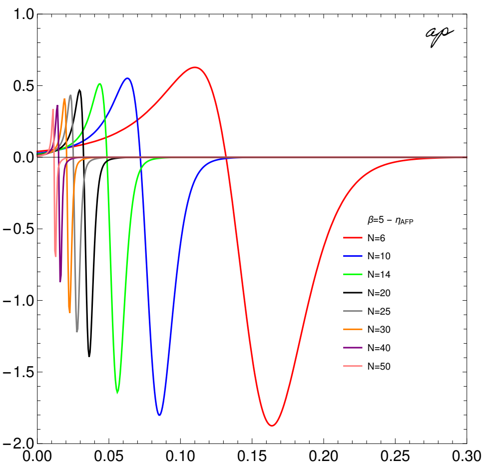

Figure 5. ηAFP plotted versus v. The effective coupling constant is v/N. Eight different N values are used. A different color is assigned to each value, as indicated in the graph.

6. Conclusions

We discussed the quantum statistics of a well-known exactly solvable and finite many-nucleon system. We worked at a low enough temperature to enable the statistical picture to adequately represent the features of the lowest-keyingenergy levels. Our main protagonists were the thermal efficiency η and the statistical complexity C.

With C-η's help, we were able to show that the AFP model displays ground-state crossovers (phase transitions) independently of the fact of using or not employing N-renormalized coupling constants.

In particular, that C exhibited double peaks is irrefutable proof of the presence of phase transitions.

This resolves the old controversy between [18,21].

Author Contributions: D.M., A.P. and A.R.P. participated in equal fashion in the research and planning of the paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Argentina Agency called Conicet via the Grant PIP 0728.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Everything that might be needed is here.

Acknowledgments: This research was partially supported by Conicet (Argentine Agency).

Conflicts of Interest: The authors declare no conflict of interest.

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