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TWO SIDED IDEALS OF OPERATORS

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§ *1.* Let X be a Banach space, and $B(X)$ the Banach algebra of all bounded linear operators in X. The closed two sided ideals of $B(X)$ (actually, of *any* Banach algebra) form a complete lattice $L(X)$. Aside from very concrete cases, $L(X)$ has not yet been determined; for instance, when $X = \ell^p$, $1 \leq p < \infty$, $L(X)$ is a chain (i.e., totally ordered) with three elements : $\{0\}$, $B(X)$ and the ideal $C(X)$ of compact operators (see (3)). On the other hand, it is known ((2), 5.23) that for $X = L^p$, $1 < p < \infty$, the lattice $L(X)$ is *not* a chain. A treatment for X a Hilbert space of arbitrary dimension can be found in (4). We aim to exhibit here a Banach space X such that $L(X)$ is both "long" and "wide".

Precisely, we have:

PROPOSITION: *There exists a real Banach space* X *with the properties:*

i) X *is separable, isometric to its dual* X^*, *and reflexive;*

ii) *it is possible to assign a closed two sided ideal* $a(F) \subset B(X)$ *to each finite set of positive integers* F, *in such a way that the mapping* $F \longrightarrow a(F)$ *is injective and inclusion preserving in both directions:* $F \subseteq G$ *if and only if* $a(F) \subseteq a(G)$.

The example is described below, in § 3.

§ *2.* In the sequel, all Banach spaces are *real* (the complex case can be dealt with similarly). If X, Y are Banach spaces, $m(Y,X)$ denotes the set of operators $T \in B(X)$ that can be factorized through Y, i.e., such that $T = SQ$ for suitable bounded linear operators $Q : X \longrightarrow Y$, $S : Y \longrightarrow X$. If Y is isomorphic (as a Banach space) to its square $Y \times Y$ ($\times$ means cartesian product), then (see (6), Prop.1.2 or (2), Th.5.13) $m(Y,X)$ is a two sided ideal of $B(X)$.

$a(Y,X)$ will denote the (uniform) closure of $m(Y,X)$; thus, if Y is isomorphic to $Y \times Y$, $a(Y,X)$ is a *closed two sided ideal* of $B(X)$.

In all that follows, *subspace* means closed lineal subspace; a subspace Y of a Banach space X is *complemented* if $X = Y + Z$ for some subspace Z satisfying $X \cap Z = \{0\}$.

We shall need the following generalization of Th.5.20, in (2).

LEMMA 1: *Let* X *be a Banach space, and* Y *a complemented subspace of* X, *isomorphic to its square* $Y \times Y$. *Then, for an arbitrary Banach space* Z, *the following conditions are equivalent:*

i) $m(Y,X) \subset a(Z,X)$

ii) Y *is isomorphic to a complemented subspace of* Z.

Proof: Let $P \in B(X)$ be a projection on Y (i.e., $P^2 = P$, $PX = Y$), $I : Y \longrightarrow Y$ the identity and $J : Y \longrightarrow X$ the canonical injection; it is clear that $P \in m(Y,X)$. Let ε be a positive real number such that $\varepsilon||P|| < 1$. Suppose now that $m(Y,X) \subset a(Z,X)$. There exist thus $S : Z \longrightarrow X$, $Q : X \longrightarrow Z$ such that $||P - SQ|| < \varepsilon$. Consider the operator $U \in B(Y)$ defined by $U = I - PSQJ$; since $I = PJ$, we see that $U = PJ - PSQJ = P(P - SQ)J$, and therefore

$$||U|| \leqq ||P|| \ ||P - SQ|| \ ||J|| \leqq ||P|| \ \varepsilon < 1 .$$

Hence $PSQJ : Y \longrightarrow Y$ is *invertible*, that is, there exists $T \in B(Y)$ such that $I = TPSQJ = VW$, where $V = TPS : Z \longrightarrow Y$ and $W = QJ : Y \longrightarrow Z$. This means that $I \in m(Z,Y)$, and from (6), Lemme 1.I (or (2), 5.12), we conclude that Y is isomorphic to a complemented subspace of Z, as desired. The converse is obvious: if Y' is a complemented subspace of Z, then $m(Y,X) = m(Y',X) \subset m(Z,X) \subset a(Z,X)$.

LEMMA 2: *Assume that* $X, Y_1, Y_2, \ldots, Y_n$ *are Banach spaces such that* Y_j *is isomorphic to* $Y_j \times Y_j$ *for* $j = 1,2,\ldots,n$.

Then $m(Y_1,X) + m(Y_2,X) + \ldots + m(Y_n,X) = m(Y_1 \times \ldots \times Y_n, X)$.

Proof: An inductive argument reduces the proof to the case $n = 2$, which is disposed of as follows. Since Y_1 and Y_2 are (isomorphic to) complemented subspaces of $Y_1 \times Y_2$, it is clear that $m(Y_1,X) \subset m(Y_1 \times Y_2, X)$ and $m(Y_2,X) \subset m(Y_1 \times Y_2, X)$, whence

$$m(Y_1,X) + m(Y_2,X) \subset m(Y_1 \times Y_2, X) .$$

Conversely, if $T = SQ \in m(Y_1 \times Y_2, X)$, where $S : Y_1 \times Y_2 \longrightarrow X$ and $Q : X \longrightarrow Y_1 \times Y_2$ with $Q(x) = (Q_1(x), Q_2(x))$, then we define $S_1 : Y_1 \longrightarrow X$, $S_2 : Y_2 \longrightarrow X$ as

$$S_1(y) = S(y,0) \quad , \quad S_2(y) = S(0,y) \quad ;$$

finally, let $T_1, T_2 \in B(X)$ be the operators $T_1 = S_1Q_1$, $T_2 = S_2Q_2$. Clearly $T_1 + T_2 = T$ with $T_j \in m(Y_j,X)$, $j = 1,2$, and therefore $T \in m(Y_1,X) + m(Y_2,X)$; the lemma follows.

Also, from (6), Lemme 1.I (or (2), 5.12) and (1), Th.7,p.205, we obtain that for $p \neq q$, $p \geq 1$, $q \geq 1$, the ideal $m(\ell^q, \ell^p)$ is not the whole of $B(\ell^p)$. Since the ideal $C(\ell^p)$ of compact operator is the largest proper two sided ideal of $B(\ell^p)$ (see (3)), it follows that:

LEMMA 3. *If* $p,q \geq 1$, $p \neq q$, *then* $m(\ell^q,\ell^p) \subset C(\ell^p)$.

§ 3. Let P be a countable set of real numbers $p \geq 1$; define Y as the product $Y = \Pi\{\ell^p; p \in P\}$, where ℓ^p is the ordinary (real) sequence space. We denote by $|x|$ the norm of an element $x \in \ell^p$, for all p. Consider now the set $\ell(P)$ of all families $\{x_p \in \ell^p ; p \in P\} \in Y$ such that $\sum \{|x_p|^2, p \in P\} < \infty$ (this is always the case, if P is finite). It can be seen that $\ell(P)$ is a linear subspace of Y and that the norm $||\{x_p\}|| = (\sum |x_p|^2)^{1/2}$ makes $\ell(P)$ a separable Banach space; if P is finite, $\ell(P) = \Pi\{\ell^p; p \in P\}$. It is clear that for each subset $Q \subset P$, the space $\ell(Q)$ can be identified to a complemented subspace of $\ell(P)$. Moreover, $\ell(P)$ is always isomorphic to its square $\ell(P) \times \ell(P)$ (see for instance (5), Prop.3,b). The dual $(\ell(P))^*$ of $\ell(P)$ can be identified to $\ell(P^*)$, where P^* is the set of conjugates p^* of elements $p \in P$, i e., $1/p + 1/p^* = 1$. In particular, if $1 \notin P$, then $\ell(P)$ is *reflexive*, and furthermore, if $P = P^*$, then $\ell(P)$ is *isometric* to its dual. Therefore, such $\ell(P)$ satisfy condition *i)* in the Proposition above.

Let P be a *fixed countably infinite set of real numbers* $p > 1$ *such that* $P = P^*$, and let X denote the space $X = \ell(P)$. For each finite subset $F \subsetneq P$, let $a(F) \subseteq B(X)$ be the ideal $a(F) = a(\ell(F),X)$. Since $\ell(F)$ is (isomorphic to) a complemented subspace of $\ell(G)$, whenever $F \subseteq G$ it is clear (Lemma 1) that the mapping $F \longrightarrow a(F)$ is inclusion preserving. On the other hand, suppose that $a(F) \subseteq a(G)$, or, equivalently, $m(\ell(F),X) \subseteq a(\ell(G),X)$.

By Lemma 2, this inequality is equivalent to:

$$m(\ell^p,X) \subseteq a(\ell(G),X) \quad , \text{for all } p \in F \ .$$

Lemma 1 applies, and we conclude that for $p \in F$, ℓ^p is isomorphic to a complemented subspace of $\ell(G)$. By (6), Lemma 1.I (or (2), 5.12) this amounts to

$$m(\ell(G), \ell^p) = B(\ell^p) \ .$$

But, again from Lemma 2,

$$m(\ell(G),\ell^p) = \sum \{ m(\ell^q, \ell^p) \ ; \ q \in G \} \ .$$

Now, if $p \notin G$, from Lemma 3 follows that $m(\ell^q, \ell^p) \subseteq c(\ell^p)$ for all $q \in G$, or $B(\ell^p) = m(\ell(G), \ell^p) \subsetneqq c(\ell^p)$, absurd. Then $p \in G$ for all $p \in F$, and this means that $F \subseteq G$. Therefore it was shown that $F \subseteq G$ *if and only if* $a(F) \subseteq a(G)$. This implies that $F \longrightarrow a(F)$ is one-to-one, and the Proposition is probed.

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