



Dualities and models in various dimensions

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ABSTRACT

Working within the path-integral framework we first establish a duality between the partition functions of two $U(1)$ gauge theories with a theta term in $d = 4$ space-time dimensions. Then, after a dimensional reduction to $d = 3$ dimensions we arrive to the partition function of a $U(1)$ gauge theory coupled to a scalar field with an action that exhibits a Dirac monopole solution. A subsequent reduction to $d = 2$ dimensions leads to the partition function of a theory in which the gauge field decouples from two scalars which have non-trivial vortex-like solutions. Finally this $d = 2$ partition function can be related to the bosonized version of the two-dimensional QED₂ (Schwinger) model.

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1. Introduction

The notion of dualities has been a source of relevant developments in the context of field theories both in high energy and condensed matter physics. In the case of gauge theories, electromagnetic duality, already identified in the absence of sources by Faraday and Maxwell, was at the root of Dirac proposal of the possible existence of magnetic monopoles.

The next step in this context was the Montonen-Olive conjecture [1] regarding the existence of two “dual equivalent” field formulations of the same theory in which electric (Noether) and magnetic (topological) quantum numbers exchange roles. This duality was then recognized as just one example of the so-called S -duality which plays a central role in supersymmetric quantum field theories and also in string theories (see for example [2] and references therein).

Concerning condensed matter physics, there has been a growing interest in applications of boson-fermion dualities to the study of the quantum Hall regime, $U(1)$ spin liquids, topological insulators and quantum phase transitions (see for example [3] and references therein).

Inspired by the duality established in Ref. [4] relating $d = 4$ dimensional Maxwell actions with a topological θ -term, we shall first rederive such connection, now in the framework of the path-integral formulation of quantum field theory starting from an interpolating partition function which allows to connect two Maxwell- θ term partition functions $Z_{M\theta}[e, \theta]$ and $Z_{M\tilde{\theta}}[\tilde{e}, \tilde{\theta}]$ related through the S -duality group $SL(2, \mathbb{Z})$. We then proceed to a series of dimensional reductions from $d = 4$ to $d = 3$ dimensions and from $d = 3$ to $d = 2$ discussing the resulting partition functions, field equations and their solutions.

2. The interpolating partition function approach

Following the approach developed in Refs. [5,6], we start by introducing an interpolating partition function $Z_I[e, \theta]$ associated to an action $S_I^{(4)}[A, B, C; e, \theta]$ in $d = 4$ euclidean space, which includes three Abelian gauge fields, one of them playing the role of a Lagrange multiplier. Within the path-integral approach we shall prove the duality discussed by Chatzistavarakidis *et al.* [4] at the level of classical actions, now for the quantum partition functions.

The action $S_I^{(4)}[A, B, C; e, \theta]$ reads

$$S_I^{(4)}[A, B, C; e, \theta] = \frac{1}{4e^2} \int \left(F_{\mu\nu}[A] - \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}[B] \right) \left(F_{\mu\nu}[A] - \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}[B] \right) d^4x$$

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$$\begin{aligned}
& + i \frac{\theta}{32\pi^2} \int \left(F_{\mu\nu}[A] - \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}[B] \right) \left(\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}[A] - i F_{\mu\nu}[B] \right) d^4x \\
& + i \frac{a}{2} \int \left(\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}[A] - i F_{\mu\nu}[B] \right) F_{\mu\nu}[C] d^4x
\end{aligned} \tag{1}$$

where $F_{\mu\nu}[A] = \partial_\mu A_\nu - \partial_\nu A_\mu$ and a is up to now, an arbitrary constant.

The interpolating partition function associated to action (1) is then defined as

$$Z_I^{(4)}[e, \theta] = \int DADBDC \exp\left(-S_I^{(4)}[A, B, C; e, \theta]\right) \tag{2}$$

Depending on which two fields one integrates out, Z_I will become the partition function for a theory with an action for the remaining one.

Integrating over C_μ leads to a delta function, $\delta[(\delta_{\mu\nu}\square - \partial_\mu\partial_\nu)B_\nu]$, imposing B_μ to vanish (up to a pure gauge). Then, integration over B_μ becomes trivial and one ends with the partition function for a Maxwell- θ -term theory

$$\begin{aligned}
Z_I^{(4)} &= \int DA \exp \left\{ - \int \left(\frac{1}{4e^2} F_{\mu\nu}[A] F_{\mu\nu}[A] + i \frac{\theta}{64\pi^2} \epsilon_{\mu\nu\alpha\beta} F_{\mu\nu}[A] F_{\alpha\beta}[A] \right) d^4x \right\} \\
&\equiv Z_{M\theta}^{(4)}[e, \theta]
\end{aligned} \tag{3}$$

We shall now proceed to obtain a dual action for the field C_μ by integrating over A_μ and B_μ . To that end we complete squares in the interpolating action $S_I^{(4)}[A, B, C; e, \theta]$, eq. (1). In fact, we can write

$$\begin{aligned}
S_I^{(4)}[A, B, C; e, \theta] &= S^{(II)}[A, B, C] + \frac{32a^2e^2\pi^4}{64\pi^4 + e^4\theta^2} \int F_{\mu\nu}[C] F_{\mu\nu}[C] d^4x \\
&\quad - i \frac{4a^2e^4\pi^2\theta}{64\pi^4 + e^4\theta^2} \int \epsilon_{\mu\nu\alpha\beta} F_{\mu\nu}[C] F_{\alpha\beta}[C] d^4x
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
S^{(II)}[A, B, C] &= \frac{d_1}{4} \int \epsilon_{\mu\nu\alpha\beta} (F_{\mu\nu}[A] - F_{\mu\nu}[B] + u F_{\mu\nu}[C]) (F_{\alpha\beta}[A] - F_{\alpha\beta}[B] + u F_{\alpha\beta}[C]) d^4x \\
&\quad + \frac{d_2}{2} \int (F_{\mu\nu}[A] - F_{\mu\nu}[B] + u F_{\mu\nu}[C]) (F_{\mu\nu}[A] - F_{\mu\nu}[B] + u F_{\mu\nu}[C]) d^4x \\
&\quad + \frac{d_3}{4} \int \epsilon_{\mu\nu\alpha\beta} (F_{\mu\nu}[A] + F_{\mu\nu}[B] + v F_{\mu\nu}[C]) (F_{\alpha\beta}[A] + F_{\alpha\beta}[B] + v F_{\alpha\beta}[C]) d^4x \\
&\quad + \frac{d_4}{2} \int (F_{\mu\nu}[A] + F_{\mu\nu}[B] + v F_{\mu\nu}[C]) (F_{\mu\nu}[A] + F_{\mu\nu}[B] + v F_{\mu\nu}[C]) d^4x
\end{aligned} \tag{5}$$

with

$$\begin{aligned}
d_1 = d_2 &= \frac{1}{4e^2} + i \frac{\theta}{32\pi^2}, & d_3 = -d_4 &= -d_1^* \\
u &= \frac{ia}{4} \frac{1}{d_1}, & v &= -u^*
\end{aligned} \tag{6}$$

Now, changing variables in the interpolating partition function (2),

$$A_\mu \rightarrow A'_\mu = A_\mu - B_\mu + u C_\mu, \quad B'_\mu = B_\mu + A_\mu + v C_\mu \tag{7}$$

$S_I^{(4)}[A, B, C; e, \theta]$ becomes

$$\begin{aligned}
S_I^{(4)}[A, B, C; e, \theta] &= S^{(II)}[A', B'] + \frac{32a^2e^2\pi^4}{64\pi^4 + e^4\theta^2} \int F_{\mu\nu}[C] F_{\mu\nu}[C] d^4x \\
&\quad - i \frac{4a^2e^4\pi^2\theta}{64\pi^4 + e^4\theta^2} \int \epsilon_{\mu\nu\alpha\beta} F_{\mu\nu}[C] F_{\alpha\beta}[C] d^4x
\end{aligned} \tag{8}$$

The terms in $S^{(II)}[A', B']$ are completely decoupled from C and integration over A'_μ and B'_μ just gives an irrelevant constant \mathcal{N} so that one ends with

$$\begin{aligned}
Z_I^{(4)}[e, \theta] &= \mathcal{N} \int DC \exp \left\{ - \frac{32a^2e^2\pi^4}{64\pi^4 + e^4\theta^2} \int F_{\mu\nu}[C] F_{\mu\nu}[C] d^4x \right. \\
&\quad \left. + i \frac{4a^2e^4\pi^2\theta}{64\pi^4 + e^4\theta^2} \int \epsilon_{\mu\nu\alpha\beta} F_{\mu\nu}[C] F_{\alpha\beta}[C] d^4x \right\}
\end{aligned} \tag{9}$$

Now, choosing $a = 1/4\pi$ we get

$$\begin{aligned} Z_I^{(4)}[e, \theta] &= \int DC \exp \left\{ -\frac{1}{4\tilde{e}^2} \int F_{\mu\nu}[C]F_{\mu\nu}[C]d^4x + i\frac{\tilde{e}\tilde{\theta}}{64\pi^2} \epsilon_{\mu\nu\alpha\beta}F_{\mu\nu}[C]F_{\alpha\beta}[C]d^4x \right\} \\ &= Z_{M\theta}^{(4)}[\tilde{e}, \tilde{\theta}] \end{aligned} \quad (10)$$

with

$$\tilde{e}^2 = \frac{64\pi^4 + e^4\theta^2}{4e^2\pi^2}, \quad \tilde{\theta} = -\frac{4\pi^2e^4}{64\pi^4 + e^4\theta^2} \quad (11)$$

Then, in view of eq. (3) one has

$$Z_{M\theta}^{(4)}[e, \theta] = Z_{M\theta}^{(4)}[\tilde{e}, \tilde{\theta}] \quad (12)$$

One can see that, working at the level of the interpolating partition function for a theory with action S_I introduced in (1), we have established a duality between two models with parameters (e, θ) and $(\tilde{e}, \tilde{\theta})$ related by eq. (11). Such result was found in Ref. [4] by relating the classical actions. Finally, after a Wick rotation to 4-d Minkowski space, defining as usual

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2} \quad (13)$$

and using relations (11) one gets the standard duality $\bar{\tau} = -1/\tau$ which, together with θ periodicity generates the $SL(2, \mathbb{Z})$ group.

3. Dimensional reductions

As it is well known, Yang-Mills self-dual instanton equations in Euclidean space become, after dimensional reduction, the first order $d = 3$ BPS monopole equations when the A_4 gauge field is identified with the Higgs field and time dependence is wiped out from all fields [7–9]. In this way, the instanton solution [10] can be connected with the 't Hooft-Polyakov monopole [11]. Also, second dimensional reduction to $d = 2$ dimensions can be seen to led to the first order vortex equations of the Abelian Higgs model [12,13]. We shall here proceed to a series of dimensional reductions of the model we discussed above with the idea of finding solutions of the reduced field equations and also discuss the resulting partition functions.

We now start to discuss dimensional reductions of the models discussed above. Let us consider the Maxwell-theta term action $S_{M\theta}^{(4)}$ associated to the partition function $Z_{M\theta}^{(4)}$ defined in eq. (3)

$$S_{M\theta}^{(4)}[A; e, \theta] = \int d^4x \left(\frac{1}{4e^2} F_{\mu\nu}[A]F_{\mu\nu}[A] + i\frac{\theta}{64\pi^2} \epsilon_{\mu\nu\alpha\beta}F_{\mu\nu}[A]F_{\alpha\beta}[A] \right) \quad (14)$$

with $\mu = \{1, 2, 3, 4\}$. In order to dimensional reduce the Lagrangian from $d = 4$ to $d = 3$ dimensions one proceeds as follows: (i) the field dependence is restricted to spatial coordinates x_i , ($i = 1, 2, 3$), and (ii) the A_4 gauge field component is identified with a scalar field ϕ ,

$$A_i(x_j; x_4) \rightarrow A_i(x_j) \quad i, j = 1, 2, 3$$

$$A_4(x_j; x_4) \rightarrow \phi(x_j) \quad (15)$$

With this the (Euclidean) interpolating partition function of the resulting reduced theory is given by

$$Z_I^{(3)}[e, \theta] = \exp \int DA_i D\phi \exp(-S_{M\theta}^{(3)}[A, \phi; e, \theta]) \quad (16)$$

with

$$S_{M\theta}^{(3)}[A, \phi; e, \theta] = \int d^3x \left(\frac{1}{2e^2} (\partial_i \phi)^2 + \frac{1}{4e^2} F_{ij}[A]F_{ij}[A] - i\frac{\theta}{16\pi^2} \partial_i \phi \epsilon_{ijk} F_{jk} \right) \quad (17)$$

Following [8], we can find solutions to this euclidean 3d model as *static solutions of 4d Minkowski model*. Notice that this requires to absorb the imaginary unit “ i ” in the constant θ (as the Minkowski version of the θ -term does not contain i). We will do something more general considering the θ parameter a complex constant and find complex field solutions. Since the equations are linear, we can always isolate the real part at the end, if necessary. Therefore, from now on, we will write the action as

$$S_{M\theta}^{(3)}[A, \phi; e, \theta] = \int d^3x \left(\frac{1}{2e^2} (\partial_i \phi)^2 + \frac{1}{4e^2} F_{ij}[A]F_{ij}[A] - \frac{\theta}{16\pi^2} \partial_i \phi \epsilon_{ijk} F_{jk} \right) \quad (18)$$

with θ complex.

As we shall see, we will find a Dirac monopole solution of the resulting partition function which can be seen, following Polyakov's idea to analyze confinement in compact QED₃, as instantons in the Euclidean theory [14].

In order to see that Dirac monopoles can also arise as classical solutions of the field equations of action $S_{M\theta}^{(3)}$, we start by integrating by parts the last term in $S_{M\theta}^{(3)}$. One has

$$-\frac{\theta}{16\pi^2} \int d^3x \partial_i \phi \epsilon_{ijk} F_{jk} = \frac{\theta}{8\pi^2} \int d^3x \phi \partial_i B_i. \quad (19)$$

where $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$ and the surface term vanishes. Then, for the case of Dirac monopole configurations, this term in the action cannot be neglected and the resulting field equation for ϕ takes the form

$$\nabla^2 \phi = \frac{e^2 \theta}{8\pi^2} \partial_i B_i \quad (20)$$

The magnetic field B_i of a monopole including the Dirac string along the z direction reads

$$\vec{B}(\vec{x}) = \frac{g}{r^2} \vec{r} - g \Theta(-z) \delta(x) \delta(y) \vec{k} \quad (21)$$

so that the field equation for ϕ takes the form

$$\nabla^2 \phi = \frac{e^2 \theta g}{8\pi^2} \delta^{(3)}(\vec{r}) \quad (22)$$

where we have used $eg = 2n\pi$. From this result we see that ϕ is given by

$$\phi_n(\vec{x}) = -\frac{ne\theta}{16\pi^2} \frac{1}{r} \quad (23)$$

Since the Dirac monopole satisfies Ampère's law (there is no electric current in the model) and $\varepsilon_{ijk} \partial_i \partial_j (1/r) = 0$, eqs. (21) and (23) for the magnetic and scalar fields provide a consistent solution for the system with action (17). Note that in contrast with the BPS monopole solution in which the scalar has asymptotically a hedgehog behavior, in the present case ϕ behaves as a Coulomb potential with a charge $ne\theta/(4\pi)$.

The field equations are linear, so we can also have multi-monopole configurations, with monopole located at positions \vec{x}_i ,

$$B_{\text{multi-m}}(\vec{x}) = \sum_i B(\vec{x} - \vec{x}_i), \quad \phi_{\text{multi-m}}(\vec{x}) = \sum_i \phi_{n_i}(\vec{x} - \vec{x}_i) \quad (24)$$

We now proceed to a second dimensional reduction of the partition function $Z_I^{(3)}$ from the $d = 3$ to $d = 2$ dimensions. In this case we shall identify A_3 with a second scalar ψ and all fields will depend just on x_a , $a = 1, 2$. Now, before this identification it will be convenient to fix the gauge in a way such that the resulting scalar ψ is massive. To this end we shall consider a gauge fixing à la 't Hooft-Feynman inserting in the path integral (18) the condition

$$\exp\left(-\frac{1}{2}\mu^2 \int d^3x A_3^2\right) \quad (25)$$

with μ a parameter with dimensions $[\mu] = 1$ since in $d = 2$ dimensions the fields A_μ , ϕ and ψ should be dimensionless, $[A_\mu] = [\phi] = [\psi] = 0$. Note that limit $\mu^2 \rightarrow \infty$ corresponds to fixing the gauge to $A_3 = 0$ while the "Feynman gauge" can be obtained for $\mu^2 = 2$.

We are now ready to identify A_3 with a scalar field ψ and dimensionally reduce partition function (18). The resulting $d = 2$ partition function $Z_I^{(2)}$ takes the form

$$Z_I^{(2)}[e, \theta] = \int DA_a D\phi D\psi \exp\left(-S_{M\theta}^{(2)}[A, \phi, \psi; e, \theta]\right) \quad (26)$$

where

$$S_{M\theta}^{(2)}[A, \phi, \psi; e, \theta] = \int d^2x \left(\frac{1}{2}(\partial_a \phi)^2 + \frac{1}{2}(\partial_a \psi)^2 + \frac{\mu^2}{2}\psi^2 + \frac{1}{4}F_{ab}[A]F_{ab}[A] - \frac{e^2 \theta}{8\pi^2} \partial_a \phi \varepsilon_{ab} \partial_b \psi \right) \quad (27)$$

Since the gauge field A_i decouples from the scalars its field equations read

$$\partial_a F_{ab} = 0 \quad (28)$$

while the scalar fields satisfy the coupled equations

$$\begin{aligned} \nabla^2 \phi &= \frac{e^2 \theta}{8\pi^2} \varepsilon_{ab} \partial_a \partial_b \psi \\ (\nabla^2 - \mu^2) \psi &= -\frac{e^2 \theta}{8\pi^2} \varepsilon_{ab} \partial_a \partial_b \phi \end{aligned} \quad (29)$$

As in the $d = 3$ dimensions, we can find classical solutions, in this case scalar global vortex-like solutions to these equations in terms of Green's functions of the operators $\nabla^2 - \mu^2$, and $\varepsilon_{ab} \partial_a \partial_b$,

$$(\nabla^2 - \mu^2) G_{r,\mu}(x) = 2\pi \delta^{(2)}(\vec{x}), \quad \varepsilon_{ab} \partial_a \partial_b G_\phi(\vec{x}) = 2\pi \delta^{(2)}(\vec{x}) \quad (30)$$

where

$$G_{r,\mu}(\vec{x}) = -K_0(\mu r) \quad (31)$$

$$G_\phi(\vec{x}) = \arctan(y/x) \quad (32)$$

with $r = \sqrt{x_1^2 + x_2^2}$, φ is the polar angle, and K_0 is the modified Bessel function. In terms of these Green's functions a solution to (29) can be written as

$$\begin{aligned}\phi &= A G_\varphi(\vec{x}) \\ \psi &= -\frac{e^2\theta}{8\pi^2} A G_{r,\mu}(\vec{x})\end{aligned}\quad (33)$$

with A an arbitrary constant. Again, due to the linearity of the field equations, we can consider global multi-vortex configurations

$$\begin{aligned}\phi &= \sum_i A_i G_\varphi(\vec{x} - \vec{x}_i) \\ \psi &= -\frac{e^2\theta}{8\pi^2} \sum_i A_i G_{r,\mu}(\vec{x} - \vec{x}_i)\end{aligned}\quad (34)$$

As a side note, if $\mu = 0$ the vortex-like solutions have a richer structure,

$$\begin{aligned}\phi &= A G_\varphi(\vec{x}) + \frac{e^2\theta}{8\pi^2} B \ln(r) \\ \psi &= -\frac{e^2\theta}{8\pi^2} A G_{r,\mu}(\vec{x}) + B G_\varphi(\vec{x})\end{aligned}\quad (35)$$

with B another arbitrary constant.

We will show that the model described by action (27) is dual, in the large μ limit, to a $2d$ massive vector field. In what follows, all the operations are assumed to be done to the partition function $\mathcal{Z} = \int D$ fields e^{-S} , however, for conciseness, we will only record the changes in the action.

The action (27) can be written as

$$\begin{aligned}\mathcal{S} &= \int d^2x \left(\frac{1}{2} (\partial_a \phi)^2 + \frac{1}{2} (\partial_a \psi)^2 + \frac{\mu^2}{2} \psi^2 + \frac{1}{4} F_{ab}[A] F_{ab}[A] - \frac{e^2\theta}{8\pi^2} B_a \partial_a \phi \right) \\ &\quad + i \int d^2x \lambda_a (B_a - \epsilon_{ab} \partial_b \psi)\end{aligned}\quad (36)$$

where λ_a is a vector (Lagrange-multiplier) field enforcing the condition

$$B_a = \epsilon_{ab} \partial_b \psi \quad (37)$$

We can then re-write action (36) as

$$\mathcal{S} = \int d^2x \left(\frac{1}{2} (\partial_a \phi)^2 + \frac{1}{2} \psi^2 (-\nabla^2 + \mu^2) \psi - i \epsilon_{ab} \partial_a \lambda_b \psi + i B_a \left(\lambda_a + i \frac{e^2\theta}{8\pi^2} \partial_a \phi \right) \right) + \mathcal{S}_A \quad (38)$$

where $\mathcal{S}_A = \frac{1}{4} \int d^2x F_{ab}[A] F_{ab}[A]$. We shall now integrate the field ψ in the partition function with action \mathcal{S} , leading to the following effective action

$$\mathcal{S} = \int d^2x \left(\frac{1}{2} (\partial_a \phi)^2 + \frac{1}{2} (\epsilon_{ab} \partial_a \lambda_b) (-\nabla^2 + \mu^2)^{-1} (\epsilon_{ab} \partial_a \lambda_b) + i B_a \left(\lambda_a + i \frac{e^2\theta}{8\pi^2} \partial_a \phi \right) \right) + \mathcal{S}_A \quad (39)$$

In the large μ limit, we have

$$(-\nabla^2 + \mu^2)^{-1} = \frac{1}{\mu^2} + \frac{1}{\mu^4} \nabla^2 + \dots \quad (40)$$

Keeping the leading order, we get

$$\mathcal{S} = \int d^2x \left(\frac{1}{2} (\partial_a \phi)^2 + \frac{1}{4\mu^2} F_{ab}[\lambda]^2 + i B_a \left(\lambda_a + i \frac{e^2\theta}{8\pi^2} \partial_a \phi \right) \right) + \mathcal{S}_A \quad (41)$$

where we have used that, in two dimensions,

$$(\epsilon_{ab} \partial_a \lambda_b)^2 = \frac{1}{2} (\partial_a \lambda_b - \partial_b \lambda_a)^2 = \frac{1}{2} F_{ab}[\lambda]^2$$

Finally, we integrate the field B_i which enforces the condition

$$\partial \phi_a = -i \frac{8\pi^2}{e^2\theta} \lambda_a$$

so we get

$$\mathcal{S} = \int d^2x \left(\frac{1}{4\mu^2} F_{ab}[\lambda]^2 - \frac{1}{2} \left(\frac{8\pi^2}{e^2\theta} \right) \lambda_a^2 \right) + \mathcal{S}_A \quad (42)$$

which correspond to a massive vector field λ_a (together with a regular gauge field A_a).

We see that action \mathcal{S} can be identified with a Proca action for a massive (spin 1) vector field in $d = 2$ dimensions which is precisely the bosonized version of the QED₂ Schwinger model [15] in which the 2-d fermion Ψ with electric charge e_{SM} is coupled to a gauge field which after bosonization acquires a mass m such that $m^2 = e_{SM}^2/\pi$ thus closing the series of dimensional reduction models that we have presented ending with a fermion-gauge field model.

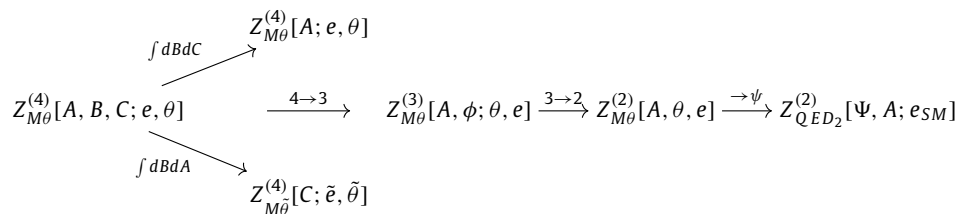
4. Summary and Discussion

Working within the path-integral framework we have established a series of dualities at the level of the partition functions. We have also proceed to a series of dimensional reductions from $d = 4$ dimensions to $d = 3$ and then to $d = 2$ and discussed the solutions of the classical field equations of the resulting theories.

Starting from the interpolating partition function $Z_I^{(4)}[A, B, C; \theta]$ in $d = 4$ dimensions introduced in eq. (2) and alternatively integrating over B_μ, C_μ or B_μ, A_μ we proved the duality between partition functions, $Z_{M\theta}^{(4)}[A; e, \theta]$ and $Z_{M\tilde{\theta}}^{(4)}[C; \tilde{e}, \tilde{\theta}]$ with parameters (e, θ) and $(\tilde{e}, \tilde{\theta})$, related according to eq. (11) which can be seen corresponds to the standard duality $\tilde{\tau} = -1/\tau$ duality so that this, together with θ periodicity generates the $SL(2; \mathbb{Z})$ modular group.

We then proceeded to a dimensional reduction from $d = 4$ to $d = 3$ and then from $d = 3$ to $d = 2$ obtaining actions of bosonic models and founding classical solutions of their field equations. In the former case we found a Dirac monopole solution for the gauge field and a $1/r$ (Coulomb potential) behavior for the scalar arising from the $A_4 = \phi$ identification, to be compared with the BPS solution for the non-Abelian case which corresponds to a 't Hooft-Polyakov monopole with a hedgehog-like scalar. Concerning the $d = 3 \rightarrow d = 2$ reduction we ended with a A_a gauge field ($a = 1, 2$) and two scalars ϕ and ψ with a partition function with action (39). We also solved the associated field equations finding non-trivial scalar solutions which correspond to global vortex-like solutions. Moreover, by integrating the scalars fields we ended with a Proca action which, via bosonization, can be finally connected between the bosonic model and QED_2 .

The results described above are summarized in the following graph:



We expect to discuss applications of the dualities that we have discussed here to problems in quantum field theory as well as in condensed matter.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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