Eur. Phys. J. A **11**, 263–270 (2001)

THE EUROPEAN PHYSICAL JOURNAL A

Application of the Dyson boson expansion method to the treatment of the pairing force

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Received: 10 April 2000 / Revised version: 27 March 2001
Communicated by P. Schuck

Abstract. The treatment of the separable pairing interaction in the context of the BRST formalism and in the Dyson boson expansion method is discussed. The approach is based on the use of the vacuum expectation value of the boson number operator to define a suitable mean field.

PACS. 21.60.-n Nuclear-structure models and methods – 21.60.Jz Hartree-Fock and random-phase approximations – 21.60.Ev Collective models

1 Introduction

The use of the BRST method in constructing solutions of non-trivial nuclear many-body Hamiltonians, has been advanced in a series of papers written by D.R. Bes and coworkers [1–5]. For a comprehensive review of the subject the reader is kindly referred to the textbook of ref. [1]. The case of the separable monopole-pairing interaction has been discussed in [3,4]. The use of the same technique for the case of nuclear rotations has been presented in [5]. As a brief introduction to the subject, we shall first remind the reader about some general features of the conventional nuclear many-body approach and then we shall proceed to discuss the essentials of the BRST method. As is customary in dealing with the microscopic description of single-particle and collective nuclear excitations (see the textbook of ref. [6]), one chooses a two-body interaction and a single-particle basis where the matrix elements of the interaction can be calculated. Then, one can attempt to perform a diagonalization of the Hamiltonian in a certain configuration basis. Naturally, a complete shell model calculation can provide us with the exact solution of the problem. Unfortunately, such a diagonalization is not feasible even for a relatively modest number of nucleons. In consequence, one is forced to introduce approximations, *i.e.*, effective single-particle and collective states which can be used to describe the solutions. Central to this picture are both the violation and the partial fulfillment of some symmetries. As an example, we can mention the diagonalization of the separable pairing Hamiltonian by using the BCS transformations and the RPA treatment of particle-hole or two quasiparticle excitations around close shells. Another example is the treatment of two quasiparticle excitations in a deformed basis. Although these approximations may work, further corrections to them are hampered by instabilities and/or divergences which originate in violations of the Pauli principle, in the overcompleteness of the basis or in the lack of an expansion parameter which can be used to formulate the corrections perturbatively. All these drawbacks stem from the fact that fermions and collective excitations are not independent. A possible solution to these problems could be to project out spurious components of the approximate solutions and to perform variations in order to determine minima in the parametric space defined by the collective variables. This is the case of the variation after projection method introduced by the Tübingen group [7]. Another view is the one introduced by the Copenhagen School (see Bohr and Mottelson [8]), which deals with intrinsic and collective variables. The unified model of nuclear rotational and vibrational degrees of freedom [8] gives us an example of the use of collective and intrinsic variables. However, perturbative expansions in the unified model scheme cannot always be performed. In this context, we referred to the method introduced by Bes [1–5], where the difficulties associated with the use of perturbation theory in a deformed basis are avoided by constructing quadratic Hamiltonians which include intrinsic, collective and auxiliary variables. Also, in dealing with fermionic Hamiltonians, the use of collective variables was shown [4] to be an alternative to other treatments, such as particle number projection [9].

The essentials of the formalism are the following:

i) the starting Hamiltonian H is rewritten in terms of intrinsic and collective variables,

ii) the mean-field treatment of H fixes a particular choice of the intrinsic frame,

iii) the symmetry broken by this choice of the intrinsic frame is restored by the inclusion of collective variables,

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through the definition of generalized constraints, as prescribed by the BRST method [1], and

iv) the perturbative treatment of the resulting Hamiltonian is performed in the intrinsic frame and it is free from infrared divergences.

So far, the known examples of the above scheme have shown that a good agreement between exact and perturbative results [1–5] can be obtained. However, for some cases, the convergence of the perturbative expansion may be slow. This is the situation found, for example, when the BRST Hamiltonian is expanded in a boson basis, as done in the previous examples [1,2,5]. The residual BRST Hamiltonian has been treated in the framework of the Nuclear Field Theory (NFT) [10] and by using the Holstein-Primakoff expansion [11].

In the present paper, we propose the use of the Dyson boson expansion method, as an alternative to the use of the Holstein-Primakoff boson expansion method, in conjunction with the already developed BRST + RPA treatment of the pairing interaction of refs. [1–5]. The perturbative expansion based on the Holstein-Primakoff boson expansion method is performed by using the shell degeneracy as expansion parameter [1]. Here, for the case of the Dyson boson expansion method [11], we shall introduce the vacuum expectation value (v.e.v.) of the boson number as a parameter. In this way, we aim at defining a suitable mean field. Corrections to mean field v.e.v. are added by performing a perturbative treatment in the subspace of states connected by the transformed Hamiltonian [12]. The v.e.v. of the boson number is determined by a variation, afterwards. In the following, we shall describe the proposed mapping and discuss the advantage of its use, for the case of the monopole pairing force of ref. [13].

2 Formalism

The conventional separable pairing Hamiltonian [13] reads

$$\tilde{H} = H_{\rm sp} - V \sum_{kk'} a_k^\dagger a_{\overline{k}}^\dagger \, a_{\overline{k'}} \, a_{k'} \,, \tag{1}$$

and, after applying the Bogoliubov transformation to the quasiparticle basis [8], it is written as the BCS Hamiltonian

$$\tilde{H} = H_{00} + H_{11} + H_{20} + H_{22} + H_{40} + H_{31} + H_{\rm qp\text{-}qp} \,. \tag{2}$$

The explicit form of each term of this equation is given in Appendix A. The BCS parameters λ, Δ and the quasiparticle occupancies, u_k and v_k, are determined from the conditions [6],

$$\begin{aligned} H_{11} &= \sum_k E_k \hat{\nu}_k, \\ H_{20} &= 0, \\ \langle \hat{N} \rangle_{\rm BCS} &= N \,, \end{aligned} \tag{3}$$

with

$$\begin{aligned} u_k^2 &= \frac{1}{2}\left(1 + \frac{\epsilon_k - \lambda}{E_k}\right), \\ v_k^2 &= \frac{1}{2}\left(1 - \frac{\epsilon_k - \lambda}{E_k}\right), \\ E_k &= \sqrt{(\epsilon_k - \lambda)^2 + \Delta^2} \,, \end{aligned} \tag{4}$$

in standard notation. In the following, we shall introduce the BRST version of the Hamiltonian of eq. (2), as it is given in refs. [1,2]. It reads

$$\begin{aligned} \hat{H}_{\rm BRST} = \hat{H} - \hat{\Omega}(2\hat{n} - \hat{N}) + i\hat{\pi}\hat{\overline{\pi}} - 2\omega^2 \hat{\overline{\eta}}[\hat{G}, \hat{n}]\hat{\eta} \\ + \omega^2 \left(\hat{G}\hat{P} - \frac{1}{2I}\hat{P}^2\right) . \end{aligned} \tag{5}$$

In the gauge treatment of the interaction we have defined $\hat{N}$ as the variable conjugate to the collective variable $\hat{\theta}$, $\hat{P}$ as the conjugate variable associated with $\hat{\Omega}$, $\hat{G}$ as the gauge-fixing function and $\hat{\overline{\pi}}$, $\hat{\pi}$, $\hat{\overline{\eta}}$, $\hat{\eta}$, as the ghost Hermitian operators [1], ω is an arbitrary constant and I is the moment of inertia for rotations in gauge space. At the RPA order of approximation, the pair contributions of the number operator, $\hat{n}_{20}$, reads

$$\hat{n}_{\rm RPA} = \sum_k n_k (\hat{\gamma}_k^\dagger + \hat{\gamma}_k) \,, \tag{6}$$

with $n_k = u_k v_k$ and where $\gamma_k^\dagger$ (γ_k) creates (annihilates) a pair of quasiparticles. The conjugate operator, $\hat{\theta}_{\rm RPA}$, is written as

$$\hat{\theta}_{\rm RPA} = i \sum_k \theta_k (\hat{\gamma}_k^\dagger - \hat{\gamma}_k) \,. \tag{7}$$

The RPA contribution to the moment of inertia, $I^{(2)}$, the RPA angle operator $\hat{\theta}_{\rm RPA}$ and the conjugate operator $\hat{n}_{\rm RPA}$ can be obtained from the system of equations [14]:

$$\begin{aligned} [H_{\rm RPA}, \hat{n}_{\rm RPA}] &= 0 \,, \\ [H_{\rm RPA}, \hat{\theta}_{\rm RPA}] &= -i \frac{\hat{n}_{\rm RPA}}{I^{(2)}} \,, \\ [\hat{\theta}_{\rm RPA}, \hat{n}_{\rm RPA}] &= i. \end{aligned} \tag{8}$$

These commutators lead to the expressions [1,2]:

$$\begin{aligned} \theta_k &= -\frac{1}{4 I^{(2)} \zeta_2}, \left(\zeta_1 \frac{\epsilon_k - \lambda}{E_k^2} + \zeta_2 \frac{\Delta}{E_k^2}\right), \\ I^{(2)} &= \frac{\Delta}{4\zeta_2} \left(\zeta_1^2 + \zeta_2^2\right), \end{aligned} \tag{9}$$

with [1,2]

$$\begin{aligned} \zeta_1 &= \sum_k [k] \frac{\epsilon_k - \lambda}{E_k^3} \,, \\ \zeta_2 &= \Delta \sum_k \frac{[k]}{E_k^3} \,. \end{aligned} \tag{10}$$

In the above equations, the quantities ϵ_k are the single-particle energies, E_k are the quasiparticle energies and $[k]$ is half the value of the single j-shell degenerancy.

The RPA Hamiltonian in its final form reads

$$H_{\rm RPA} = H_0 + \frac{\hat{n}^2_{\rm RPA}}{2I^{(2)}} + \sum_{n(\omega_n\neq 0)} \omega_n \hat{\Gamma}^\dagger_n \hat{\Gamma}_n \,, \qquad (11)$$

where

$$\Gamma^\dagger_n = \sum_k (X_{nk}\gamma^\dagger_k - Y_{nk}\gamma_k) \qquad (12)$$

are the operators which create physical phonons with energies ω_n.

The spurious phonons sector of the Hamiltonian is taken from [1,2] and it reads

$$\hat{H}^{\rm (spurious)}_{\rm BRST} = \omega(\overline{a}a + \overline{b}b) + \omega({\Gamma_1}^\dagger\Gamma_1 - {\Gamma_0}^\dagger\Gamma_0)\,, \qquad (13)$$

where

$$\begin{aligned}
\Gamma^\dagger_1 &= \sqrt{\frac{1}{2I^{(2)}\omega}}\hat{n}_{\rm RPA} + i\sqrt{\frac{I^{(2)}\omega}{2}}\hat{\theta}_{\rm RPA} - \sqrt{\frac{I^{(2)}}{2\omega}}\hat{\Omega}\,,\\
\Gamma^\dagger_0 &= -i\sqrt{\frac{I^{(2)}\omega}{2}}\hat{\theta}_{\rm RPA} + \sqrt{\frac{I^{(2)}}{2\omega}}\hat{\Omega} + i\sqrt{\frac{\omega}{2I^{(2)}}}\hat{P} \qquad (14)
\end{aligned}$$

are the spurious phonons of energy ω and the operators a and b are linear combinations of the ghost operators, with [1,2]:

$$\begin{aligned}
a &= \frac{1}{\sqrt{2\omega}}\hat{\overline{\pi}} - i\sqrt{\frac{\omega}{2}}\hat{\eta}\,,\\
b &= \frac{1}{\sqrt{2\omega}}\hat{\pi} + i\sqrt{\frac{\omega}{2}}\hat{\overline{\eta}}\,,\\
\overline{a} &= i\frac{1}{\sqrt{2\omega}}\hat{\pi} + \sqrt{\frac{\omega}{2}}\hat{\overline{\eta}}\,,\\
\overline{b} &= -i\frac{1}{\sqrt{2\omega}}\hat{\overline{\pi}} + \sqrt{\frac{\omega}{2}}\hat{\eta}\,, \qquad (15)
\end{aligned}$$

where

$$[a,\overline{a}]_+ = [b,\overline{b}]_+ = 1\,. \qquad (16)$$

The operators $\Gamma^\dagger_n$, Γ_n, $\hat{n}_{\rm RPA}$ and $\hat{\theta}_{\rm RPA}$ form a complete set of operators which create (annihilate) states with finite energy. By inverting the above equations, we can write the operators $\gamma^\dagger_k$ and γ_k as

$$\begin{aligned}
\gamma^\dagger_k = &\sum_{n>1(\omega_n\neq 0)} (X_{nk}\Gamma^\dagger_n + Y_{nk}\Gamma_n)\\
&-\theta_k\sqrt{[k]}\hat{n}_{\rm RPA} + iu_kv_k\sqrt{[k]}\hat{\theta}_{\rm RPA}\,,\\
\gamma_k = &\sum_{n>1(\omega_n\neq 0)} (Y_{nk}\Gamma^\dagger_n + X_{nk}\Gamma_n)\\
&-\theta_k\sqrt{[k]}\hat{n}_{\rm RPA} - iu_kv_k\sqrt{[k]}\hat{\theta}_{\rm RPA}\,. \qquad (17)
\end{aligned}$$

The corresponding RPA amplitudes are given by

$$\begin{aligned}
X_{1k} &= \left(\frac{u_kv_k}{I^{(2)}\omega} - \theta_k\right)\sqrt{\frac{I^{(2)}\omega}{2}}\,,\\
Y_{1k} &= -\left(\frac{u_kv_k}{I^{(2)}\omega} + \theta_k\right)\sqrt{\frac{I^{(2)}\omega}{2}}\,,\\
X_{0k} &= -\theta_k\sqrt{\frac{I^{(2)}\omega}{2}}\,,\\
Y_{0k} &= -\theta_k\sqrt{\frac{I^{(2)}\omega}{2}}\,. \qquad (18)
\end{aligned}$$

The quantities X_{nk} (Y_{nk}), with $n \neq 0,\ 1$, are the forward (backward) amplitudes of the non-zero energy modes of the RPA Hamiltonian.

Up to this point, we have followed literally the procedure outlined in [1,2]. For the benefit of the readers who are unfamiliar with the BRST method, and in order to preserve the meaning of the introduced BRST degrees of freedom, we have strictly copied the notation of [1,2]. The perturbative treatment of the residual interactions between intrinsic and collective degrees of freedom has been presented in [2], where $[k] = j_k + 1/2$ was taken as the order parameter. The ordering of the terms (see [1]) is then uniquely defined and one has to add all possible terms. This may be a rather difficult task, mostly because at each order in the expansion parameter several groups of different diagrams can contribute.

Let us now introduce the Dyson boson expansion [11] as alternative to the use of NFT [10]. We introduce the ideal boson operators $b^\dagger_k$ (b_k),

$$\begin{aligned}
\hat{\gamma}^\dagger_k &= \frac{1}{\sqrt{[k]}}\hat{b}^\dagger_k([k] - \hat{b}^\dagger_k\hat{b}_k)\,,\\
\hat{\gamma}_k &= \sqrt{[k]}\hat{b}_k\,,\\
\hat{\nu}_k &= 2\hat{b}^\dagger_k\hat{b}_k\,, \qquad (19)
\end{aligned}$$

with $[\hat{b}_k, \hat{b}^\dagger_k] = 1$.

After the Dyson boson mapping is performed on the operators of eqs. (11)-(15) the residual BRST Hamiltonian in the Dyson boson basis reads

$$\begin{aligned}
\hat{H}^{\rm (res)}_{\rm BRST} = &-\sum_{kk'} r_{kk'}\left(\sqrt{\frac{[k']}{[k]}}\hat{b}^\dagger_k\hat{b}^\dagger_k\hat{b}_k\hat{b}_{k'} + \sqrt{\frac{[k]}{[k']}}\hat{b}^\dagger_{k'}\hat{b}^\dagger_{k'}\hat{b}_{k'}\hat{b}_k\right)\\
&-\sum_{kk'} s_{kk'}\left(\sqrt{\frac{[k']}{[k]}}\hat{b}^\dagger_k\hat{b}^\dagger_k\hat{b}_k\hat{b}^\dagger_{k'} + \sqrt{\frac{[k]}{[k']}}\hat{b}^\dagger_k\hat{b}^\dagger_{k'}\hat{b}^\dagger_{k'}\hat{b}_{k'}\right)\\
&+\sum_{kk'}\frac{s_{kk'}}{\sqrt{[k'][k]}}\hat{b}^\dagger_k\hat{b}^\dagger_k\hat{b}_k\hat{b}^\dagger_{k'}\hat{b}^\dagger_{k'}\hat{b}_{k'}\\
&+\sum_{kk'} h_{kk'}\sqrt{[k]}(\hat{b}^\dagger_k\hat{b}^\dagger_{k'}\hat{b}_{k'} + \hat{b}^\dagger_{k'}\hat{b}_{k'}\hat{b}_k)\\
&-\sum_{kk'}\frac{h_{kk'}}{\sqrt{[k]}}\hat{b}^\dagger_k\hat{b}^\dagger_k\hat{b}_k\hat{b}^\dagger_{k'}\hat{b}_{k'} - V\left(\sum_k 2n_k\hat{b}^\dagger_k\hat{b}_k\right)^2\\
&-\hat{\Omega}2\hat{n}_{11} - \omega^2\hat{\overline{\eta}}[\hat{\theta}_{\rm RPA}, 2\hat{n}_{11}]\hat{\eta}\,, \qquad (20)
\end{aligned}$$

with

$$r_{kk'} = -\frac{V}{2}(u_k^2 u_{k'}^2 + v_k^2 v_{k'}^2)\,,$$
$$s_{kk'} = \frac{V}{2}(u_k^2 v_{k'}^2 + v_k^2 u_{k'}^2)\,,$$
$$h_{kk'} = -2V u_{k'} v_{k'}(u_k^2 - v_k^2)\,, \qquad (21)$$

and

$$\hat{n}_{11} = \frac{1}{2}(u_k^2 - v_k^2)\hat{b}_k^\dagger \hat{b}_k\,. \qquad (22)$$

In order to treat the Hamiltonian of eq. (20), we shall introduce a two-step procedure. First, we shall define the expectation value of the operator $\hat{b}_k^\dagger \hat{b}_k$ on the RPA vacuum

$$\langle |\, \hat{b}_k^\dagger \hat{b}_k \,| \rangle \equiv \phi_k\,, \qquad (23)$$

as a parameter in eq. (20), before transforming to the phonon basis. Next, by using the Wick theorem, we shall write the residual Hamiltonian of eq. (20) in the phonon basis. The final result reads

$$\begin{aligned} H_{\rm BRST}^{\rm (res)} &= V_0(\phi) \\ &+ \sum_n (V_1(n,\phi)\Gamma_n^\dagger + V_1'(n,\phi)\Gamma_n) \\ &+ \sum_{nn'} V_2(n,n',\phi)\Gamma_n^\dagger \Gamma_{n'} \\ &+ \sum_{nn'} (V_3(n,n',\phi)\Gamma_n^\dagger \Gamma_{n'}^\dagger + V_3'(n,n',\phi)\Gamma_{n'}\Gamma_n) \\ &+ \sum_{mnn'} (V_4(n,n',m,\phi)\Gamma_m^\dagger \Gamma_{n'}^\dagger \Gamma_n^\dagger \\ &+ V_4'(n,n',m,\phi)\Gamma_n \Gamma_{n'}\Gamma_m) \\ &+ \sum_{nn'm} (V_5(n,n',m,\phi)\Gamma_n^\dagger \Gamma_{n'}^\dagger \Gamma_m \\ &+ V_5'(n,n',m,\phi)\Gamma_m^\dagger \Gamma_{n'}\Gamma_n) \\ &+ \sum_n V_6(n,\phi) \\ &\times(\overline{a}a + \overline{b}b - 1 - i\overline{a}\overline{b} - iab)(\Gamma_n^\dagger + \Gamma_n)\,, \end{aligned} \qquad (24)$$

with n, n' and m running over all possible one phonon states. The expressions for the vertex functions V_i are given in Appendix B. The spectrum of the Hamiltonian $H_{\rm BRST}$ has non-zero eigenvalues. At this point, we can proceed, as done in [1], to calculate perturbatively the corrections to the ground-state energy and to the energy of the one-phonon state. The vertex functions V of eq. (24) contain different powers of the parameter ϕ_k and the Hamiltonian of eq. (24) is acting in the subspace which includes states up to three phonons. In order to treat $H_{\rm BRST}^{\rm (res)}$ perturbatively, the left and right basis must contain the same states. In the next section, we shall show that the use of the parameter ϕ_k of eq. (23) may simplified the perturbative treatment of the Hamiltonian of eq. (20). In eq. (24) terms which are proportional to powers of $\hat{b}_k^\dagger \hat{b}_k$ have been replaced by powers of the parameter ϕ_k. In this way, we are introducing a sort of density-dependent approximation, similar to the one used in some Green functional approaches of boson correlations [15]. As said before, the actual value of ϕ_k can be fixed by a variation restricted to the perturbative corrections of eq. (24).

Table 1. Contributions to the energy of the ground state. The values of n, listed in the first column, indicate the number of phonons included in each vertex. The notation is explained in Appendix B.

$n=0$	$V_0(\phi)$
1	$\frac{V_1(r,\phi)V_1'(r,\phi)}{E_0-\omega_r}$
2	$\frac{2V_3(r,r,\phi)V_3'(r,r,\phi)}{E_0-2\omega_r} + \frac{2V_3(1,1,\phi)V_3'(1,1,\phi)}{E_0-2\omega}$ $+\frac{2V_3(0,0,\phi)V_3'(0,0,\phi)}{E_0+2\omega} + \frac{V_3(0,1,\phi)V_3'(0,1,\phi)}{E_0}$
3	$\frac{6V_4(r,r,r',\phi)V_4'(r,r,r',\phi)}{E_0-3\omega_r} + \frac{2V_4(r,1,1,\phi)V_4'(r,1,1,\phi)}{E_0-\omega_r-2\omega}$ $+\frac{2V_4(r,0,0,\phi)V_4'(r,0,0,\phi)}{E_0-\omega_r+2\omega} + \frac{V_4(r,1,0,\phi)V_4'(r,1,0,\phi)}{E_0-\omega_r}$

Table 2. Contributions to the energy of the one-phonon state. The values of n, given in the first column, correspond to the number of phonons entering in each vertex function. The notation is explained in Appendix B.

$n=0$	$V_0(\phi)$
1	$-4\frac{V_1(r,\phi)V_1'(r,\phi)}{\omega_r}$
2	$V_2(r,r,\phi)$ $-\frac{2V_3(r,1,\phi)V_3'(r,1,\phi)}{\omega_r+\omega} - \frac{2V_3(r,0,\phi)V_3'(r,0,\phi)}{\omega_r-\omega}$ $-\frac{2V_3(r,r,\phi)V_3'(r,r,\phi)}{2\omega_r}$
3	$-\frac{V_5(r,r,r\phi)V_5'(r,r,r,\phi)}{\omega_r} + \frac{V_5(0,1,r,\phi)V_5'(0,1,r,\phi)}{\omega_r}$ $+\frac{V_5(1,1,r,\phi)V_5'(1,1,r,\phi)}{\omega_r-2\omega} + \frac{V_5(0,0,r,\phi)V_5'(0,0,r,\phi)}{\omega_r+2\omega}$ $-\frac{V_4(r,r,r,\phi)V_4'(r,r,r,\phi)}{3\omega_r} - \frac{V_4(r,1,1,\phi)V_4'(r,1,1,\phi)}{\omega_r+2\omega}$ $+\frac{V_4(r,0,0,\phi)V_4'(r,0,0,\phi)}{-\omega_r+2\omega} - \frac{V_4(r,1,0,\phi)V_4'(r,1,0,\phi)}{\omega_r}$

3 Results and discussions

In this section, we shall discuss the use of the above-presented formalism in the case of a two-level model. Furthermore, we shall assume that the parameter ϕ_k is in-

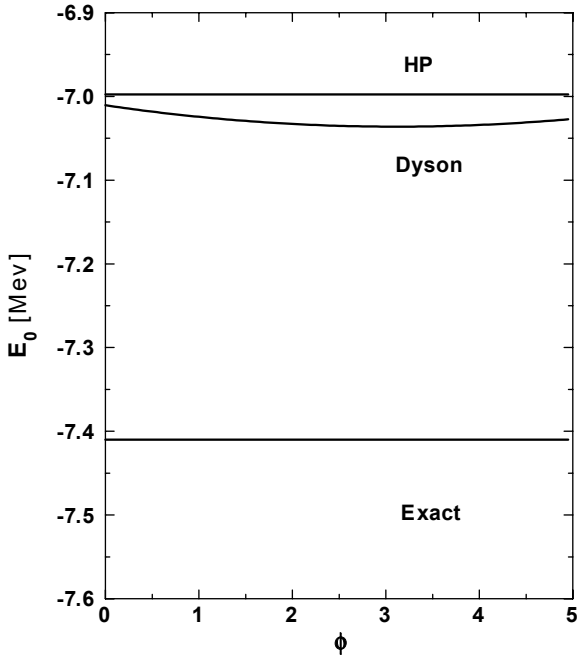

Fig. 1. Contribution to the energy of the ground state, as a function of the parameter ϕ, for the case $\Omega = 5$ and $x = 0.5$. With HP and Dyson we are denoting the results of the Holstein-Primakoff and Dyson boson expansion methods, respectively.

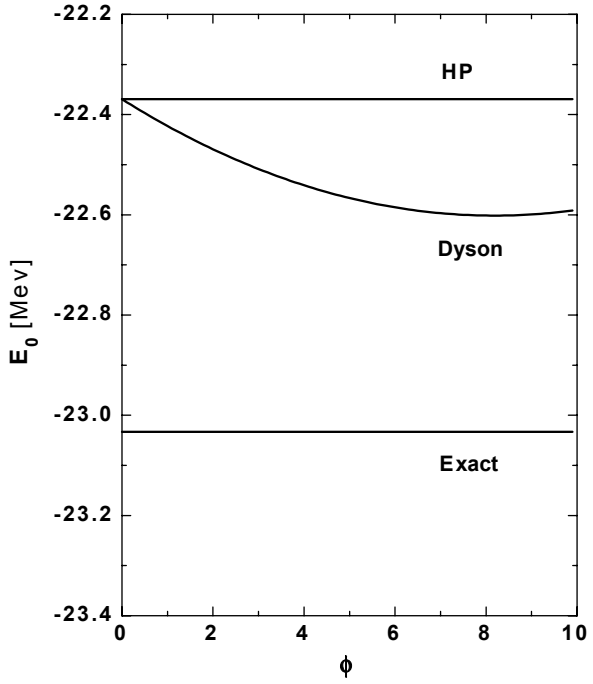

Fig. 2. Same as fig. 1, for the case $\Omega = 10$ and $x = 0.35$.

dependent of the single-particle index k. This approximation is an acceptable one in the spirit of the RPA method. This approximation is justified if one thinks that the ideal bosons represent collective bosons. Otherwise, the single variation of the the perturbative corrections generated by eq. (24) would have to be replaced by a multi-variation. The fact that the perturbative corrections can be minimized by choosing ϕ appropriately, as we shall see from the results, seems to indicate the feasibility of the procedure. In Appendix B, we have specified the expressions of sect. 2 for the case of a single-particle space consisting of two shells. In the numerical applications we have solved the BCS equations and we have constructed the vertex functions of eq. (24). Next, we have calculated the zero-order spectrum of $H_{\rm BRST}$ and performed perturbative calculations of the ground-state and one-phonon excited-state energies. The explicit form of the contributions to the ground-state energy is given in table 1, where each diagram is written in terms of the vertex functions V_n. The corresponding expressions for V_n are given in Appendix B. The corrections to the energy of the one-phonon state are listed in table 2. As done for the case of the corrections to the ground-state energy, each vertex function of table 2 is written in terms of the Hamiltonian amplitudes of Appendix B. In order to compare the results of the different approximations, we have calculated the diagrams of tables 1 and 2 for the case of a reduced model space consisting of two single-particle shells with degeneracy $[k] = 5$ (case 1) and $[k] = 10$ (case 2). For both model spaces we have calculated exact solutions and perturbed ones in the Holstein-Primakoff and Dyson approaches. For the case of the Holstein-Primakoff we have considered only diagrams at leading order in $1/[k]$ [1].

The exact solution has been obtained as shown in [13], for $N = 2[k]$ and $x \equiv G\,[k]/(2\epsilon)$. In figs. 1, 2 and figs. 3, 4, the results corresponding to the energy of the ground state and to the first excited state, respectively, are shown as a function of the parameter ϕ. These results have been obtained by fixing the parameters in the superfluid phase of the model. For the case of the ground-state energy the Dyson mapping gives a better result than the Holstein-Primakoff approximation and it saturates. For the case of the first excited state, the results of the Dyson approach are better, as compared with the exact results, than the Holstein-Primakoff ones and the agreement is restricted to a narrow interval of the values of ϕ. For these cases, figs. 3 and 4, the results of the Dyson approach reach a minimum at values of ϕ of the order of 2-3.

Finally, with the aim of comparing the degree of accuracy of the different approximations discussed in this work, we have calculated the ground-state energy using the projection method of [9]. The results, corresponding to the two-level model space, are shown in table 3. As expected, the results obtained with the BRST approach, are close to the results obtained by applying the projection technique of Hara *et al.* [9]. It is seen that the agreement improves when the shell degenerancy Ω increases. For $\Omega = 10$ the BRST + Dyson approach and the particle number projection yield the same result, which differs in $\simeq 2\%$ from the exact value.

From these results, we can conclude that the use of the Dyson boson expansion method can improve the convergence of the BRST perturbative expansion.

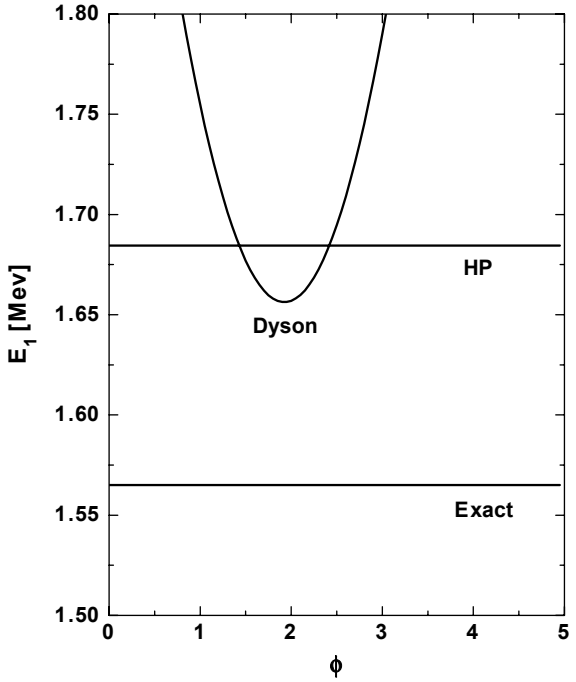

Fig. 3. Contribution to the energy of the one-phonon state as a function of the order parameter, for the case $\Omega = 5$ and $x = 0.5$.

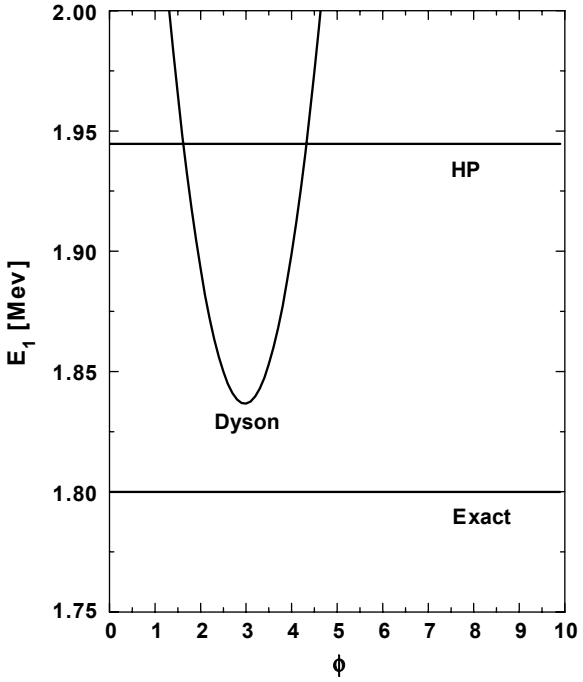

Fig. 4. The same as fig. 3 for the case $\Omega = 10$ and $x = 0.35$.

Naturally, in performing these approximations, we are incorporating higher-order terms in the $1/[k]$ expansion. The Holstein-Primakoff expansion allows for fixing an order in $1/[k]$ but it may be rather cumbersome to implement at higher orders. At this point, we would like to remind the reader that the conventional perturbative approach applied to the Hamiltonian H of eq. (1) would simply yield divergent results.

Table 3. Ground-state energy, E_0, calculated in the approximations discussed in the text. The values are given in units of MeV and the meaning of the columns is the following: standard BCS approach (BCS), BRST + Holstein-Primakoff boson mapping (HP), BRST + Dyson boson mapping (Dyson), particle number projection (P_N) and exact results (Exact). The values shown in the first and second row have been obtained by using the parameters Ω and x given in the captions of figs. 1 and 2, respectively.

E_0 (BCS)	E_0 (HP)	E_0 (Dyson)	E_0 (P_N)	E_0 (Exact)
−6.87	−7.00	−7.05	−7.28	−7.41
−22.20	−22.39	−22.60	−22.60	−23.03

4 Conclusions

In this paper we have applied the BRST method to treat the schematic pairing force Hamiltonian, as it was first done by Bes *et al.* [1–4]. We have compared the results of the combined BRST + boson expansion for two different boson expansion methods, namely the Holstein-Primakoff and the Dyson one. From the comparison between these results, for the ground-state and first-excited-state energies, we can conclude that the use of the Dyson boson expansion method can improve the agreement with exact solutions. As a matter of fact, these conclusions apply to the considered two-level model situation but this may be also the case of the other known applications of the BRST method [5]. More work along this line will be done in the future.

This work has been partially supported by the CONICET of Argentina. (M.R.) acknowledges the hospitality of the INT at the University of Washington. (O.C.) thanks Drs P. Hess, H. Geyer and J. Dobaczewski, for suggestions and comments about the manuscript, and the INT at the University of Washington, Seattle, for its hospitality during the INT-00-3 program.

Appendix A.

We can write the pairing Hamiltonian in the quasiparticle basis as

$$\tilde{H} = \hat{H} - \lambda 2\hat{n}\,,$$
$$\tilde{H} = H_{00} + H_{11} + H_{20} + H_{22} + H_{40} + H_{31} + H_{\text{qp-qp}}\,,$$

$$H_{00} = \sum_k 2{v_k}^2(\epsilon_k - \lambda) - \frac{\Delta^2}{V}\,,$$
$$H_{11} = \sum_k((\epsilon_k - \lambda)({u_k}^2 - {v_k}^2) - \Delta 2u_k v_k)\hat{\nu}_k\,,$$

$$H_{20} = \sum_k ((\epsilon_k - \lambda)2u_k v_k - \Delta({u_k}^2 - {v_k}^2))(\hat{\gamma}_k^\dagger + \hat{\gamma}_{\overline{k}}^\dagger)\,,$$

$$H_{22} = -V \sum_{kk'} \frac{1}{2}({u_k}^2 {u_{k'}}^2 + {v_k}^2 {v_{k'}}^2)(\hat{\gamma}_k^\dagger \hat{\gamma}_{k'} + \hat{\gamma}_{k'}^\dagger \hat{\gamma}_k)\,,$$

$$H_{40} = V \sum_{kk'} \frac{1}{2}({u_k}^2 {v_{k'}}^2 + {v_k}^2 {u_{k'}}^2)(\hat{\gamma}_k^\dagger \hat{\gamma}_{k'}^\dagger + \hat{\gamma}_{k'} \hat{\gamma}_k)\,,$$

$$H_{31} = -V \sum_{kk'} u_{k'} v_{k'} ({u_k}^2 - {v_k}^2)(\hat{\gamma}_k^\dagger \hat{\nu}_{k'} + \hat{\nu}_{k'} \hat{\gamma}_k)\,,$$

$$H_{\text{qp-qp}} = -V \left(\sum_k u_k v_k \hat{\nu}_k \right)^2 .$$

In these expressions the operators $\hat{\nu}_k$ is the quasiparticle number operator and $\hat{\gamma}_k^\dagger$ ($\hat{\gamma}_k$) create (annihilate) a pair of quasiparticles.

Appendix B.

The vertex functions of the BRST Hamiltonian are given in this Appendix. We have adopted the notation of indexes corresponding to the two-level model limit. The argument of each vertex function V_n will contain one or more indexes describing the type of phonons that can be connected at the vertex. The index r represents real phonons, the indexes 0 and 1 represent spurious 0 and 1 phonons (see eq. (14)), respectively. As an illustration, we provide here vertex functions V_n, up to two phonons. Similar expressions are obtained for higher values of the number of phonons:

$$V_0(\phi) = 4V\phi^2 \left(1 - \frac{3}{2}(2uv)^2\right),$$

$$V_1(r,\phi) = 2h_{31}V \left(\frac{E}{2\omega_r}\right)^{1/2} \times \left[\left(3\phi - \frac{2\phi(1+4\phi)}{[k]}\right)\left(1 + \frac{\omega_r}{2E}\right) + 3\phi\left(1 - \frac{\omega_r}{2E}\right)\right] + 2\theta\omega \left(\frac{2E}{\omega_r}\right)^{1/2},$$

$$V_1'(r,\phi) = 2h_{31}V \left(\frac{E}{2\omega_r}\right)^{1/2} \times \left[\left(3\phi - \frac{2\phi(1+4\phi)}{[k]}\right)\left(1 - \frac{\omega_r}{2E}\right) + 3\phi\left(1 + \frac{\omega_r}{2E}\right)\right] + 2\theta\omega \left(\frac{2E}{\omega_r}\right)^{1/2},$$

$$V_2(r,r,\phi) = \frac{EV}{\omega_r}\left[f_1(\phi)\left(1 - \frac{\omega_r^2}{4E^2}\right) + 2f_2(\phi)\right],$$

$$V_2(0,0,\phi) = I^{(2)}\omega\theta^2 V\left[f_3(\phi) + 2f_4(\phi)\right],$$

$$V_2(1,1,\phi) = I^{(2)}\omega V \times \left[f_3(\phi)\left(\theta^2 - \frac{(uv)^2}{(I^{(2)}\omega)^2}\right) + f_4(\phi)\left(\theta^2 + \frac{(uv)^2}{(I^{(2)}\omega)^2}\right)\right],$$

$$V_2(0,1,\phi) = I^{(2)}\omega\theta V \times \left[f_3(\phi)\left(\theta + \frac{uv}{I^{(2)}\omega}\right) + f_4(\phi)\theta\right],$$

$$V_2(1,0,\phi) = I^{(2)}\omega\theta V \times \left[f_3(\phi)\left(\theta - \frac{uv}{I^{(2)}\omega}\right) + f_4(\phi)\theta\right],$$

$$V_3(r,r,\phi) = \frac{EV}{2\omega_r} \times \left[f_1(\phi)\left(1 + \frac{\omega_r}{2E}\right)^2 + 2f_2(\phi)\left(1 - \frac{\omega_r^2}{(2E)^2}\right)\right],$$

$$V_3'(r,r,\phi) = \frac{EV}{2\omega_r} \times \left[f_1(\phi)\left(1 - \frac{\omega_r}{2E}\right)^2 + 2f_2(\phi)\left(1 - \frac{\omega_r^2}{(2E)^2}\right)\right],$$

$$V_3(0,0,\phi) = \frac{1}{2}V_2(0,0,\phi)\,,$$

$$V_3'(0,0,\phi) = V_3(0,0,\phi)\,,$$

$$V_3(1,1,\phi) = \frac{I^{(2)}\omega V}{2} \times \left[f_3(\phi)\left(\theta - \frac{uv}{I^{(2)}\omega}\right)^2 + 2f_4(\phi)\left(\theta^2 - \frac{(uv)^2}{(I^{(2)}\omega)^2}\right)\right],$$

$$V_3'(1,1,\phi) = \frac{I^{(2)}\omega V}{2} \times \left[f_3(\phi)\left(\theta + \frac{uv}{I^{(2)}\omega}\right)^2 + 2f_4(\phi)\left(\theta^2 - \frac{(uv)^2}{(I^{(2)}\omega)^2}\right)\right],$$

$$V_3(0,1,\phi) = V_2(1,0,\phi)\,,$$

$$V_3'(0,1,\phi) = V_2(0,1,\phi)\,,$$

with

$$f_1(\phi) = -2(uv)^2(1+6\phi)\left(1 - \frac{2phi}{[k]}\right) + (1 - 2u^2v^2)4\phi\left(1 - \frac{\phi}{[k]}\right),$$

$$f_2(\phi) = (1 - 2u^2v^2)4\phi - (2uv)^2(1+6\phi) - (2uv)^2\phi\,,$$

$$f_3(\phi) = -2(uv)^2(1+6\phi)\left(1 - \frac{2\phi}{[k]}\right) - (1 - 2u^2v^2)4\phi\left(1 - \frac{\phi}{[k]}\right),$$

$$f_4(\phi) = (1 - 2u^2v^2)4\phi - (2uv)^2(1+6\phi) + (2uv)^2\phi\,,$$

$$h_{31} = [k]^{1/2}2uv(u^2 - v^2).$$

In these expression we have introduced the factors

$$\theta = -\frac{E}{2\Delta[k]}\,,$$

and

$$I^{(2)} = \frac{\Delta^2[k]}{2E^3}\,.$$

The rest of the notation is the one given in the text.

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