Results in Physics 27 (2021) 104485

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.elsevier.com/locate/rinp

Two-sites' spin chain as a good statistical representative of an infinite one

F. Pennini [a,b,*], A. Plastino [c,d]

[a] Departamento de Física, Universidad Católica del Norte, Av. Angamos 0610, Antofagasta, Chile
[b] Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de La Pampa, CONICET, Av. Peru 151, 6300, Santa Rosa, La Pampa, Argentina
[c] Instituto de Física La Plata–CCT-CONICET, Universidad Nacional de La Plata, C.C. 727, 1900, La Plata, Argentina
[d] SThAR – EPFL, Lausanne, Switzerland

ARTICLE INFO

Keywords:
Complexity studies
Symbolic quantifiers
Few-fermions' system

ABSTRACT

We apply information-theory based symbolic quantifiers, related to complexity studies, in order to investigate interacting spin chains, in which there is competition between one-body and two-body quantum interactions. Our system of reference is a well-known *XY* model with *N*-spins. We show that most of its thermal features can already be detected with $N = 2$.

Preliminaries

Goals and motivation

Consider a system subject to a one-body external field's (intensity h) action. The system itself is bonded by two-body forces of strength k. We wish to study the correlation $k - h$ via symbolic quantifiers for just a few fermions. In other words, we aspire to provide a study of *N*-spins dynamical features using just symbolic tools of mostly statistical nature. This kind of analysis has been extensively developed in the extant literature –see, for instance, Refs. [1–8] and references therein– and provides deep comprehension regarding the correlation between purely dynamic and exclusively statistical aspects of some physical problems. To go further in such direction is our present motivation, in particular into what respects the influence of the number of particles in the complexity of the concomitant physics. We show that most thermal properties become apparent with just $N = 2$.

The novelty here resides in that:

- The system under scrutiny is a many body quantum one, the well known one-dimensional *XY* model (one finds hundreds of references. See, for instance, Refs. [9–15], and references therein).
- Much work on it revolves around the thermodynamic limit (very large number of particles). Here we will deal with only a few spin less fermions. The case of just two of them will reveal notable features.
- We appeal to complexity-quantifiers [16–27]. Interesting insights will be shown to be gained.
- It becomes transparent here the way in which average occupation values of the *XY*-sites (and their derivatives) determine thermal *XY* features.

The relevant symbolic information quantifiers

Even if one has some knowledge regarding the degree of unpredictability and randomness of a system, this does not entail that one can adequately grasp the extant correlation-structures (CS) that, in turn, strongly influence the prevailing density matrix $\widehat{\rho}$. One would want to capture these CS in the manner in which the entropy captures disorder.

The opposite extremal situations of (a) perfect order and (b) maximal randomness should not be endowed with important SC [16]. In between (a) and (b) a variegated range of SC-degrees can exist, that should be reflected by the features of $\widehat{\rho}$. Giving recipes on how this could be done presents serious difficulties.

Crutchfield is quoted as stating in 1994 that "Physics does have the tools for detecting and measuring complete order equilibria and fixed point or periodic behavior and ideal randomness via temperature and thermodynamic entropy or, in dynamical contexts, via the Shannon entropy rate and Kolmogorov complexity [17,18]. What is still needed, though, is a definition of structure and a way to detect and to measure it". Seth Lloyd enumerated as many as 40 ways of defining complexity, none of them satisfactory enough [17,18].

More specifically, we wish that some adequate functional of $\widehat{\rho}$ may enable one to grasp correlations as entropy captures randomness. A breakthrough was reached by the complexity-definition of López-Ruiz, Mancini and Calbet [16]. LMC's complexity did separate and quantify

* Corresponding author.

https://doi.org/10.1016/j.rinp.2021.104485
Received 6 July 2020; Accepted 19 June 2021
Available online 28 June 2021

contributions coming from Shannon's entropy or information S and structure. The structural portion was encoded by the concept of disequilibrium, denoted by D. This quantity measures, in Hilbert space, the distance from the actual $\hat{\rho}$ to an almost-uniform density matrix. If from $\hat{\rho}$ we can extract a, M-components probability distribution (PD) P, $D = D(P)$ is given by [16]

$$D = \sum_{i=1}^{M} (p_i - 1/M)^2, \tag{1}$$

where $p_1, p_2, \ldots, p_M$ are the corresponding probabilities, with the condition $\sum_{i=1}^{M} p_i = 1$. From now onwards, we always assume that $\hat{\rho}$ yields such a PD P.

Organization

The paper is organized as follows. In Section "Background: symbolic quantifiers and thermal ones" we introduce our symbolic quantifiers. In Section "Few spin-sites system illustrating the 1-body-2-body competition" we recall details of the XY-model and of the essential thermodynamic quantifiers employed to describe it. Sections "The model for $N = 2$" and "Case $N = 3$" are devoted to the quantum many-body model to be used here, with applications given in Section "Cases $N = 4$ and $N = 1000$". Finally, some conclusions are drawn in Section "Conclusions".

Background: symbolic quantifiers and thermal ones

We assume a system in equilibrium at temperature T whose normalized probability distribution is given by

$$p_i = \frac{e^{-\beta E_i}}{\sum_{i=1}^{M} e^{-\beta E_i}}, \tag{2}$$

with $\beta = 1/k_B T, k_B$ is the Boltzmann constant, and E_i denotes the eigenvalues of the pertinent Hamiltonian.

It is well known that the Helmholtz' free energy F is written as [28]

$$F(T) = -k_B T \ln \sum_{i=1}^{M} e^{-\beta E_i}. \tag{3}$$

Following the same route that López-Ruiz traverses in Ref. [21], we can demonstrate that

$$D = \exp(2\beta(F(T) - F(T/2))) - 1/M. \tag{4}$$

In addition, the LMC-Statistical Complexity is defined as [16]

$$C = DS. \tag{5}$$

Few spin-sites system illustrating the 1-body-2-body competition

Hamiltonian of the one dimensional XY model

Our Hamiltonian contains N fermions influenced by two competing interactions [11].

- a one body external magnetic field along the z-axis, of strength h, plus
- a spin–spin two body interaction between pairs of neighboring particles of spin $\hat{\mathbf{S}}$,so that the ensuing quantum Hamiltonian is

$$\hat{H} = \sum_{m=1}^{N-1} k(\hat{S}_m^x \hat{S}_{m+1}^x + \hat{S}_m^y \hat{S}_{m+1}^y) - h\hat{S}_z^m - h\hat{S}_N^z, \tag{6}$$

with $\hat{\mathbf{S}}$ the Pauli operators for spin 1/2 [11]. The Hamiltonian describes N interacting spins located at N sites.

Our interest here lies in the correlations $h - k$. Smith showed how to diagonalize $\hat{H}$ [11] and find then the N energy-eigenvalues ϵ_m. We assume here from that we have at our disposal the pertinent values of the ϵ_m. After appropriate manipulation Smith [11] arrives at

$$\hat{H} = \frac{Nh}{2} + \sum_{\alpha=1}^{N} \lambda_\alpha \hat{\mu}_\alpha^\dagger \hat{\mu}_\alpha, \tag{7}$$

where $\hat{\mu}_\alpha$ are non-interacting fermion destruction operators and λ_α are the eigenvalues of the $N \times N$ matrix $\hat{A} = \hat{B} - h\hat{I}$. $\hat{I}$ is the $N \times N$ identity matrix. Moreover, the eigenvalues of the matrix $\hat{B}$ are given by $B_{m,m+1} = B_{m+1,m} = k/2$, otherwise $B_{m,n} = 0$ [11]. For our present purposes, as we will see in the next Section, we rename $\lambda_\alpha = -\epsilon_\alpha$.

Thermodynamic quantifiers and the energy spectrum

Following Ref. [11], the free energy of the system resulting of the Hamiltonian (7) is

$$F = \frac{h}{2} - \frac{k_B T}{N} \sum_{\alpha=1}^{N} \ln(1 + \exp(-\beta\epsilon_\alpha)), \tag{8}$$

where ϵ_α are the eigenvalues of the tridiagonal matrix $\hat{A}$ whose analytical forms are given by [29]

$$\epsilon_\alpha = h - k\cos\left(\frac{\alpha\pi}{N+1}\right), \quad \alpha = 1, \ldots, N. \tag{9}$$

The energy-difference between neighbor eigenvalues α and $\alpha + 1$, that we call $\Delta\epsilon_\alpha = \epsilon_{\alpha+1} - \epsilon_\alpha$, is plotted in Fig. 1 versus α for $N = 1000$ and different coupling constants k. It is seen that it tends to vanish as α grows. This fact will be of great importance for our purposes. We will see that the zero-eigenvalues tend to accumulate themselves within a small k-range of values.

After some algebraic manipulation, we can now rewrite Eq. (8) as follows

$$F = F_0 - \frac{k_B T}{N} \sum_{\alpha=1}^{N} \ln(2\cosh(-\beta\epsilon_\alpha/2)), \tag{10}$$

where we have defined $F_0 = h/2 + (1/(2N))\sum_{\alpha=1}^{N} \epsilon_\alpha$. Note that Eq. (10) was called a measure of the disorder of the system in Ref. [12]. This makes it natural to connect the free energy with the disequilibrium D, as explicitly reported at Eq. (4).

The entropy is obtained from the thermodynamic relation $S = -(\partial F/\partial T)_N$. Thus, we immediately get

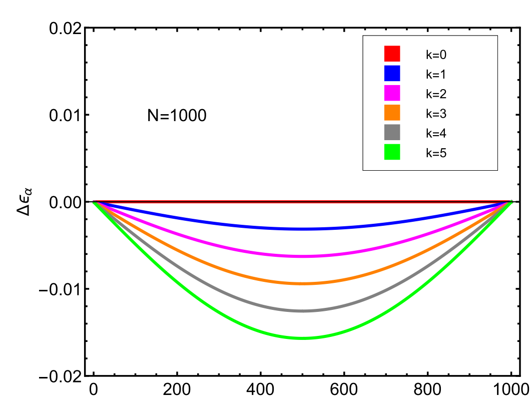

Fig. 1. Energy-differences $\Delta\epsilon_\alpha$ between neighboring-eigenvalues ϵ_α versus α, for several k-values and $N = 1000$. We take $h = 1$.

$$S = \frac{1}{NT}\sum_{\alpha=1}^{N}\frac{\epsilon_\alpha}{1+\exp(\beta\epsilon_\alpha)} + \frac{k_B}{N}\sum_{\alpha=1}^{N}\ln(1+\exp(-\beta\epsilon_\alpha)). \tag{11}$$

Recall that we call our pertinent eigenvalues $-\epsilon_\alpha$. From the connection $F = U - TS$ one easily obtains for the mean energy U the following expression

$$U = \frac{1}{N}\sum_{\alpha=1}^{N}\frac{\epsilon_\alpha}{1+\exp(\beta\epsilon_\alpha)} + \frac{h}{2}. \tag{12}$$

The magnetization is obtained from $M = -(\partial F/\partial h)_T$ [11]. Thus, we have

$$M = -\frac{1}{2} - \frac{1}{N}\sum_{\alpha=1}^{N}\frac{1}{1+\exp(\beta\epsilon_\alpha)}, \tag{13}$$

where, in view of Eq. (9), we have considered that $(\partial\epsilon_\alpha/\partial h)_T = 1$ for all α.

The magnetic susceptibility is defined according to $\chi = (\partial M/\partial h)_T$. Thus we have,

$$\chi = \frac{1}{Nk_BT}\sum_{\alpha=1}^{N}\frac{\exp(\beta\epsilon_\alpha)}{(1+\exp(\beta\epsilon_\alpha))^2}, \tag{14}$$

where we have used again that $(\partial\epsilon_\alpha/\partial h)_T = 1$ for all α. Finally, the specific heat at constant external field is obtained from $C_h = (\partial U/\partial T)_h$ which reads

$$C_h = \frac{1}{Nk_BT^2}\sum_{\alpha=1}^{N}\frac{\epsilon_\alpha^2\exp(\beta\epsilon_\alpha)}{(1+\exp(\beta\epsilon_\alpha))^2}. \tag{15}$$

Statistical complexity

We get the disequilibrium from Eqs. (4) and (8). After some of algebra we arrive at

$$D = \prod_{\alpha=1}^{N}\left\{\frac{1+\exp(-2\beta\epsilon_\alpha)}{(1+\exp(-\beta\epsilon_\alpha))^2}\right\}^{1/N} - 1/N. \tag{16}$$

In addition, the statistical complexity is given by Eq. (5).

Thermodynamic quantifiers and Fermi occupation numbers at zero chemical potential

The ϵ_α can be regarded as "single-fermions energies" and the associated, effective occupation numbers become (zero chemical potential)

$$\langle n_\alpha\rangle = \frac{1}{1+\exp(\beta\epsilon_\alpha)}. \tag{17}$$

They depend upon the temperature, and on the two competing coupling constants k, h. Their sum would be a mean total fermion number $\langle N\rangle$. Notice that two useful relations ensue

$$e^{\beta\epsilon_\alpha} = \frac{1-\langle n_\alpha\rangle}{\langle n_\alpha\rangle};\quad e^{-\beta\epsilon_\alpha} = \frac{\langle n_\alpha\rangle}{1-\langle n_\alpha\rangle}. \tag{18}$$

The free energy is now

$$F = F_0 + \frac{k_BT}{N}\sum_{\alpha=1}^{N}\ln\langle n_\alpha\rangle. \tag{19}$$

One also can recast S as

$$S = \frac{1}{NT}\sum_{\alpha=1}^{N}\epsilon_\alpha(\langle n_\alpha\rangle - 1) - \frac{k_B}{N}\sum_{\alpha=1}^{N}\ln\langle n_\alpha\rangle, \tag{20}$$

while the mean energy becomes

$$U = \frac{1}{N}\sum_{\alpha=1}^{N}\epsilon_\alpha\langle n_\alpha\rangle + \frac{h}{2}. \tag{21}$$

Also, we have the new relations

$$M = -\frac{1}{2} - \frac{1}{N}\sum_{\alpha=1}^{N}\langle n_\alpha\rangle, \tag{22}$$

$$\chi = \frac{1}{k_BTN}\sum_{\alpha=1}^{N}\langle n_\alpha\rangle(1-\langle n_\alpha\rangle). \tag{23}$$

$$C_h = \frac{1}{Nk_BT^2}\sum_{\alpha=1}^{N}\epsilon_\alpha^2\langle n_\alpha\rangle(1-\langle n_\alpha\rangle), \tag{24}$$

and

$$D = \prod_{\alpha=1}^{N}\left[(1-\langle n_\alpha\rangle)^2 + \langle n_\alpha\rangle^2\right]^{1/N} - 1/N. \tag{25}$$

All relevant quantities above been expressed in terms of the occupation numbers, as befits a quantum many-body treatment [30].

The model for $N = 2$

This is a paradigmatic case. It will be shown that its complexity does not differ much from that of $N\to\infty$. We will also see that all many body features of the one-dimensional XY-model for discrete, finite N emerge already at this simple level. Moreover, this case can be straightforwardly treated in purely analytic fashion, which greatly facilitates its analysis, because we deal with a 2×2-Hamiltonian matrix $\widehat{A}$ whose form is

$$\widehat{A} = \begin{pmatrix} -h & k/2 \\ k/2 & -h \end{pmatrix}. \tag{26}$$

The diagonalization of the matrix $\widehat{A}$ leads to eigenvalues $\epsilon_1 = -k/2 + h$, $\epsilon_2 = k/2 + h$, and $\epsilon_2 > \epsilon_1$. We transparently appreciate in this simple instance just how straightforwardly the competition k-h is manifested. Also, there is an obvious $k, -k$ symmetry. If N augments, the energy matrix will always be tridiagonal. The diagonal elements will always equal $-h$, while, likewise, the off diagonal elements will keep equaling $k/2$. Thus, the $k-h$ competition will be always clearly reflected by the energy matrix.

Occupation numbers $\langle n_1\rangle$, $\langle n_2\rangle$.

One has a mean total fermion number $\langle N\rangle = \langle n_1\rangle + \langle n_2\rangle$, where the occupation numbers are of the fermion-appearance

$$\langle n_1\rangle = \frac{1}{1+\exp(\beta(-h+k/2))}, \tag{27}$$

and

$$\langle n_2\rangle = \frac{1}{1+\exp(-\beta(h+k/2))}. \tag{28}$$

At $T = 0$ we see that $\langle n_1\rangle = 1$ if $k < 2$ and $\langle n_2\rangle = 1$ if $k > -2$. There are phase transitions at $k_{crit} = \pm 2$. In our two first graphs in Fig. 2 we depict $\langle n_1\rangle$ and $\langle n_2\rangle$. The very low temperature dynamics becomes transparent now. For $k < k_{crit}$ both occupation numbers equal unity. For larger $|k|$', one or the other of the chain-sites becomes empty. Figs. 3 displays $\langle N\rangle$.

Most typical XY features derive from the behavior of the occupation numbers depicted in these graphs. This should be evident from the fact that all our thermal quantifiers can be expressed in terms of the $\langle n_\alpha\rangle$. The drop from $\langle N\rangle = 2$ to unity is the origin of most of the physical effects one uncovers in this system.

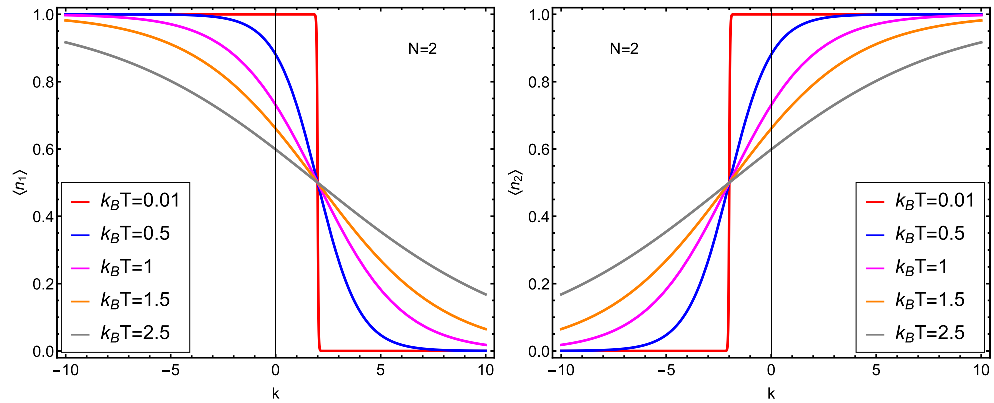

Fig. 2. Left panel: $\langle n_1\rangle$ versus k for several temperatures. Right panel: $\langle n_2\rangle$ versus k for several temperatures. We take $h=1$. There is a nice phase transition at very low T taking place at $k_{crit}=\pm2$.

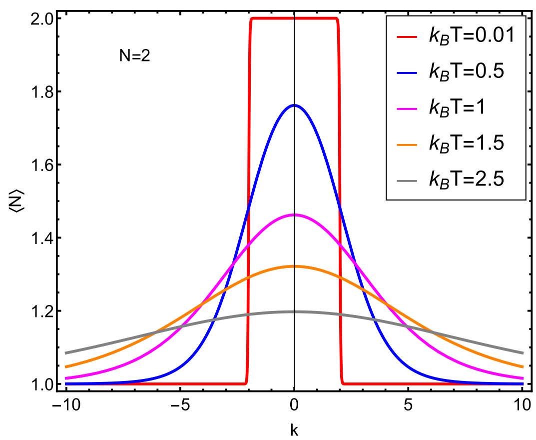

Fig. 3. Mean total fermion number $\langle N\rangle$ versus k for different values of k_BT/h. We take $h=1$ We see again that $k_{crit}=\pm2$.

Mean energy, entropy, and specific heat

The mean energy is here

$$U=\sum_{\alpha=1}^{2}\langle n_\alpha\rangle\epsilon_\alpha+\frac{h}{2}. \tag{29}$$

In Fig. 4 we depict the behavior of U as a function of the strength k. There is a phase transition (at $T\sim0$) when the coupling constant $k_{crit}=\pm2$, which becomes a crossover for finite temperatures, a typical fermion-statistics effect. Also, as one would expect, the system becomes the more bounded the larger $|k|$. At low enough T, one appreciates the k_{crit} influence on the $\langle U\rangle$-behavior.

The entropy now reads

$$S=\frac{1}{2T}\sum_{\alpha=1}^{2}\frac{\epsilon_\alpha}{1+\exp(\beta\epsilon_\alpha)}+\frac{k_B}{2}\sum_{\alpha=1}^{2}\ln(1+\exp(-\beta\epsilon_\alpha)). \tag{30}$$

We detect peaks at $k_{crit}=\pm2$, attenuated as T augments, as seen in Fig. 5. Also, note that $S=0$ at $T=0$.

The specific heat reads

$$C_h(T)=\frac{1}{2}k_BT^2\sum_{\alpha=1}^{2}\frac{\epsilon_\alpha^2\exp(\beta\epsilon_\alpha)}{(1+\exp(\beta\epsilon_\alpha))^2}. \tag{31}$$

We note that, at zero temperature $C_h=0$ for all values of h and k. Indeed, when T tends to infinity, C_h tends to zero. The Schottky anomaly is clearly visible, thus reproducing an effect seen in solid-state physic. Our specific heat at low temperature has a peak, and, at high T's, our two levels are equally populated. No change in entropy for small changes in temperature ensue, which results in low C_h. See the disequilibrium D-graph below for high T. Since $D\to0$ as $T\to\infty$, the uniform distribution prevails in these circumstances, and this mathematically entails $C_h=0$ (see Fig. 6).

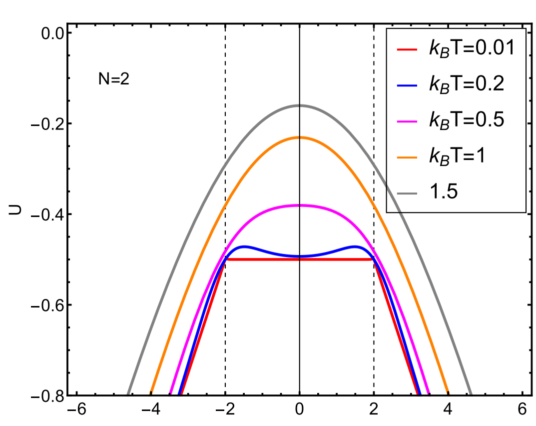

Fig. 4. Mean energy U versus k for different values of k_BT/h. We take $h=1$. We see that $k_{crit}=\pm2$.

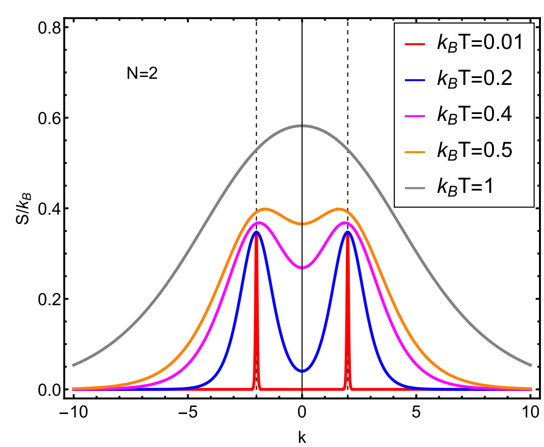

Fig. 5. Entropy S/k_B versus k for different values of k_BT/h. We take $h=1$. We see that $k_{crit}=\pm2$. Note the sharp peaks at k_{crit} that ensue at low enough temperatures.

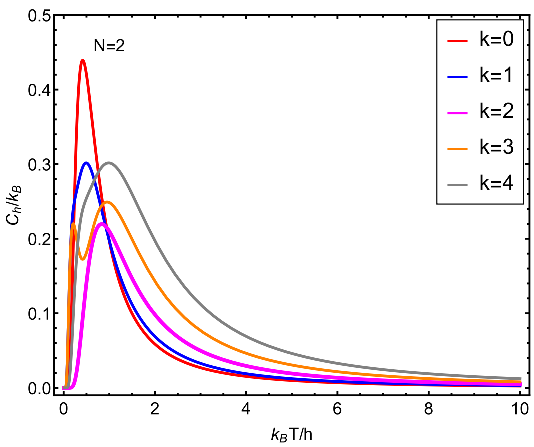

Fig. 6. Specific heat C_h/k_B versus k_BT/h for different values of k. We take $h = 1$. The typical over-shooting peak is observed.

Magnetization

In this occasion we have that

$$M = -\frac{1}{2} - \frac{1}{2}\sum_{\alpha=1}^{2}\langle n_\alpha\rangle. \tag{32}$$

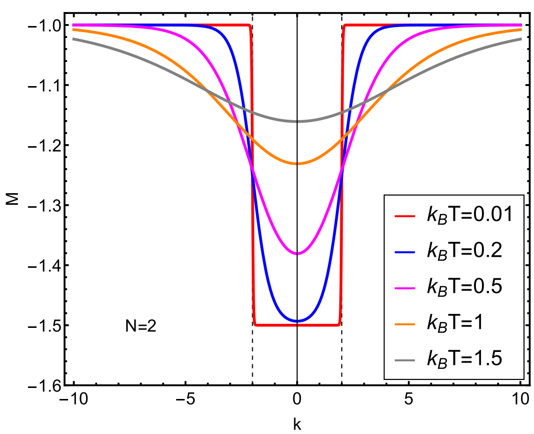

Fig. 7. Magnetization M versus k for different values of k_BT/h. We take $h = 1$. Here, $N = 2$. Note that the spin–spin interaction diminishes (the absolute value of) the degree of magnetization M if it increases beyond k_{crit} at low enough T.

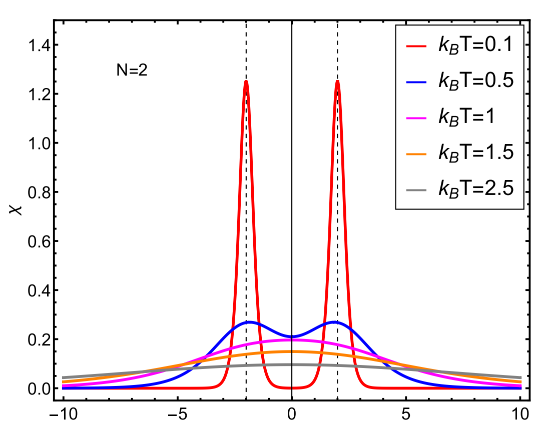

Fig. 8. Susceptibility χ versus k for different values of k_BT/h. We take $h = 1$ and $N = 2$. We detect peaks at $k_{crit} = \pm 2$.

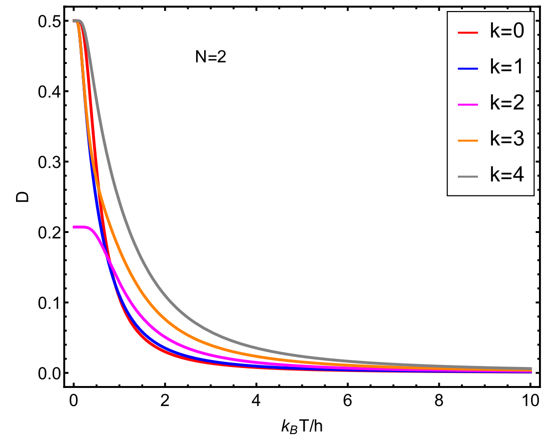

Fig. 9. Disequilibrium D versus k_BT/h for different values of k. See that at high T the uniform distribution prevails, since D measures the distance to it.

Again, we re-encounter the above phase transition at $T = 0$, that becomes a crossover at finite T. We clearly see *how* the spin–spin interaction destroys magnetic alignment, an illustration of our $k-h$ competition. We show this in Fig. 7.

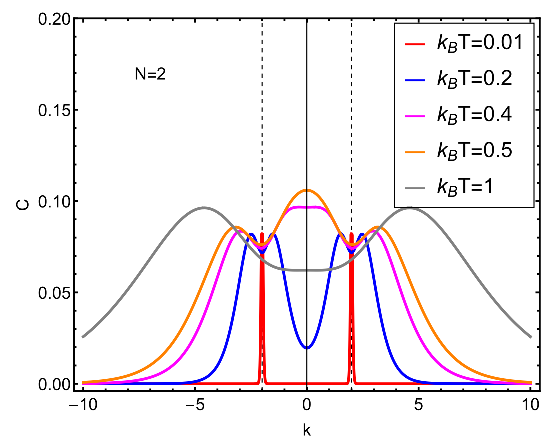

Fig. 10. Statistical Complexity C versus k for different values of k_BT. Here, $h = 1$ and $N = 2$.

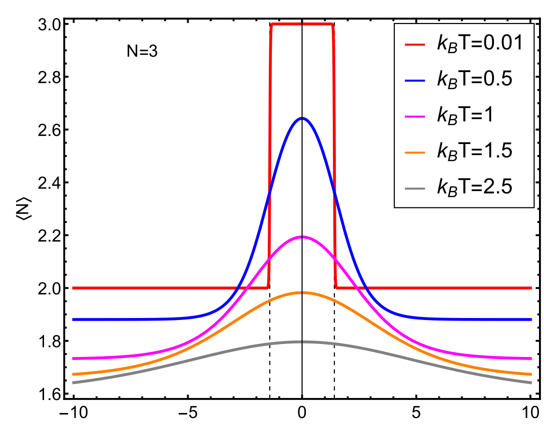

Fig. 11. $\langle N\rangle$ versus k for several values of k_BT/h for $N = 3$. We take $h = 1$. The dashed vertical lines correspond to $k_{crit} = \pm\sqrt{2}$.

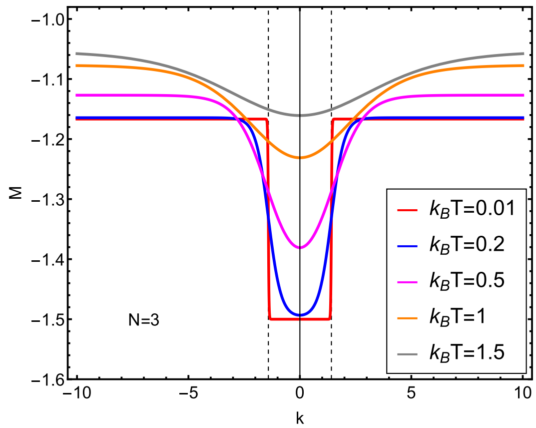

Fig. 12. Magnetization M versus k for several values of k_BT/h. Here $N = 3$ and we take $h = 1$. We see that $k_{crit} = \pm\sqrt{2}$ visualized at vertical dashed lines.

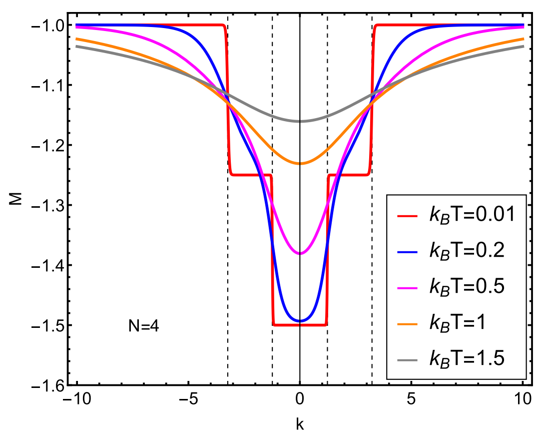

Fig. 13. N = 4: magnetization M versus k for several values of k_BT/h. We take $h = 1$. The vertical dashed lines are located in $k_{crit} = -3.24, -1.24, 1.24$, and 3.24. The number of abrupt changes increase with respect to those for the $N = 2$ situation, on account of the associated complexity growth.

Magnetic susceptibility

The magnetic susceptibility for $N = 2$ follows from Eq. (14) and acquires the appearance

$$\chi = \frac{1}{2k_BT}\sum_{\alpha=1}^{2}\frac{\exp(\beta\epsilon_\alpha)}{(1+\exp(\beta\epsilon_\alpha))^2}. \tag{33}$$

In Fig. 8 we plot χ versus k for several values of k_BT/h. As we know, χ indicates the degree of sensitivity to magnetization. Therefore, we observe peaks at $k_{crit} = \pm 2$. Temperature effects attenuate this sensitivity.

Additionally, in the limit $h\to 0$, the susceptibility is given by

$$\chi = \frac{\mathrm{Sech}^2(\beta k/4)}{4k_BT}, \tag{34}$$

and series-expanding up to first order we obtain $\chi \approx C_\chi/4k_BT$, where $C_\chi = 1/4k_B$ is the Curie constant.

Disequilibrium and Statistical Complexity

The disequilibrium is, for $N = 2$, of the form

$$D = \prod_{\alpha=1}^{2}\left\{\frac{1+\exp(-2\beta\epsilon_\alpha)}{(1+\exp(-\beta\epsilon_\alpha))^2}\right\}^{1/2} - 1/2, \tag{35}$$

which is shown in Fig. 9. We appreciate that D attains the maximum value equal 0.5 for the temperature $T = 0$ and it tends to zero when T goes to infinity.

The physical quantity $C = DH$ (in k_B-units)–with $H = S/k_B$– is our main present quantifier. At low T it displays two nice peaks whenever the eigenvalues of our matrix A vanish. They take place at k_{crit}. Thus, maximum complexity is attained then the competition $k - h$ is even and the two competing interactions exactly match each other, as be shown in Fig. 10.

Case $N = 3$

This case deserves some scrutiny because one of the eigenvalues of the matrix $\widehat{A}$ becomes independent of the coupling strength k and we are studying its competition with the strength h. The eigenvalues are: $\epsilon_1 = k/\sqrt{2} - h, \epsilon_2 = -k/\sqrt{2} - h$, and $\epsilon_3 = -h$. Overall, the features found for $N = 2$ are re-encountered here. These considerations are reflected in Figs. 11 and 12. Note that the absolute value of the magnetization is not "diminished" by the spin–spin interaction to the same extent as for $N = 2$, on account of the odd-sites effect caused by the third spin.

Cases $N = 4$ **and** $N = 1000$

Me mention that for $N = 4$ the matrix $\widehat{A}$ has four k-dependent eigenvalues: $-h \pm (1+\sqrt{5})k/4$ and $-h \pm (1-\sqrt{5})k/4$. If they vanish, C exhibits four peaks when plotted against k, just at the points of zero eigenvalue. For $N = 5$, or for any odd N, one of the eigenvalues is just $-h$. This means that only $2N-1$ C-peaks ensue for both $N = 2n$ and for

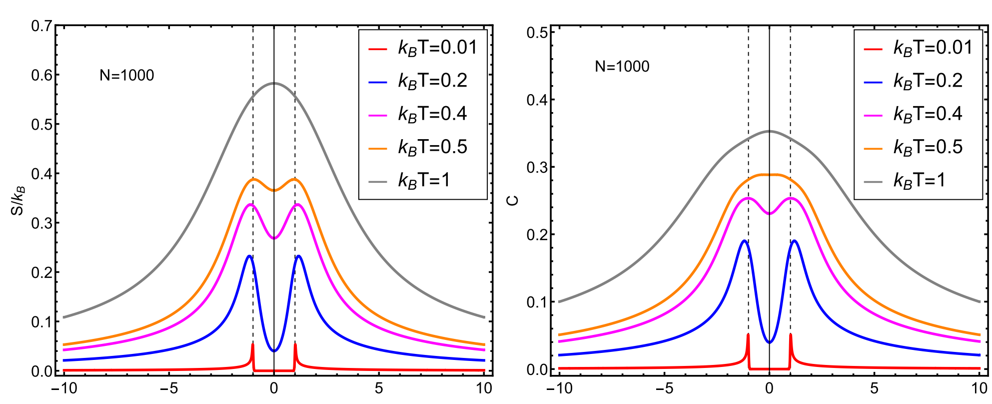

Fig. 14. Left panel: Entropy S/k_B versus k for several values of k_BT/h. Right panel: Statistical Complexity C versus k for several values of k_BT/h. We take $h = 1$ and $N = 1000$. Note that there are only two peaks in $k_{crit} = \pm 1$, just as it happens for $N = 2$.

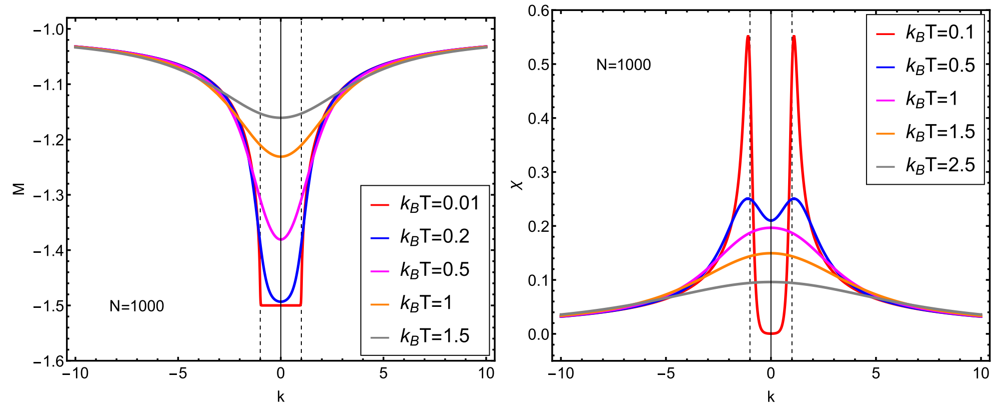

Fig. 15. Left panel: Magnetization M versus k for several values of k_BT/h. Right panel: Susceptibility χ versus k for several values of k_BT/h. We take $h = 1$ and $N = 1000$. Note that the two graphs quite resemble those for $N = 2$.

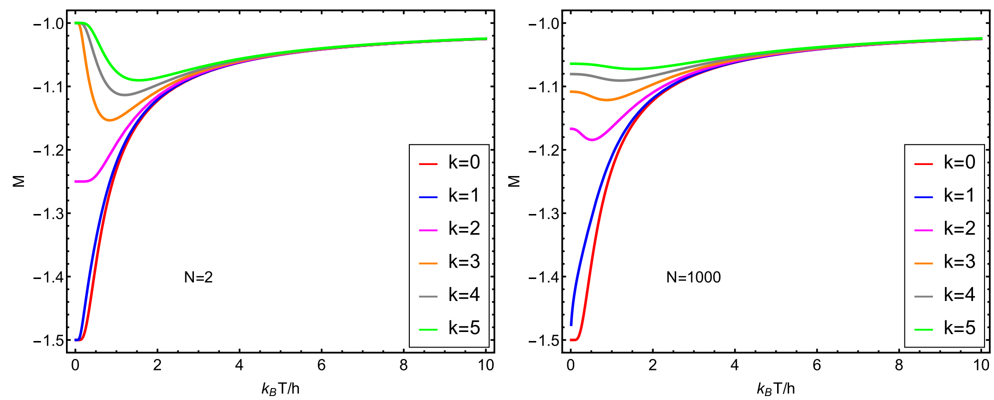

Fig. 16. $N = 2$ (left panel) and $N = 1000$ (right panel) associated magnetization M versus k_BT/h for several values of k. We take $h = 1$.

$N = 2n + 1$. However, as we saw above in Fig. 1, as N grows, the zero-eigenvalues become very close together. As $N \to \infty$, only one zero eigenvalue manifest itself as a single C-peak. This is clearly seen for $N = 1000$. We can confidently assert that our system has a more complex structure with 4-sites than with 1000 ones, a rather counter-intuitive, unexpected result. Such discussion is clearly shown in the Figs. 13–15.

We finish our presentation with a striking resemblance between the results for $N = 2$ (left) and $N = 1000$ (right), in what refers to the M-behavior, as seen in Fig. 16. One appreciates that the spin–spin interaction works against the magnetic alignment.

Conclusions

We have performed an exhaustive study of a system of N interacting spins (neighbor-neighbor, intensity k) located at N-sites of a one dimensional chain subjected to an external magnetic field of strength h. We investigated manifold facets of the competition k, h, as reflected by several thermal quantifiers. We can emphasize several interesting features. This are:

- Surprisingly enough, the cases $N = 2$ and the infinite chain display just two complexity peaks, at the sane k-location (see Ref. [13] for the thermodynamic limit).
- The case $N = 2$ already displays all the physical effects encountered for spin-chains with arbitrary N values.
- The zero-eigenvalues, for which the complexity is maximal, tend to accumulate themselves within a small k-range of values. This is the basic reason that allows the $N = 2$ system to become a good statistical representative of systems of much larger N values.
- As for the competition k, h, its main effect is the destruction of magnetic alignment at high enough k.
- All the physics of the chain can be expressed in terms of the site-occupation numbers.
- Of course, all effects uncovered at low T become attenuated as the temperature increases.
- Our main conclusion emphasizes the utility of $N = 2$ models, widely used in quantum many-body physics. We just saw that here they faithfully display the main features of their large N counterparts.

CRediT authorship contribution statement

F. Pennini: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing - original draft, Writing - review & editing. **A. Plastino:** Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing - original draft, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

Research was partially supported by FONDECYT, grant 1181558 and by CONICET (Argentine Agency).

References

[1] Lamberti PW, Martin MT, Plastino A, Rosso OA. Instensive entropic non-triviality measure. Physica A 2004;334:119.
[2] Kolmogorov AN. A new metric invariant oftransient dynamical systems and automorphisms in Lebesgue spaces. Dokl Akad Nauk SSSR 1958;119:861–4.
[3] Sinai YG. On the notion of entropy of a dynamical system. Dokl Akad Nauk SSSR 1959;124:768.
[4] Mischaikow K, Mrozek M, Reiss J, Szymczak A. Construction of Symbolic Dynamics from Experimental Time Series. Phys Rev Lett 1999;82:1144.
[5] Powell GE, Percival IC. A spectral entropy method for distinguishing regular and irregular motion of Hamiltonian systems. J Phys A: Math Gen 1979;12:2053.
[6] Rosso OA, Mairal ML. Characterization of time dynamical evolution of electroencephalographic epileptic records. Physica A 2002;312:469.
[7] Bandt C, Pompe B. Permutation Entropy: A Natural Complexity Measure for Time Series. Phys Rev Lett 2002;88:174102.
[8] Kowalski AM, Martín MT, Plastino A, Rosso OA. Bandt-Pompe approach to the classical-quantum transition. Physica D 2007;233:21.
[9] Radgohar R, Montakhab A. Global entanglement and quantum phase transitions in the transverse XY Heisenberg chain. Phys Rev B 2018;97:024434.
[10] Sachdev S. Quantum phase transitions. UK: Cambridge University Press; 2011.
[11] Smith ER. One-dimensional X-Y model with random coupling constants. I. Thermodyn mics Solid State Phys 1970;3:1419.
[12] Bunder JE, McKenzie RH. Effect of disorder on quantum phase transitions in anisotropic XY spin chains in a transverse field. Phys Rev B 1999;60:344.
[13] Apel VM, Mundarain D, Pennini F, Plastino A. Correlations between Complexity and Entanglement in a One-Dimensional XY Model. Quantum Rep 2020;20:305.
[14] Osborne TJ, Nielsen T. Entanglement in a simple quantum phase transition. Phys Rev A 2002;66:032110.
[15] Niemeijer Th. Some exact calculations on a chain of spins. Physica 1967;36:377–419.
[16] López-Ruiz R, Mancini HL, Calbet X. A statistical measure of complexity. Phys Lett A 1995;209:321–6.
[17] Crutchfield JP. The calculi of emergence: computation, dynamics and induction. Physica D 1994;75:11–54.
[18] Feldman DP, Crutchfield JP. Measures of statistical complexity: Why? Phys Lett A 1998;238:244–52.
[19] Martin MT, Plastino A, Rosso OA. Statistical complexity and disequilibrium. Phys Lett A 2003;311(126–132):126–32.
[20] Rudnicki L, Toranzo IV, Sánchez-Moreno P, Dehesa JS. Monotone measures of statistical complexity. Phys Lett A 2016;380:377–80.
[21] López-Ruiz R. A Statistical Measure of Complexity in Concepts and recent advances in generalized infpormation measures and statistics. In: Kowalski A, Rossignoli R, Curado EMC, editors. Bentham Science Books. New York; 2013. p. 147–68.
[22] Sen KD, editor. Statistical Complexity. Springer, Berlin: Applications in elctronic structure; 2011.
[23] Mitchell M. Complexity: A guided tour. Oxford, England: Oxford University Press; 2009.
[24] Martin MT, Plastino A, Rosso OA. Generalized statistical complexity measures: geometrical and analytical properties. Physica A 2006;369:439–62.
[25] López-Ruiz R. Complexity in some physical systems. Int J Bifurcation Chaos 2001;11:2669–73.
[26] Sañudo J, López-Ruiz R. Calculation of statistical entropic measures in a model of solids. Phys Lett A 2012;376:2288–91. https://doi.org/10.1016/j.physleta.2012.05.052.
[27] Catalán RG, Garay J, López-Ruiz R. Features of the extension of a statistical measure of complexity to continuous systems. Phys Rev E 2002;66:011102.
[28] Pathria RK. Statistical Mechanics, 2nd. Oxford, UK: Butterworth-Heinemann; 1996.
[29] Kulkarni Devadatta, Schmidt Darrell, Tsui Sze-Kai. Eigenvalues of tridiagonal pseudo-Toeplitz matrices. Linear Algebra Appl 1999;297:63–80.
[30] Fetter AL, Walecka JD. Quantum Theory of Many-Particle Systems. NY: McGraw-Hill Book Company; 1971.