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Minimal vertex separators and new characterizations for dually chordal graphs

Pablo De Caria [1] and Marisa Gutierrez [2]

CONICET, Departamento de Matemática
Universidad Nacional de La Plata
C.C. 172, (1900), La Plata, Argentina

Abstract

Many works related to dually chordal graphs, their cliques and neighborhoods were published. We will undertake a similar study but by considering minimal separators and their properties. Moreover we find new characterizations of dually chordal graphs.

Keywords: Graph theory, dually chordal, separator.

1 Introduction

The class of chordal graphs has been widely investigated and several of its characteristics are very useful in the resolution of many problems.

Clique graphs of chordal graphs form a class considered in many senses as dual to chordal graphs, hence the name dually chordal graphs. A good variety of researches were carried out and as a result many characterizations of dually chordal graphs were discovered, mainly involving cliques and neighborhoods but not much has been revealed about their minimal vertex separators. For

[1] Email: pablodc72@hotmail.com
[2] Email: marisa@mate.unlp.edu.ar

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that reason, the purpose of this paper is to study minimal separators of dually chordal graphs to determine if the properties known about cliques and neighborhoods have their counterparts dealing with minimal separators.

First we relate minimal separators and neighborhoods and some conditions under which a minimal separator is contained in a neighborhood are found. Then we prove that a graph is dually chordal if and only if it has a spanning tree such that every minimal separator induces a subtree, which was known if instead of minimal separators we consider cliques or neighborhoods.

Recall that a good description of cliques of dually chordal graphs is known in the sense that a graph G is dually chordal if an only if it is Helly and $K(G)$ is chordal. We will be able to show that a characterization of dually chordal graphs through minimal separators in similar terms is possible being just necessary to add one condition. We prove that a graph is dually chordal if and only if their minimal separators induce connected subgraphs, they satisfy the Helly property and the intersection graph of them is chordal.

2 Some graph terminology

A *complete* is a subset of pairwise adjacent vertices of $V(G)$. A *clique* is a maximal complete. The subgraph *induced* by $A \subseteq V(G)$, $G[A]$, has A as vertex set and two vertices are adjacent in $G[A]$ if they are adjacent in G.

Given two vertices v and w of a graph G, the *distance* between v and w, or $d(v, w)$, is the length of any shortest path connecting v and w in G. For a vertex $v \in V(G)$ its *closed neighborhood*, $N[v]$, is the set comprised by v and all the vertices adjacent to it. The *disk* centered at vertex v with radius k is the set $N^k[v] := \{w \in V(G),\ d(v, w) \leq k\}$.

Given two vertices u and v, a *uv-separator* is a set $S \subseteq V(G)$ such that $G - S$ is not connected and u and v are in different connected components of it. It is *minimal* if no proper subset of S has the same property.

Let G be a connected graph and let T be a spanning tree of G; for all $v, w \in V(G)$, $T[v, w]$ will denote the path in T from v to w.

Let F be a family of nonempty sets. The *intersection graph* of F has the elements of F as vertices, being two of them adjacent if they are not disjoint. The *clique graph* $K(G)$ of a graph G is the intersection graph of the cliques of G. A graph whose cliques have the Helly property (any subfamily of pairwise intersecting cliques has a nonempty intersection) is called a *Helly graph*.

3 Basic notions and properties

A *chord* of a cycle is an edge containing two vertices not adjacent in the cycle. *Chordal* graphs are those without chordless cycles of length at least four.

A vertex w is a *maximum neighbor* of v if $N^2[v] \subseteq N[w]$. A linear ordering $v_1 ... v_n$ of the vertices of G is a *maximum neighborhood ordering* of G if, for $i = 1, ..., n$, v_i has a maximum neighbor in $G[\{v_i, ..., v_n\}]$. *Dually chordal graphs* can be defined as those possessing a maximum neighborhood ordering.

Besides, more characterizations of dually chordal graphs have been given [1]. In fact, G is a connected dually chordal graph if and only if :

(i) *There is a spanning tree T of G such that any clique of G induces a subtree in T.*

(ii) *There is a spanning tree T of G such that any closed neighborhood of G induces a subtree in T.*

(iii) *There is a spanning tree T of G with any disk inducing a subtree in T.*

(iv) *G is Helly and $K(G)$ is chordal.*

It is even true that any spanning tree fulfilling (i), (ii) or (iii) automatically fulfills the other two. Such a tree will be said to be *compatible* with G.

Condition (ii) could be rewritten as follows:

Theorem 3.1 *[4] A connected graph G is dually chordal if and only if there is a spanning tree T of G such that, for all $x, y, z \in V(G)$, $xy \in E(G)$ and $z \in T[x, y]$ implies that $xz \in E(G)$ and $yz \in E(G)$.*

4 On separators of dually chordal graphs

A typical characterization of chordal graphs is given by the fact that any minimal vertex separator induces a complete subgraph. Obviously it does not work for dually chordal graphs because many of them are not chordal but we check that weaker conditions are fulfilled.

Theorem 4.1 *Let u and v be two nonadjacent vertices of a dually chordal graph G. Then there is a vertex w, $w \neq u$ and $w \neq v$, such that $N[w] - \{u, v\}$ separates u and v.*

Proof. Let T be a tree compatible with G and $w \in T[u, v] - \{u, v\}$. If P is a path in G joining u and v then there will be two consecutive vertices x_1 and

x_2 in P such that $w \in T[x_1, x_2]$ and since T is compatible with G we claim that $\{x_1, x_2\} \subseteq N[w]$. As u and v are nonadjacent x_1 and/or x_2 belong to $N[w] - \{u, v\}$ and consequently $N[w] - \{u, v\}$ separates u and v. □

However not all the minimal separators of a dually chordal graph are contained in a neighborhood and an example of this fact will be shown later.

Notwithstanding, that property becomes true under additional conditions.

Lemma 4.2 *If G is a dually chordal graph and $A \subseteq V(G)$ is such that $d(x, y) \leq 2$ for all $x, y \in A$ then there is a vertex w with $A \subseteq N[w]$.*

Theorem 4.3 *If G is a dually chordal graph without chordless cycles of length greater than five then any minimal separator of two nonadjacent vertices is contained in the neighborhood of some vertex.*

Sketch of proof. It is an adaptation of the proof that any minimal vertex separator of a chordal graph induces a complete subgraph, given by Dirac [2].

In fact, let S be a minimal separator of two vertices u and v and let $x, y \in S$. By considering a cycle C containing x and y constructed as in Dirac's Theorem we can conclude that $d(x, y) \leq 2$. And by Lemma 4.2 we conclude that S is contained in a closed neighborhood. □ □

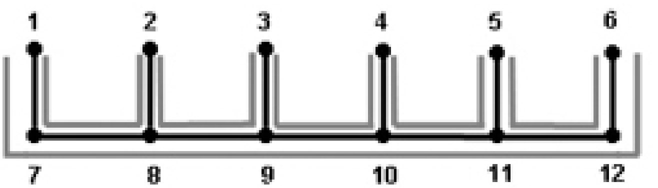

Fig. 1. A representation for a dually chordal graph. 1,2,3,4,5,6,1 is a chordless cycle. $\{2, 5, 8, 9, 10, 11\}$ is a minimal separator of 1 and 4 and it is not contained in the neighborhood of any vertex.

The property is not always valid for dually chordal graphs with a chordless cycle of length at least six. In the counterexample given by Figure 1 instead of drawing the graph we include a compatible tree, being two vertices adjacent if among the paths attached to the tree there is at least one containing them.

It is known to us that if T is compatible with a dually chordal graph then cliques and neighborhoods of G induce subtrees in T. We see now that the same is true for minimal vertex separators.

Lemma 4.4 *Let G be a dually chordal graph, T a tree compatible with G, u and v two nonadjacent vertices and S a minimal uv-separator in G. If*

$w \in S - T[u,v]$ then the path in T from w to the vertex of $T[u,v]$ closest to w (with respect to T), excepting u and v if necessary, is contained in S.

Theorem 4.5 *A graph G is dually chordal if and only if there is a spanning tree T of G such that any minimal separator of two nonadjacent vertices induces a subtree in T.*

Sketch of proof. Assume that T is a spanning tree like the above described. Let $x, y \in V(G)$ be two adjacent vertices and $z \in T[x,y] - \{x,y\}$. If x and z were not adjacent, let S be a minimal xz-separator. As S must induce a subtree and must contain a vertex in $T[x,z]$ we conclude that y and z are in the same connected component of $G - S$, but x is also in that component because it is adjacent to y, contradicting that S is an xz-separator. Therefore x and z are adjacent. Similarly y is adjacent to z and consequently, by Theorem 3.1, T is compatible with G and the graph is dually chordal.

Conversely, if G is dually chordal, take a tree T compatible with G. Let u and v be two nonadjacent vertices and S a minimal uv-separator. Assume that $x, y \in S$ and that x and y are not adjacent in T. In order to conclude that S induces a subtree it will suffice to prove that $(T[x,y] - \{x,y\}) \cap S \neq \emptyset$.

It is true in case $x \notin T[u,v]$ or $y \notin T[u,v]$ because of the previous lemma. Now assume that $\{x,y\} \subseteq T[u,v]$, with $x \in T[u,y]$. Let C be the connected component of $T - \{x,y\}$ which contains the internal vertices of $T[x,y]$, $D = \{z \in V(G),\ x \in T[y,z]\}$ and $E = \{z \in V(G),\ y \in T[x,z]\}$.

Suppose that $(T[x,y] - \{x,y\}) \cap S = \emptyset$. Then $C \cap S = \emptyset$ because of the previous lemma. Let $G[A]$ and $G[B]$ be the connected components of $G - S$ containing u and v respectively. If C were not contained in A the fact that T is compatible with G and $y \in S$ would lead to a contradiction. We have a similar contradiction if we suppose that C is not contained in B. Then C would be contained in both A and B, which is impossible.

This contradiction is avoided if $(T[x,y] - \{x,y\}) \cap S \neq \emptyset$. □

As we have seen above, a graph is dually chordal if and only if it is Helly and $K(G)$ is chordal. Helped by the previous theorem we get a similar result involving minimal separators, giving a new characterization for dually chordal graphs. Only one more condition will be added to get an equivalence.

Theorem 4.6 *A graph is dually chordal if and only if each minimal separator induces a connected subgraph, the family of minimal separators satisfies the Helly property and its intersection graph is chordal.*

Sketch of proof. It suffices to prove it for connected graphs. If a graph is dually chordal there is a spanning tree such that all the minimal separators induce subtrees. Then every minimal separator induces a connected subgraph, they satisfy the Helly Property [3] and their intersection graph is chordal [6].

Conversely, if the minimal separators satisfy the Helly property and the intersection graph is chordal there is at least one tree whose vertex set is $V(G)$ with each minimal separator inducing a subtree [5]. Of all those trees, choose T such that $s(T) := \sum_{vw \in E(T)} d(v,w)$ is minimum. We claim that T is a spanning tree of G. Otherwise, let v and w be two vertices adjacent in T but not in G, with $d(v,w) = k$. Consider the family formed by all the minimal separators containing v and w, if any, one minimal vw-separator contained in $N[v]$ and one minimal vw-separator contained in $\{u \in V(G),\ d(u,w) = k-1\}$. This family is pairwise intersecting, so there is a vertex u belonging to every member. Set $A := \{x \in V(G),\ w \in T[v,x]\}$ and $B := \{x \in V(G),\ v \in T[w,x]\}$. If $u \in A$, let T' be a tree obtained from T by removing vw and adding uv. Then any minimal vertex separator in G induces a subtree in T' and $s(T') < s(T)$, contradicting our choice of T. If $u \in B$, we can remove vw and add uw to T and a similar contradiction arises. Thus T is a spanning tree of G and due to the previous theorem G is dually chordal. $\Box$

A simple example illustrating that the condition of having connected separators is required is given by a cycle of length four.

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