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Electronic Notes in Discrete Mathematics 44 (2013) 287–292

Electronic Notes in DISCRETE MATHEMATICS

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# Introducing subclasses of basic chordal graphs

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**Abstract**

Basic chordal graphs arose when comparing clique trees of chordal graphs and compatible trees of dually chordal graphs. They were defined as those chordal graphs whose clique trees are exactly the compatible trees of its clique graph.

In this work, we consider some subclasses of basic chordal graphs, like hereditary basic chordal graphs, basic $DV$ and basic $RDV$ graphs, we characterize them and we find some other properties they have, mostly involving clique graphs.

*Keywords:* Chordal graph, dually chordal graph, clique tree, compatible tree.

## 1 Introduction

### 1.1 Definitions

For a graph $G$, $V(G)$ is the set of its vertices and $E(G)$ is the set of its edges. A *clique* is a maximal set of pairwise adjacent vertices. The family of cliques of $G$ is denoted by $\mathcal{C}(G)$. For $v \in V(G)$, $\mathcal{C}_v$ will denote the family of cliques containing $v$. All graphs considered will be assumed to be connected.

A set $S \subseteq V(G)$ is a *uv-separator* if vertices $u$ and $v$ are in different connected components of $G-S$. It is *minimal* if no proper subset of $S$ has the

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http://dx.doi.org/10.1016/j.endm.2013.10.045

same property. $S$ is a *minimal vertex separator* if there exist two nonadjacent vertices $u$ and $v$ such that $S$ is a minimal $uv$-separator. $\mathcal{S}(G)$ will denote the family of all minimal vertex separators of $G$.

Let $\mathcal{F}$ be a family of sets of vertices of a graph. $\mathcal{F}$ is *separating* if, for all $v \in \bigcup_{F\in\mathcal{F}} F$, $\bigcap_{v\in F} F = \{v\}$. The *intersection graph* of $\mathcal{F}$, $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint. The *clique graph* $K(G)$ is the graph $L(\mathcal{C}(G))$. The *two section* of $\mathcal{F}$, $S(\mathcal{F})$, is a graph whose vertex set is $\bigcup_{F\in\mathcal{F}} F$, where $u$ and $v$, $u \neq v$, are adjacent in $S(\mathcal{F})$ if and only if there exists $F \in \mathcal{F}$ such that $\{u, v\} \subseteq F$.

### *1.2 Basic chordal graphs and goals*

*Chordal graphs* were defined as those graphs for which every cycle of length greater than or equal to four has a chord. They can also be described in many other ways. A *clique tree* of $G$ is a tree $T$ whose vertex set is $\mathcal{C}(G)$ and such that for every $v \in V(G)$, $\mathcal{C}_v$ induces a subtree of $T$. A graph is chordal if and only if it has a clique tree [4].

The clique graphs of chordal graphs, i.e., *dually chordal graphs*, also have a tree (the compatible tree) that characterizes them. A *compatible tree* of $G$ is a spanning tree $T$ of $G$ such that every clique of $G$ induces a subtree of $T$.

Compatible trees are not hard to find. In fact, every clique tree of a chordal graph $G$ is a compatible tree of $K(G)$. However, it is not necessarily true that every compatible tree of $K(G)$ is a clique tree of $G$. *Basic chordal graphs* appeared in this context [2] and were defined as those chordal graphs whose clique trees are exactly the compatible trees of its clique graph.

One of the aspects about basic chordal graphs that were studied is the recognition problem. Let $S$ be a minimal vertex separator of a graph $G$. Denote by $\mathcal{C}_S$ the family of cliques of $G$ containing $S$ and let $B_S$ be the family consisting of every clique of $G$ that intersects every $C \in \mathcal{C}(G)$ such that $C \cap S \neq \emptyset$. The following characterization enables a polynomial-time recognition of basic chordal graphs.

**Theorem 1.1** *[2] A chordal graph $G$ is basic chordal if and only if for all $S \in \mathcal{S}(G)$, $B_S = \mathcal{C}_S$.*

Another important fact about basic chordal graphs is that, despite being a strict subclass of chordal graphs, their clique graphs are the same.

**Theorem 1.2** *[2] The class of clique graphs of basic chordal graphs is equal to the class of dually chordal graphs.*

In this paper, we define and study some subclasses of basic chordal graphs. In Section 2, we introduce the class of hereditary basic chordal graphs, find several characterizations for them and show that this class is equivalent to some others, like strictly chordal graphs and (4,6)-leaf powers. We also show that the clique graphs of hereditary basic chordal graphs are the block graphs. In Section 3, we study the correspondence between special classes of clique trees and compatible trees, thus giving rise to basic $DV$ and basic $RDV$ graphs. We also characterize these classes and find their clique graphs.

## 2 Hereditary basic chordal graphs

A graph $G$ is said to be *hereditary basic chordal* if $G$ and all its induced subgraphs are basic chordal. These graphs clearly form a hereditary class and hence they have a family of minimal forbidden induced subgraphs. Other than the cycles $C_n$ $(n \geq 4)$, the only two minimal forbidden induced subgraphs are the dart and the gem, as it will be shown later. As a consequence, we will first explore some properties of gem-free graphs and of dart-free graphs.

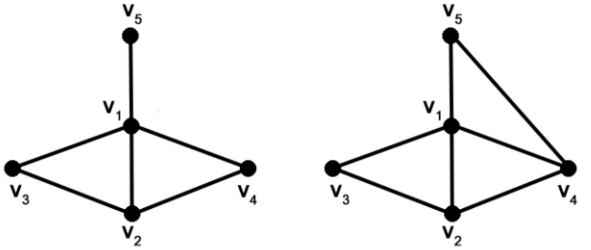


Fig. 1. A dart (left) and a gem (right).

**Proposition 2.1** *Let $G$ be a chordal graph. Then, $G$ is gem-free if and only if every edge of $K(G)$ is in some clique tree of $G$.*

**Proposition 2.2** *Let $G$ be a chordal dart-free graph. Then, no minimal vertex separator of $G$ contains another.*

A conjunction of the previous two propositions gives another characterization for the class of hereditary basic chordal graphs. Some other characterizations can also be found in the next result.

**Theorem 2.3** *Let $G$ be a chordal graph. The following are equivalent:*

***(i)*** *$G$ is hereditary basic chordal.*

***(ii)*** *$G$ is a (dart,gem)-free graph.*

***(iii)*** *Every edge of $K(G)$ is in some clique tree of $G$ and no minimal vertex separator of $G$ contains another.*

***(iv)*** *For every triple $C_1, C_2, C_3$ of pairwise intersecting cliques of $G$, $C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3$.*

***(v)*** *For all $C \in \mathcal{C}(G)$ and $S \in \mathcal{S}(G)$, $S \cap C \neq \emptyset$ implies $S \subseteq C$.*

***(vi)*** *The minimal vertex separators of $G$ are pairwise disjoint.*

The characterization of hereditary basic chordal graphs via minimal forbidden induced subgraphs reveals that this class appeared before in some other works under very different contexts and names.

First, in 2005, William Kennedy studied strictly chordal graphs in his Master Thesis [3]. There, a graph $G$ is said to be *strictly chordal* if $G$ is chordal and for every subfamily $\mathcal{F} \subseteq \mathcal{C}(G)$, $I(\mathcal{F}) := \bigcap_{C \in \mathcal{F}} C \neq \emptyset$ and $\mathcal{F} = \{C \in \mathcal{C}(G) : C \cap I(\mathcal{F}) \neq \emptyset\}$ imply that $C \cap C' = I(\mathcal{F})$ for all $C, C' \in \mathcal{F}$. Moreover, the thesis includes a proof of the fact that strictly chordal graphs are exactly the (dart-gem)-free chordal graphs.

A connection with leaf powers was found later. Let $G$ be a graph and $k, l$ be integers such that $2 \leq k < l$. $G$ is defined to be a *(k,l)-leaf power* if there exists a tree $T$ whose set of leaves is $V(G)$ and such that $d_T(u, v) \leq k$ for all $uv \in E(G)$ and $d_T(u, v) \geq l$ for all $uv \notin E(G)$ ($d_T$ denotes distance in $T$). Brandstädt and Wagner [1] proved that strictly chordal graphs are exactly the (4,6)-leaf powers. Therefore,

**Theorem 2.4** *Let $G$ be a graph. The following are equivalent:*

(i) *$G$ is hereditary basic chordal.*

(ii) *$G$ is strictly chordal.*

(iii) *$G$ is a (4,6)-leaf power.*

### *2.1 Clique graphs of hereditary basic chordal graphs*

We will show here that the class of clique graphs of hereditary basic chordal graphs is the class of block graphs. Note first that hereditary basic chordal graphs are strongly chordal because the fact that they are gem-free implies that they are sun-free. Since the clique graph of a strongly chordal graphs is also strongly chordal, the clique graph of every hereditary basic chordal graph is chordal. In order to proceed, we need the following proposition:

**Proposition 2.5** *Let $G$ be a hereditary basic chordal graph and $C, C'$ be two cliques of $G$ such that $C \cap C' \neq \emptyset$. Then, the edge $CC'$ is contained in only* *one clique of $K(G)$.*

The graphs whose edges are contained in only one clique are the diamond-free graphs. The class of chordal diamond-free graphs is equal to the class of block graphs. Therefore, the clique graphs of hereditary basic chordal graphs are block graphs. Furthermore, trees form a subclass of hereditary basic chordal graphs and their clique graphs are all block graphs. This outlines the proof of the initial claim of the section, which we now state as a theorem.

**Theorem 2.6** *Let $G$ be a graph. Then, $G$ is the clique graph of a hereditary basic chordal graph if and only if $G$ is a block graph.*

## 3 Basic DV and basic RDV graphs

This section focuses on the existence of special types of clique trees and compatible trees and their connection with basic chordal graphs. Three subclasses of clique trees, described below, have been studied for more than twenty years.

A *UV clique tree* of a graph $G$ is a clique tree such that, for every $v \in V(G)$, $T[\mathcal{C}_v]$ is a path of $T$. A *DV clique tree* of $G$ is a clique tree such that its edges have been directed and, for every $v \in V(G)$, $T[\mathcal{C}_v]$ is a directed path of $T$. An *RDV clique tree* of $G$ is a $DV$ clique tree that is rooted at a vertex $w$. A chordal graph is $UV/DV/RDV$ if it has a $UV/DV/RDV$ clique tree.

The clique graphs of $UV$ graphs are all dually chordal graphs. This and other reasons justify that $UV$ graphs do not have a dual class. On the contrary, $DV$ and $RDV$ graphs do have each a dual class. The *dually DV(RDV)* graphs are the clique graphs of $DV(RDV)$ graphs. This duality is also reflected by the existence of characteristic trees. The *DV(RDV) compatible tree* is a (rooted) directed spanning tree with every clique of the graph inducing a directed path.

We had mentioned before that every clique tree of a chordal graph $G$ is compatible with $K(G)$. There is a similar result for $DV$ and $RDV$ graphs.

**Proposition 3.1** *Let $G$ be a $DV(RDV)$ graph. Then, every $DV(RDV)$ clique tree of $G$ is a $DV(RDV)$ compatible tree of $K(G)$.*

Once again, the converse is not necessarily true. Given a $DV(RDV)$ graph, there could be a compatible $DV(RDV)$ tree of $K(G)$ that is not a $DV(RDV)$ clique tree of $G$. We define a *basic DV(RDV)* graph as a $DV(RDV)$ basic chordal graph whose $DV(RDV)$ clique trees are the $DV(RDV)$ compatible trees of its clique graph. Basic $DV$ and basic $RDV$ graphs are not difficult to identify, as the following theorem shows.

**Theorem 3.2** *Let $G$ be a chordal graph. Then,*

***(a)*** *$G$ is basic $DV$ if and only if $G$ is basic chordal and $DV$.*

***(b)*** *$G$ is basic $RDV$ if and only if $G$ is basic chordal and $RDV$.*

The importance of Theorem 3.2 lies on showing that, for a basic chordal graph $G$, the correspondence is not only between the clique trees of $G$ and the compatible trees of $K(G)$, but also between the $DV(RDV)$ clique trees of $G$ and the $DV(RDV)$ compatible trees of $K(G)$, when they exist.

Finally, we study the clique graphs of these classes. Given a dually chordal graph $G$, denote by $\mathcal{SDC}(G)$ the family of subsets of $V(G)$ inducing a subtree of every compatible tree of $G$. If $G$ is dually $DV$, define $\mathcal{X}(G)$ to be the family of sets of vertices of $G$ inducing a directed path of every $DV$ compatible tree of $G$. If $G$ is dually $RDV$, $\mathcal{Y}(G)$ will denote the family of sets of vertices of $G$ inducing a directed path of every $RDV$ compatible tree of $G$. The connection between these families and clique graphs is the following:

**Theorem 3.3** *Let $H$ be a chordal graph.*

***(a)*** *Let $G$ be dually $DV$. Then, $K(H) = G$ and $H$ is basic $DV$ if and only if $H$ is the intersection graph of a separating family $\mathcal{F}$ contained in $\mathcal{SDC}(G) \cap \mathcal{X}(G)$ and such that $\mathcal{S}(\mathcal{F}) = G$.*

***(b)*** *Let $G$ be dually $RDV$. Then, $K(H) = G$ and $H$ is basic $RDV$ if and only if $H$ is the intersection graph of a separating family $\mathcal{F}$ contained in $\mathcal{SDC}(G) \cap \mathcal{Y}(G)$ and such that $\mathcal{S}(\mathcal{F}) = G$.*

One family $\mathcal{F}$ satisfying the conditions of Theorem 3.3 consists of the cliques of $G$ and its unit sets of vertices. Thus, much like the clique graphs of basic chordal graphs are all dually chordal graphs, we have:

**Theorem 3.4** *$K(Basic\, DV) = Dually\, DV$ and $K(Basic\, RDV) = Dually\, RDV$.*

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