\textbf{t-Pebbling in $k$-connected graphs with a universal vertex}

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\section*{Abstract}

The $t$-pebbling number is the smallest integer $m$ so that any initially distributed supply of $m$ pebbles can place $t$ pebbles on any target vertex via pebbling moves. The 1-pebbling number of diameter 2 graphs is well-studied. Here we investigate the $t$-pebbling number of diameter 2 graphs under the lens of connectivity.

\section{Introduction}

Graph pebbling models the transportation of consumable resources. It has an interesting history, with many challenging open problems, and with applications to zero-sum theory in abelian groups. Calculating pebbling numbers of graphs is a well known computationally difficult problem. See [4, 5] for more background.

A \textit{configuration} $C$ of pebbles on the vertices of a connected graph $G$ is a function $C : V(G) \rightarrow \mathbb{N}$ (the nonnegative integers), so that $C(v)$ counts the number of pebbles placed on the vertex $v$. We write $|C|$ for the \textit{size} $\sum_v C(v)$ of $C$; i.e. the number of pebbles in the configuration. A

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pebbling step from a vertex $u$ to one of its neighbors $v$ reduces $C(u)$ by two and increases $C(v)$ by one. Given a specified root vertex $r$ we say that $C$ is $t$-fold $r$-solvable if some sequence of pebbling steps starting from $C$ places $t$ pebbles on $r$. We are concerned with determining $\pi_t(G, r)$, the minimum positive integer $m$ such that every configuration of size $m$ on the vertices of $G$ is $t$-fold $r$-solvable. The $t$-pebbling number of $G$ is defined to be $\pi_t(G) = \max_{r \in V(G)} \pi_t(G, r)$. We omit $t$ when $t = 1$. Clearly, $\pi_t(G) \leq t \pi(G)$.

Pebbling number of diameter 2 graphs was solved and characterized by the following theorem. For the purpose of the present work, it is enough to know that a pyramidal graph has no universal vertex (a vertex adjacent to every other vertex) and has connectivity 2.

**Theorem 1.** [2, 6] For a diameter 2 graph $G$ with connectivity $k$ and $n$ vertices, $\pi(G) = n + 1$ if and only if $k = 1$ or $G$ is pyramidal. Otherwise (i.e. $k = 2$ and $G$ is not pyramidal, or $k \geq 3$), $\pi(G) = n$.

In contrast, other than the following bound, little is known about the $t$-pebbling number of diameter 2 graphs.

**Theorem 2.** [3] If $G$ is a diameter 2 graph on $n$ vertices then $\pi_t(G) \leq \pi(G) + 4t - 4$. Moreover, $\liminf_{t \to \infty} \pi_t(G)/t = 4$.

The goal of the present paper is to determine the exact $t$-pebbling number of a large subfamily of diameter 2 graphs by considering their connectivity. Define $G(n, k)$ to be the set of all $k$-connected graphs on $n$ vertices having a universal vertex. Set $f_t(n, k) = n + 4t - k - 2$ and $h_t(n) = n + 2t - 2$. Notice that $h_t(n) \geq f_t(n, k)$ if and only if $k \geq 2t$. Define $p_t(n, k) = \max\{f_t(n, k), h_t(n)\}$. The main result is the following theorem which is proved in Section 3.

**Theorem 3.** If $G \in G(n, k)$ then $\pi_t(G) = p_t(n, k)$.

We observe from our result that, for any fixed $t$, in the family of graphs with universal vertex, there are graphs whose $t$-pebbling number is much
lower than the bound given by Theorem 2, and also that there are graphs
reaching that bound: when \( k \geq 2t \) we have \( \pi_t(n, k) = (n + 4t - 4) - 2(t-1) \);
when \( k < 2t \) \( \pi_t(n, k) = (n + 4t - 4) - (k - 2) \).

It will be useful to take advantage of the following version of Menger’s
Theorem ([7], exercise 4.2.28).

Theorem 4. (Menger’s Theorem) [7] Let \( G \) be a \( k \)-connected graph
and \( S = \{v_1, \ldots, v_k\} \) be a multiset of vertices of \( G \). For any \( r \notin S \) there
are \( k \) pairwise-internally-disjoint paths, one from each \( v_i \) to \( r \).

2 Technical Lemmas

We begin with a lemma that is used to prove lower bounds on the
pebbling number of a graph by helping to show that certain configurations
are unsolvable.

For a vertex \( v \), define its open neighborhood \( N(v) \) to be the set of vertices
adjacent to \( v \), and its closed neighborhood \( N[v] = N(v) \cup \{v\} \). We say that
a vertex \( y \) is a junior sibling of a vertex \( x \) (or, more simply, junior to \( x \))
if \( N(y) \subseteq N[x] \), and that \( y \) is a junior if it is junior to some vertex \( x \).

Lemma 5. (Junior Removal Lemma) [1] Given the graph \( G \) with root
\( r \) and \( t \)-fold \( r \)-solvable configuration \( C \), suppose that \( y \neq r \) is a junior with
\( C(y) = 0 \). Then \( C \) (restricted to \( G - y \)) is \( t \)-fold \( r \)-solvable in \( G - y \).

Given a configuration \( C \) of pebbles, we say that a path \( Q = (r, q_1, \ldots, q_j) \)
with \( j \geq 1 \) is a slide from \( q_j \) to \( r \) if no \( q_i \) is empty and \( q_j \) has at least two
pebbles.

A potential move is a pair of pebbles sitting on the same vertex. To say
that \( C \) has \( j \) potential moves means that the \( j \) pairs are pairwise disjoint.
For example, any configuration on 5 vertices with values 0,1,1,2, and 7
has 4 potential moves. The potential of \( C \), \( \text{pot}(C) \), is the maximum \( j \) for
which \( C \) has \( j \) potential moves; i.e., \( \text{pot}(C) = \sum_{v \in V} \lfloor (C(v)/2) \rfloor \). Because
every solution that requires a pebbling move uses a potential move, the
following fact is evident.
Fact 6. If $C$ is a configuration with $C(r) + \text{pot}(C) < t$ then $C$ is not $t$-fold $r$-solvable.

Basic counting yields the following lemma.

Lemma 7. (Potential Lemma) Let $G$ be a graph on $n$ vertices. If $C$ is a configuration on $G$ of size $n + y$ ($y \geq 0$) having $z$ zeros, then $\text{pot}(C) \geq \lceil \frac{y + z}{2} \rceil$.

A nice application of the Potential Lemma is the following result, which we will use repeatedly in the arguments that follow.

Lemma 8. (Slide Lemma) Let $r$ be a vertex of a $k$-connected graph $G$. Let $C$ be a configuration on $G$ of size $n + y$ ($y \geq 0$) with $z$ zeros. If $\lceil \frac{y + 3z}{2} \rceil \leq k$ then $C$ is $\lceil \frac{y + z}{2} \rceil$-fold $r$-solvable.

Proof. Set $p = \lceil \frac{y + z}{2} \rceil$. By Lemma 7 we can choose a set $P$ of $p$ potential moves. Note that the hypothesis implies that $p \leq k - z$. Delete all non-root zeros to obtain $G'$. Since $G$ is $k$-connected, $G'$ is $p$-connected. Thus Menger’s Theorem 4 implies that there are $p$ pair-wise disjoint slides in $G'$ from $P$ to $r$, which implies that $C$ is $p$-fold $r$-solvable.

\[ \Box \]

3 Proof of Theorem 3

The proof will follow from Lemmas 9 and 10, below. Let $u$ be a universal vertex of a graph $G \in \mathcal{G}(n, k)$. If $C$ is a configuration of size $n + 2t - 3$ with $C(u) = 0$ and every other vertex odd then $\text{pot}(C) = t - 1$, and so $C$ is not $t$-fold $u$-solvable. Hence $\pi_t(G, u) \geq n + 2t - 2$. On the other hand, if $|C| \geq n + 2t - 2$ then $\text{pot}(C) \geq t$ when $u$ is empty, and $\text{pot}(C) \geq t - 1$ when $u$ is not; either way $C$ is $t$-fold $u$-solvable because $u$ is universal. Thus $\pi_t(G, u) = n + 2t - 2$, which is at most $p_t(n, k)$ always.

3.1 Lower bound

Clearly, $\pi_t(G) \geq \pi_t(G, u) = h_t(n)$. Now let $r$ be any non-universal vertex of $G$, and let $s$ be a vertex at distance 2 from $r$. Let $X$ be any
(r, s)-cutset of size k (in particular, u ∈ X) and define the configuration

\[
\begin{array}{cccccccc}
  \hline
  t & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  \hline
  2 & 0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 \\
  3 & 0 & 4 & 7 & 11 & 15 & 19 & 23 & 27 \\
  4 & 0 & 2 & 6 & 10 & 14 & 18 & 22 & 26 \\
  5 & 0 & 2 & 5 & 9 & 13 & 17 & 21 & 26 \\
  6 & 0 & 2 & 4 & 8 & 12 & 16 & 20 & 24 \\
  7 & 0 & 2 & 4 & 7 & 11 & 15 & 19 & 23 \\
  8 & 0 & 2 & 4 & 6 & 10 & 14 & 18 & 22 \\
  9 & 0 & 2 & 4 & 6 & \circ (9) & 13 & 17 & 21 \\
 10 & 0 & 2 & 4 & 6 & 8 & 12 & 16 & 20 \\
11 & 0 & 2 & 4 & 6 & \circ (1) & 15 & 19 & \\
\hline
\end{array}
\]

Figure 1: The values \(m\) for which \(\pi_t(G) = |V(G)| + m\).

\(F_t(n, k)\) by placing 0 on \(r\) and on every vertex in \(X\), \(4t − 1\) on \(s\), and 1 on each vertex of \(V(G) - (X \cup \{r, s\})\); then \(|F_t(n, k)| = (4t − 1) + (n − k − 2) = f_t(n, k) − 1\).

Since the vertices of \(X - \{u\}\) have 0 pebbles and all them are juniors to \(u\), Lemma 5 states that if \(t\) pebbles can reach \(r\) then \(2t\) pebbles can reach \(u\). But, with exactly \(2t − 1\) potential moves in \(F\), by Fact 6, we can place at most \(2t − 1\) pebbles on \(u\). Therefore \(\pi_t(G, r) \geq f_t(n, k)\), implying \(\pi_t(G) \geq f_t(n, k)\).

We record these results as

Lemma 9. For \(G \in \mathcal{G}(n, k)\) we have \(\pi_t(G) \geq p_t(n, k)\).

3.2 Upper bound

We will prove that any configuration of size \(f_t(n, k)\) when \(k \leq 2t\), and of size \(h_t(n)\) when \(k \geq 2t\), is \(t\)-fold \(r\)-solvable for any \(r \in V(G)\).

Lemma 10. For \(k \geq 2\), let \(G \in \mathcal{G}(n, k)\) be a graph with a universal vertex \(u\), and let \(r\) be any root vertex. Then \(\pi_t(G, r) \leq p_t(n, k)\).
Proof. First note that the lemma is true when $t = 1$. Indeed, in this case we have $k \geq 2t$, and so $p_t(n, k) = h_t(n) = n + 2t - 2 = n$. On the other hand, because no pyramidal graph has a universal vertex, we have from Theorem 1 that $\pi(G) = n$, hence $\pi(G, r) \leq n$.

In addition, the lemma holds for $k = 2$. Indeed, in this case we have $k \leq 2t$, and so $p_t(n, k) = f_t(n, k) = n + 4t - k - 2 = n - 4t - 4$. Also, we have by Theorem 2 that $\pi_t(G, r) \leq n + 4t - 4$.

Hence, we may assume that $t \geq 2$ and $k \geq 3$. Figure 1 shows the structure of this proof. As was noted above, the grey section has been proven before. We continue by proving the dashed-bordered, lower left section and diagonal circled entries together, and then the solid-bordered, upper right section by induction.

**Base case.**

We will simultaneously address the case $k = 2t - 1$ (the circled entries), for which $|C| = f_t(n, k) = n + 2t - 1$, and the case $k \geq 2t$ (the dashed-bordered section), for which $|C| = h_t(n) = n + 2t - 2$, by writing $k \geq 2t - 1$ and considering a configuration of size $|C| = n + 2t - 2 + \phi$, where $\phi = 1$ if $2t - 1 = k$ and 0 otherwise. The natural idea we leverage here is repeating the argument that increased zeros force increased potential, which, combined with connectivity, yields either more solutions or more zeros.

Let $x \geq 0$ such that $k = 2t - 1 + x$. By Lemma 7, since we may assume that $C(r) = 0$ (otherwise apply induction on $t$), we have at least $\lceil (2t - 2 + 1)/2 \rceil = t$ potential moves. Therefore, we have at least $t$ solutions if there are at least $t$ different slides from them to $r$.

Thus we consider the case in which there are at most $t - 1$ slides; that is, from some of the vertices in which a potential move is sitting, say $v$, there is no path to $r$ without an internal zero after considering the remaining $t - 1$ slides. Since $G$ is $k$-connected, that implies that $C$ has at least $k - (t - 1)$ zeros between $v$ and $r$ and so, because of $r$, $C$ has at least $k - (t - 1) + 1 = t + 1 + x$ zeros.
Assume that there are exactly \( z = t + 1 + j \) zeros, for some \( j \geq x \). Then, by Lemma 7, \( C \) has at least

\[
\left\lfloor \frac{(2t - 2) + (t + 1 + j)}{2} \right\rfloor = t + \left\lfloor \frac{t - 1 + j}{2} \right\rfloor
\]
potential moves. If there are at least \( t - \left\lfloor \frac{t - 1 + j}{2} \right\rfloor \) slides from them to \( r \), then we can use those slides for that many solutions. Then, the other \( \left\lfloor \frac{t - 1 + j}{2} \right\rfloor \) solutions can be obtained from the remaining \( 2 \left\lfloor \frac{t - 1 + j}{2} \right\rfloor \) potential moves, putting \( 2 \left\lfloor \frac{t - 1 + j}{2} \right\rfloor \) pebbles on the universal vertex \( u \) and then \( \left\lfloor \frac{t - 1 + j}{2} \right\rfloor \) on \( r \).

Otherwise, there are at most \( t - \left\lfloor \frac{t - 1 + j}{2} \right\rfloor - 1 \) slides, from which we find, using \( k = 2t - 1 + x \), at least

\[
k - \left( t - \left\lfloor \frac{t - 1 + j}{2} \right\rfloor - 1 \right) + 1 = t + x + \left\lfloor \frac{t - 1 + j}{2} \right\rfloor + 1
\]
zeros. Clearly, this number cannot exceed the total number of zeros \( z = t + 1 + j \); therefore \( j \geq x + \left\lfloor \frac{t - 1 + j}{2} \right\rfloor \geq x + \frac{t - 1 + j}{2} \), and so \( j \geq t - 1 + 2x \).

Let \( j = t - 1 + 2x + i \) for some \( i \geq 0 \); then \( z = t + 1 + j = t + 1 + t - 1 + 2x + i = 2t + 2x + i \). Applying Lemma 7 again, there are at least

\[
\left\lfloor \frac{(2t - 2) + (2t + 2x + i)}{2} \right\rfloor = 2t + x - 1 + \lceil i/2 \rceil
\]
potential moves.

If either \( x \geq 1 \) or \( i \geq 1 \), then we can move \( 2t \) pebbles to the universal vertex \( u \), and then \( t \) to \( r \).

Hence, we consider the case for which \( x = i = 0 \); i.e. \( k = 2t - 1 \), \( z = 2t \), and \( |C| = n + 2t - 1 \) (because \( \phi = 1 \) in such a case). We let \( T \) be the star centered on \( u \), having leaves \( r \) and the nonzero vertices of \( G \). Clearly, \( T \) is a subgraph of \( G \) with \( n + 2t - 1 \) pebbles on it and with either \( 2 + (n - z) \) or \( 1 + (n - z) \) vertices, depending on whether \( u \) is empty or not. In either case \( n(T) \leq 2 + n - z = 2 + n - 2t \). Therefore, since

\[
\pi_t(T, r) = n(T) + 4t - 3 \leq (2 + n - 2t) + 4t - 3 = n + 2t - 1 = |C(T)|,
\]
we see that $C$ is $r$-solvable.

**Induction step.**

Finally, we consider the case $k < 2t - 1$ (the solid-bordered section); so $|C| = f_t(n, k) = n + 4t - k - 2$. Since $2(t - 1) = 2t - 1 - 1 \geq k$, we have $\pi_{t-1}(G, r) = f_{t-1}(n, k) = n + 4(t - 1) - k - 2 = n + 4t - k - 2 - 4 = |C| - 4$. Hence, if $C$ has a solution of cost at most 4, we are done. Otherwise, there is at most one vertex $v$ having two or more pebbles, and on such a vertex there are at most 3 pebbles. This implies the contradiction $|C| \leq 3 + (n - 2)$, which completes the proof. 

In future work we intend to study $k$-connected diameter 2 graphs without a universal vertex, and use that work as a base step toward studying graphs of larger diameter.

**References**


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