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LECTURE

# Dynamical chaos in the Solar system

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**Abstract.** Methods and results of a new major part of science on the dynamics of the Solar system bodies, namely, the part devoted to researches of dynamical chaos in the motion of celestial bodies, are described and analyzed. The dynamical chaos (non-determined dynamical behavior in the absence of any random perturbations) in the motion of celestial bodies is caused by interaction of nonlinear resonances, either orbital or spin-orbit. The resonances and chaos in the motion of minor and large bodies of the Solar system, — planetary satellites, asteroids, comets, and planets, are considered. Special attention is given to the problem of observability of the chaotic behavior; in particular, methods of analytical estimating the Lyapunov time, specifying the "predictability horizon" of the motion, are described.

## 1. Introduction

Within many centuries nothing had seemed to humankind to be farther from chaos and accident than the repeated, measured motion of the Solar system bodies. Integer relations between the orbital periods of some of them, first discovered in the 18th century, seemed to be an obvious confirmation for the reigning order and harmony. However, in sharp contrast with this seeming evidence, these were just such relations (named *commensurabilities* or *resonances*) that appeared to be the cause of *dynamical chaos* — the phenomenon not at all related to any random perturbations on the moving bodies, but concealed in the very nature of the motion.

The idea of harmonious order based on geometrical relations inspired Johannes Kepler in his first researches of the motion of planets. In 1593 he noticed, that the radii of the circle inscribed in an equilateral triangle and the circle described around it have approximately the same ratio as the radii of the orbits of Jupiter and Saturn (Fig. 1). Under impression of this observation he developed a model of the Solar system as a concentric sequence of five principal polyhedra. The Solar system in this model, named "Kepler's goblet" (Fig. 2), obeyed to a certain static geometrical order.

In 1784 Pierre Simon Laplace paid attention to another, not geometrical but dynamical, relation between the orbits of Jupiter and Saturn: the orbital periods of these planets are close to integer commensurability (resonance) 2/5. Taking this relation into account, he explained the observed anomalies in the motion of Jupiter and Saturn and revealed their periodic (instead of monotonous

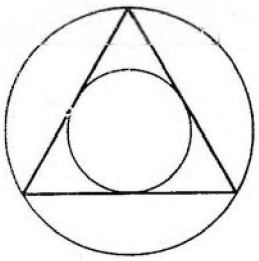

Figure 1. The orbits of Jupiter and Saturn, and Kepler's triangle.

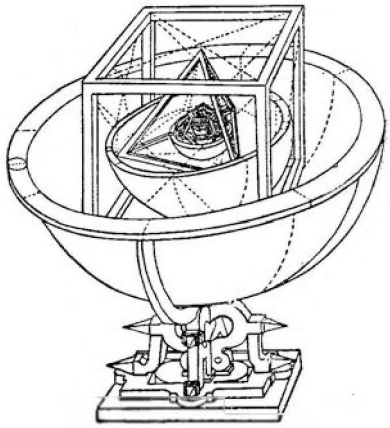

Figure 2. "Kepler's goblet".

— *secular*) character. This triumph of celestial mechanics, alongside with other outstanding achievements (in particular, with Alexis Claude Clairaut's rather exact prediction of the date of returning of the Halley comet in 1759), allowed Laplace to put forward an idea that the world is completely predetermined: if, at any moment of time, the positions and velocities of all the world-comprising particles are set, then the world's subsequent history is predetermined. As an embodiment of this idea, a mechanical device — an *orrery* — can serve, representing the motions and phases of the Solar system planets and their satellites by means of a clockwork. As A. Pannekoek (1961) wrote, "the Solar system was thought to be a huge mechanism brought and pushed in the motion solely by the force of universal gravitation. It was a fully cognizable and calculable clockwork which kept the motion forever".

Thus, at the end of the 18th century the consideration of a *single resonance* in Laplace's theory resulted in the formulation and general scientific recognition of determinism. The full predefiniteness of the motion of both large and minor bodies of he Solar system was not put into question. It was believed that the more accurate are observations and the more perfect is the theory, then the greater is the time interval for which it is possible to predict the motion

of any celestial body. It is paradoxical, but two centuries later an analysis of *interaction of resonances* in works by Boris Chirikov, Jack Wisdom and other scientists destroyed the deterministic concept. The refusal of it was promoted, in particular, by returning of the Halley comet (the third returning after that predicted by Clairaut) in 1986: now its orbit was considered not as an example of a completely calculable predetermined motion but as an example of manifestation of *dynamical chaos*. Thus the Laplace determinism had existed on the time span of three orbital revolutions of the Halley comet. It is curious that even in the days when the Laplace determinism dominated in the scientific thought absolutely, the educated community as a whole did hardly perceive the cosmic world as an ideally predictable mechanism: e. g. the apparitions of comets were generally perceived as sudden and random events.

## 2. Pendulum, resonances and chaos

*Resonance* represents the central concept of nonlinear dynamics. Chirikov (1982) defines it so: "Resonance is understood as such situation when some frequencies of a non-perturbed system are close to each other or to frequencies of an external perturbation". How one can be convinced in the existence of resonance in the motion of any particular celestial bodies? In fact, any observed ratio of frequencies can be approximated by some rational number with any degree of precision; but does resonance actually exist? To solve this problem, a *resonant phase* (named also *resonant* or *critical angle*, or *resonant* or *critical argument*) is defined. This is a linear combination (algebraic sum) of angular variables of the system with integer coefficients, the choice of which defines the resonant relation between frequencies. If the amplitude of variation of the resonant phase is limited, i. e. this angle librates, similarly to librations of a pendulum, then the system is in resonance; if it increases or decreases unlimitedly, i. e. rotates, then resonance is absent. The trajectory at the boundary between libration and rotation carries the name of the *separatrix*. So, the dynamics of a rigid pendulum (shown in Fig. 3) provides a model of resonance. In certain sense this model of resonance is universal (Chirikov 1979, 1982).

In celestial mechanics one deals, as a rule, with *nonlinear resonances*: the frequency of the phase oscillations on resonance depends on the amplitude (energy) of oscillations, as in the pendulum example.[1] In case of linear resonance the frequency does not depend on the amplitude.

A slightest external push of the rigid pendulum, placed near the upper position of equilibrium ($\varphi = \pm\pi = \pm 180°$; the angle $\varphi$ is defined in Fig. 3), is capable to change the motion character considerably (e. g. to replace oscillation with rotation). This phenomenon is nothing but the so-called essential dependence on initial conditions. What would occur, if the pendulum or any other system with a separatrix were subject to periodic perturbation? The system motion near the separatrix in the typical case, i. e. for the majority of the initial conditions and parameter values, becomes most unusual. Now this is well-known, but for the

[1]The properties of nonlinear resonance are described in detail in Chirikov's general review (Chirikov 1982), where the fundamental concepts of nonlinear dynamics are explained in most accessible and, at the same time, rigorous way.

first time the confusing and intricate behavior of the trajectories close to the *perturbed separatrix* (in the celestial-mechanical *three-body problem*) was pointed out by Henri Poincaré in 1899 (Poincaré 1899). However, it was not supposed at that time that the character of this intricate motion is in any sense "random".

Figure 3. The pendulum.

For a long time the studies of the motion near the separatrix did not concern chaotic behavior at all, but were limited to analysis of special cases where the traditional approach was possible. In 1908 Andrew Stephenson published a paper (Stephenson 1908) on the dynamics of the inverted ($\varphi = \pi = 180°$) rigid pendulum with the vibrating point of suspension. He found that the vertical vibration of the suspension point with particular values of frequency and amplitude is capable to stabilize the inverted pendulum. Later on, at the end of the forties of the 20th century, Peter Kapitsa (1951, 1954) demonstrated this effect in real experiments. Modern opportunities of computer techniques enable one to look at this effect in a new aspect: if one constructs a *section of the phase space* for such a pendulum, it becomes evident that the domain of stability represents only a tiny island in an extensive chaotic "sea" formed by the trajectories with obviously irregular behavior. Up to the middle of the 20th century this chaotic motion was not subject to scientific research, neither the chaotic motion of any other dynamical system.

In 1959 Chirikov for the first time described dynamical chaos as a phenomenon arising due to *interaction of resonances*. For a criterion of occurrence of chaos he offered the criterion of *overlapping of resonances* (Chirikov 1959, 1979). Let us give an explaining example. The phase space in case of the non-perturbed rigid pendulum has two dimensions, defined by two variables: the angle $\varphi$ of the pendulum deviation from the vertical, and the momentum $p = ml\dot{\varphi}$, where $m$ is the mass of the pendulum, $l$ is its length, $\dot{\varphi}$ is the rate of variation of the angle $\varphi$. In the well-known *phase portrait* "$\varphi$–$p$" of the non-perturbed pendulum (Fig. 4), a single domain ("cell") of librations, bound by the non-perturbed separatrix, is present. In other words, the pendulum model of resonance in the non-perturbed case describes single resonance. If one "switches on" a periodic perturbation, namely, vibration of the suspension point, then the phase space of this dynamical system is not two-dimensional anymore, and for

comparison with the phase portrait in the non-perturbed case it is necessary to construct the *phase space section*. It is built as follows: one plots the values of the system variables on the graph not continuously but "stroboscopically", i. e. discretely at constant intervals of time equal to the period of perturbation. On the section constructed in this way one discovers not one but three domains of librations — three resonances (Fig. 5). If the perturbation frequency is relatively large, the separation of resonances in the momentum is large and they almost do not interact. On reducing the frequency of perturbation, the resonances approach each other and appreciable chaotic layers emerge in the vicinity of the separatrices, where, as it is well visible in Fig. 5, the motion is irregular; on reducing further the frequency of perturbation, the layers merge into a single chaotic layer — the result of interaction of resonances at their strong mutual approach in the phase space.

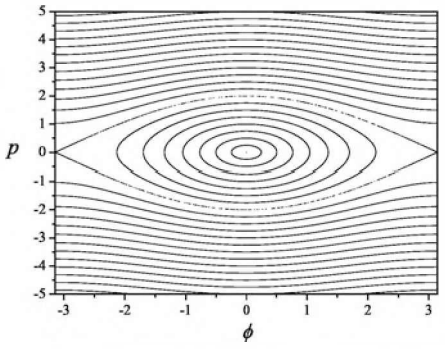


Figure 4. A phase portrait of the non-perturbed pendulum.

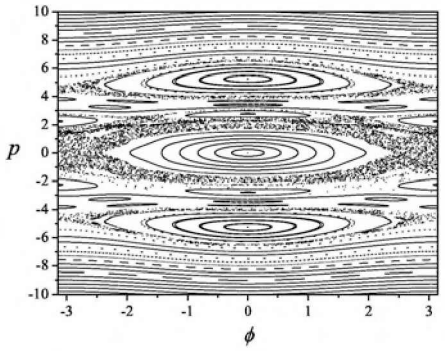


Figure 5. A triplet of interacting resonances at a moderate relative frequency of perturbation.

In Chirikov's saying, "... the physicist first of all tries to find out which resonances play role in this or that system and how do they interact with each other" (Chirikov 1982). It is just the presence of resonances, often regarded to be embodiments of order, leads to the unpredictable, chaotic character of the

motion. In other words, the presence of resonances in the phase space causes the presence of the chaotic component in this space. However, as we have just seen, for chaos to exist, the presence of not one but two or more resonances in the phase space is required, because their interaction is necessary.

## 3. Separatrix maps and chaotic layers

The motion in the vicinity of the perturbed separatrices of resonances is chaotic. For weakly perturbed systems the *chaotic layer* concept is convenient to apply for description of the chaotic component. The chaotic layer is a zone in the vicinity of the separatrices in the phase space where the dynamical system moves chaotically. The chaotic layer theory has applications in various areas of physics, mechanics and, in particular, in celestial mechanics (Chirikov 1979; Shevchenko 2007a, 2007b, 2010). The key role in this theory is played by the *separatrix maps*. They represent the motion of a system close to separatrices in a discrete way ("stroboscopically", in the same manner as in the described above construction of the phase space sections): system's state, set by the "time" and "energy" variables, is mapped discretely at the moments of passage of the positions of equilibrium by the pendulum describing the resonance. A rather general example of the separatrix map is

$$\begin{aligned} y' &= y + \sin x, \\ x' &= x + \lambda \ln |y'| + c, \end{aligned} \tag{1}$$

where $x$, $y$, and $x'$, $y'$ are the "time" and "energy" variables (in the normalized unitless form) at two moments of time following one after another; $\lambda$, $c$ are the constants expressed through physical parameters of the original system.

In deriving the equations of the separatrix map in classical form (Chirikov 1979), the *perturbed pendulum* model considered above, i. e. the dynamical system describing the pendulum with a periodic perturbation, serves for the model of nonlinear resonance. The main parameter of the separatrix map is the so-called *adiabaticity parameter* $\lambda$ equal to the ratio of the frequency of perturbation to the frequency of small phase oscillations on resonance. The separatrix map is applicable for the description of the motion at any values of this parameter (Shevchenko 2000). An example of the chaotic layer described by the separatrix map is shown in Fig. 6. The separatrix map in case of asymmetric perturbation represents a more complicated algorithm: it contains conditional transfer statements (Shevchenko 1999).

## 4. Lyapunov exponents

For dynamical chaos to emerge, three basic conditions must be satisfied (Devaney 1986; Meiss 1992): (1) *Sensitive dependence on initial conditions*, implying that nearby trajectories diverge in time exponentially. (2) The boundedness of the motion, implying that the exponential divergence is not just a smooth infinite expansion. (3) The set of trajectories with such behavior must have non-zero measure.

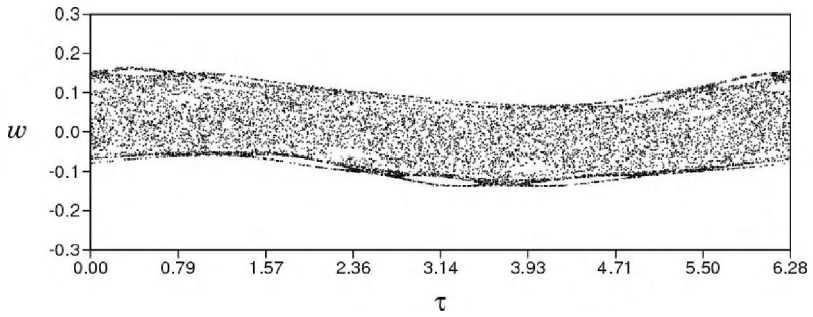


Figure 6. The chaotic layer in the "time–energy" coordinates, an example (Shevchenko 1998b).

The rate of divergence of the close trajectories (in the phase space and in the logarithmic scale of distances) is characterized by the *maximum Lyapunov exponent*. A distinction of the maximum Lyapunov exponent from zero testifies that the motion is chaotic (Chirikov 1979; Lichtenberg & Lieberman 1992). The inverse of this quantity, $T_{\mathrm{L}} = L^{-1}$, is the *Lyapunov time*. It represents the characteristic time of predictable dynamics. The importance of this timescale for celestial mechanics is provided by the fact that any exact theory of the motion of any celestial-mechanical system cannot be constructed on any times much greater than its Lyapunov time.

The art of calculation of the Lyapunov exponents on computers has more than a thirty-year history and during this time it has become an extensive part of applied mathematics; see reviews in (Froeschlé 1984; Lichtenberg & Lieberman 1992). Modern numerical methods provide calculation of the Lyapunov exponents which is effective and precise. On the other hand, methods of analytical estimating the Lyapunov exponents appeared only recently (Holman & Murray 1996; Murray & Holman 1997; Shevchenko 2002a). The method of estimating the maximum Lyapunov exponent (Shevchenko 2002a; Shevchenko 2008b), based on the theory of separatrix maps, allows one to obtain analytical estimates of the Lyapunov exponents (in accord with the numerical-experimental ones) in a number of problems on dynamics of the Solar system bodies; see the review by Shevchenko (2007a).

Generally, estimating the Lyapunov exponents is one of the most important tools in the study of chaotic motion (Lichtenberg & Lieberman 1992), in particular in celestial mechanics.

## 5. Analytical estimating the Lyapunov exponents

Let us review the problem of analytical estimating the Lyapunov exponents, following (Shevchenko 2007b). In (Shevchenko 2002a), an analytical method for estimating the maximum Lyapunov exponent of the chaotic motion in the vicinity of the separatrices of perturbed nonlinear resonance was derived in the framework of the separatrix map theory. Following this general approach, we represent the maximum Lyapunov exponent $L$ of the motion in the main chaotic layer of the system as the ratio of the maximum Lyapunov exponent $L_{\mathrm{sx}}$ of its separatrix map and the average half-period of libration (or, equivalently, the

average period $T$ of rotation) of the resonance phase $\varphi$ inside the layer. For convenience, we introduce a non-dimensional quantity $T_{\rm sx} = \Omega T$. Then the general expression for $L$ is

$$L = \Omega \frac{L_{\rm sx}}{T_{\rm sx}}. \tag{2}$$

The quantity $T_{\rm L} \equiv L^{-1}$, by definition, is the Lyapunov time.

### 5.1. The model of perturbed resonance

For the model of nonlinear resonance we take the Hamiltonian:

$$H = \frac{\mathcal{G}p^2}{2} - \mathcal{F}\cos\varphi + a\cos(\varphi - \tau) + b\cos(\varphi + \tau). \tag{3}$$

Under general conditions (Chirikov 1979; Lichtenberg & Lieberman 1992), a model of nonlinear resonance is provided by the Hamiltonian (3). The first two terms in Eq. (3) represent the Hamiltonian $H_0$ of the unperturbed pendulum; $\varphi$ is the pendulum angle (the resonance phase angle), $p$ is the momentum. The periodic perturbations are given by the last two terms; $\tau$ is the phase angle of perturbation: $\tau = \Omega t + \tau_0$, where $\Omega$ is the perturbation frequency, and $\tau_0$ is the initial phase of the perturbation. The quantities $\mathcal{F}$, $\mathcal{G}$, $a$, $b$ are constants. The frequency of the pendulum small-amplitude oscillations $\omega_0 \equiv (\mathcal{F}\mathcal{G})^{1/2}$.

An example of the phase space section of the Hamiltonian (3) at $\tau = 0 \bmod 2\pi$ is shown in Fig. 5 ($\Omega = 5$, $\omega_0 = 1$, $a = b$, $\varepsilon \equiv \frac{a}{\mathcal{F}} = 0.5$). This is a chaotic resonance triplet.

The motion near the separatrices of Hamiltonian (3) is described by the separatrix algorithmic map (Shevchenko 1999):

$$\begin{aligned}
&\text{if } w_n < 0 \text{ and } W = W^- \text{ then } W = W^+,\\
&\text{if } w_n < 0 \text{ and } W = W^+ \text{ then } W = W^-;\\
&w_{n+1} = w_n - W\sin\tau_n,\\
&\tau_{n+1} = \tau_n + \lambda \ln\frac{32}{|w_{n+1}|} \pmod{2\pi};
\end{aligned} \tag{4}$$

with the parameters

$$\lambda = \frac{\Omega}{\omega_0}, \tag{5}$$

$$\begin{aligned}
W^+(\lambda,\eta) &= \varepsilon\lambda\left(A_2(\lambda) + \eta A_2(-\lambda)\right),\\
W^-(\lambda,\eta) &= \varepsilon\lambda\left(\eta A_2(\lambda) + A_2(-\lambda)\right),
\end{aligned} \tag{6}$$

where $\varepsilon = \frac{a}{\mathcal{F}}$, $\eta = \frac{b}{a}$. The Melnikov–Arnold integral $A_2(\lambda)$ is given by the relation

$$A_2(\lambda) = 4\pi\lambda\frac{\exp(\pi\lambda/2)}{\sinh(\pi\lambda)}, \tag{7}$$

see (Chirikov 1979; Shevchenko 1998b; Shevchenko 2000).

The quantity $w$ denotes the relative (with respect to the separatrix value) pendulum energy: $w \equiv \frac{H_0}{\mathcal{F}} - 1$. The variable $\tau$ keeps its meaning of the phase

angle of perturbation. One iteration of map (4) corresponds to one half-period of pendulum's libration or one period of its rotation.

If $a = b$ (symmetric case), the separatrix algorithmic map reduces to the ordinary separatrix map

$$\begin{aligned} w_{i+1} &= w_i - W \sin \tau_i, \\ \tau_{i+1} &= \tau_i + \lambda \ln \frac{32}{|w_{i+1}|} \pmod{2\pi}, \end{aligned} \tag{8}$$

first written in the given form in (Chirikov 1979); the expression for $W$ (Shevchenko 1998b; Shevchenko 2000) is

$$W = \varepsilon\lambda \left(A_2(\lambda) + A_2(-\lambda)\right) = 4\pi\varepsilon \frac{\lambda^2}{\sinh \frac{\pi\lambda}{2}}. \tag{9}$$

An equivalent form of Eqs. (8), used e. g. in (Chirikov & Shepelyansky 1984; Shevchenko 1998a), is

$$\begin{aligned} y_{i+1} &= y_i + \sin x_i, \\ x_{i+1} &= x_i - \lambda \ln |y_{i+1}| + c \pmod{2\pi}, \end{aligned} \tag{10}$$

where $y = w/W$, $x = \tau + \pi$; and

$$c = \lambda \ln \frac{32}{|W|} \tag{11}$$

(compare with Eqs. (1)).

The applicability of the theory of separatrix maps for description of the motion near the separatrices of the perturbed nonlinear resonance in the full range of the relative frequency of perturbation, including its low values, was discussed and shown to be legitimate in (Shevchenko 2000).

The half-width $y_b$ of the main chaotic layer of the separatrix map (10) in the case of the least perturbed border of the layer is presented as a function of $\lambda$ in fig. 1 in (Shevchenko 2004a). The observed dependence follows the piecewise linear law with a transition point at $\lambda \approx 1/2$. This transition takes place not only in what concerns the width of the layer, but also in other characteristics of the motion, in particular, in the maximum Lyapunov exponent. The clear sharp transition at this point manifests a qualitative distinction between two types of dynamics, namely, "slow" and "fast" chaos.

The parameter $\lambda = \Omega/\omega_0$ of the separatrix map measures the separation of the perturbing and guiding resonances in the units of one quarter of the width of the guiding resonance, because the resonances separation in frequency space is equal to $\Omega$, while the guiding resonance width is equal to $4\omega_0$ (Chirikov 1979). Therefore, $\lambda$ can be regarded as a kind of the resonance overlap parameter (Shevchenko 2011). In the asymptotic limit of the adiabatic (slow) case the resonances in the multiplet strongly overlap, while in the asymptotic limit of the "fast" case the resonances are separated. However note that the border $\lambda \approx 1/2$ (Shevchenko 2008a) between slow and fast chaos does not coincide with the borderline between the cases of overlapping and non-overlapping of resonances: the latter borderline lies much higher in $\lambda$. E.g., in phase space of

the standard map the integer resonances start to overlap (on decreasing $\lambda$) at $\lambda \approx 2\pi/0.97 \approx 6.5$ (Chirikov 1979).

We consider four generic resonance types: the fastly chaotic resonance triplet, fastly chaotic resonance doublet, slowly chaotic resonance triplet, slowly chaotic resonance doublet (we call them, respectively, the "ft", "fd", "st", "sd" resonance types).

### 5.2. Fast chaos. Resonance triplet

Consider the case $a = b$ and $\lambda > 1/2$. There is a symmetric (due to the first condition) triad of interacting resonances, and chaos is fast (due to the second condition).

The case of the fastly chaotic triad is completely within the range of applicability of the method presented in (Shevchenko 2002a). The perturbed nonlinear resonance is modelled by Hamiltonian (3) with $a = b$. Following (Shevchenko 2002a), we take the dependence of the maximum Lyapunov exponent of the separatrix map (10) upon $\lambda$ in the form

$$L_{\rm sx}(\lambda) \approx C_h \frac{2\lambda}{1+2\lambda}, \tag{12}$$

where $C_h \approx 0.80$ is Chirikov's constant (Shevchenko 2004b).

The average increment of $\tau$ (proportional to the average libration half-period, or rotation period) inside the chaotic layer is (Chirikov 1979; Shevchenko 2002a):

$$T_{\rm sx}(\lambda, W) \approx \lambda \ln \frac{32e}{\lambda |W|}, \tag{13}$$

where $e$ is the base of natural logarithms. From Eq. (2), one has for the Lyapunov time for the "ft" resonance type:

$$T_{\rm L} = \frac{T_{\rm pert}}{2\pi} \frac{T_{\rm sx}}{L_{\rm sx}} \approx T_{\rm pert} \frac{(1+2\lambda)}{4\pi C_h} \ln \frac{32e}{\lambda |W|}, \tag{14}$$

where $T_{\rm pert} = 2\pi/\Omega$ is the period of perturbation.

### 5.3. Fast chaos. Resonance doublet

The previous analysis of the symmetric case $a = b$ sets a foundation for an analysis of the more general asymmetric case $a \neq b$, since the Lyapunov exponents in the asymmetric case can be found by averaging the contributions of the separate components of the chaotic layer.

Calculation of the average constitutes a complicated problem. In particular, one should know the relative average times of residence of the system in three different components of the layer corresponding to direct rotation, reverse rotation, and libration of the pendulum. The relative times of residence depend on the asymmetry of perturbation. A simple heuristic method of averaging was proposed in (Shevchenko 2002a). We consider the limit case of $a$ or $b$ equal to zero. It means that one of the two perturbing resonances simply does not exist, and instead of the resonance triad we have a duad.

If $\lambda > 1/2$, the equality $b = 0$ implies $|W^-| \ll |W^+|$, and, vice versa, $a = 0$ implies $|W^-| \gg |W^+|$. We designate the dominating quantity by $W$.

Consider first the libration side of the chaotic layer. Then $W^-$ and $W^+$ alternate (replace each other) at each iteration of the separatrix algorithmic map (4). It is straightforward to show that, if $W^-$ or $W^+$ is equal to zero, the separatrix algorithmic map (4) on the doubled iteration step reduces to the ordinary separatrix map (8) with the doubled value of $\lambda$ and the same non-zero value of $W$. One iteration of the new map corresponds to two iterations of the old one. Since the half-width of the chaotic layer of map (8) in the case of fast chaos is $\approx \lambda W$ (Chirikov 1979, Shevchenko 2008a), the layer's extent in $w$ on the side of librations doubles and becomes $\approx 2\lambda W$. Note that the parameters $\lambda$ and $W$ are considered here as independent from each other.

Consider then the circulation sides of the chaotic layer. The side corresponding to reverse (or direct) rotations does not exist, if $W^-$ (or, respectively, $W^+$) is equal to zero; its measure is zero. The other side, corresponding to direct (or reverse) rotations is described by the ordinary separatrix map (8) with the parameters $\lambda$, $W$; its extent in $w$ is $\approx \lambda W$.

The averaged (over the whole layer) value of the maximum Lyapunov exponent is the sum of weighted contributions of the layer components corresponding to the librations, direct rotations and reverse rotations of the pendulum. The weights are directly proportional to the times that the trajectory spends inside the components, and, via supposed approximate ergodicity, to the relative measures of the components in the phase space. Taking into account the just made estimates of the widths of the chaotic layer's components in the duad case, one can expect that the relative weights of librations and circulations in the "fd" case are respectively 4 and 1.

Hence the formula for the Lyapunov exponent for the "fd" resonance type is

$$L = \frac{\Omega}{\mu_{\rm libr} + 1} \left( \mu_{\rm libr} \frac{L_{\rm sx}(2\lambda)}{T_{\rm sx}(2\lambda, W)} + \frac{L_{\rm sx}(\lambda)}{T_{\rm sx}(\lambda, W)} \right), \tag{15}$$

and

$$T_{\rm L} = \frac{T_{\rm pert}}{2\pi} \cdot \frac{\mu_{\rm libr} + 1}{\mu_{\rm libr} \frac{L_{\rm sx}(2\lambda)}{T_{\rm sx}(2\lambda, W)} + \frac{L_{\rm sx}(\lambda)}{T_{\rm sx}(\lambda, W)}}, \tag{16}$$

where $\mu_{\rm libr} \approx 4$, and $W$, $L_{\rm sx}$, $T_{\rm sx}$ are given by formulas (9, 12, 13).

### 5.4. Slow chaos. Resonance triplet

In the case of $\lambda < 1/2$, the diffusion across the layer is slow, and on a short time interval the phase point of the ordinary separatrix map (10) follows close to some current curve. We call this curve guiding. Let us derive an analytical expression for the guiding curve with an irrational winding number far enough from the main rationals. We approximate the winding number by the rationals $m/n$. Thus $c \approx 2\pi m/n$. Noticing that at an iteration $n$ of the map the phase point hits in a small neighborhood of the starting point, one obtains for the derivative:

$$\frac{dy}{dx} = \frac{1}{nc-2\pi m}\sum_{k=0}^{n-1}\sin(x+kc) = \\ = \frac{1}{nc-2\pi m}\sin\frac{nc}{2}\,\mathrm{cosec}\frac{c}{2}\sin\left(x+\frac{n-1}{2}c\right). \quad (17)$$

Integrating and passing to the limit $n \to \infty$, one obtains:

$$y = -\frac{1}{2}\,\mathrm{cosec}\frac{c}{2}\cos\left(x-\frac{c}{2}\right)+\mathcal{C}, \quad (18)$$

where $\mathcal{C}$ is an arbitrary constant of integration.

The motion is chaotic only when curve (18) crosses the singular line $y = 0$. Hence the half-width of the chaotic layer is $y_b = \left|\mathrm{cosec}\frac{c}{2}\right|$. Averaging (by taking an integral analytically) the quantity $-\ln|y_{i+1}|$ (equal to $(\langle\Delta x\rangle - c)/\lambda$, Eqs. (10)), where $y_{i+1}$ is substituted by $y$ of Eq. (18), over the chaotic layer in the derived boundaries, we find the approximate analytical expression for $\Theta \equiv (T_{\mathrm{sx}} - c)/\lambda$:

$$\Theta \approx \ln\left|4\sin\frac{c}{2}\right|. \quad (19)$$

Then, we need an expression for $L_{\mathrm{sx}}(\lambda)$. We explore the $\lambda$ dependence of $L_{\mathrm{sx}}$ in a numerical experiment. At each step in $\lambda$ (namely, $\Delta\lambda = 0.005$, $\lambda \geq 0.005$) we find the value of $c$ corresponding to the case of the least perturbed layer and plot the value of $L_{\mathrm{sx}}$. At $\lambda < 0.3$, the dependence turns out to be practically linear. The linear fit $L_{\mathrm{sx}}(\lambda) = a\lambda$ gives $a = 1.01132 \pm 0.00135$, and the correlation coefficient $R = 0.9998$.

We set $L_{\mathrm{sx}} \approx \lambda$ for the generic (non-resonant) values of $c$ and for $\lambda < 1/2$. Then, from Eqs. (2, 19) one has the following approximate formula for the maximum Lyapunov exponent:

$$L \approx \frac{\Omega}{\ln\left|4\sin\frac{c}{2}\right| + \frac{c}{\lambda}}, \quad (20)$$

where $c = \lambda \ln\frac{32}{|W|}$.

For $\lambda \ll 1$ one has $W \approx 8\varepsilon\lambda$, hence the formula for the Lyapunov time for the "st" resonance type:

$$T_{\mathrm{L}} \approx \frac{T_{\mathrm{pert}}}{2\pi}\ln\left|\frac{16}{\varepsilon\lambda}\sin\left(\frac{\lambda}{2}\ln\frac{4}{|\varepsilon|\lambda}\right)\right|. \quad (21)$$

### 5.5. Slow chaos. Resonance doublet

Utilizing the approximation of the Melnikov–Arnold integral $A_2(\lambda) \approx 2\pi\lambda + 4$ at $\lambda \ll 1$, $\eta = 0$, one has: $W^{\pm} \approx \varepsilon\lambda(4 \pm 2\pi\lambda) \approx 4\varepsilon\lambda$. So, in the "sd" case, the separatrix algorithmic map (4) degenerates to the ordinary separatrix map (8) with $W \approx 4\varepsilon\lambda$, i. e. mathematically the case is equivalent to the "st" case, but with a different (halved) value of $W$.

Following the lines of the previous Subsection, it is then straightforward to write down the formula for the Lyapunov time for the "sd" resonance type:

$$T_{\rm L} \approx \frac{T_{\rm pert}}{2\pi} \ln \left| \frac{32}{\varepsilon\lambda} \sin\left( \frac{\lambda}{2} \ln \frac{8}{|\varepsilon|\lambda} \right) \right|. \qquad (22)$$

### 5.6. Separatrix map theory versus numerical experiment

To check the theory, the Lyapunov exponents of the chaotic motion near the separatrices of Hamiltonian (3) have been directly computed in (Shevchenko 2007b) by means of the program package (Shevchenko & Kouprianov 2002; Kouprianov & Shevchenko 2003) utilizing the HQRB method by von Bremen, Udwadia, & Proskurowski (1997). The power of this method is far greater than that necessary in the present computation. It allows one to evaluate the full Lyapunov spectrum of a multidimensional system. The spectrum of our system consists of a sole pair of Lyapunov exponents — the maximum one and its negative counterpart. The integration of the equations of motion has been performed by the integrator by Hairer, Nørsett, & Wanner (1987). It is an explicit 8th order Runge–Kutta method due to Dormand and Prince, with the step size control. We choose the time unit in such a way that $\omega_0 = 1$.

The results of the computations for $\lambda = 0.1$ and 2 are shown in Fig. 7 in decimal logarithmic scales. The integration time interval has been chosen to be equal to $10^6$. This is sufficient for the computed values of the Lyapunov exponents to saturate in each case. Local wave-like patterns represent prominent features of the constructed dependences; they are conditioned by the process of encountering resonances, while $\varepsilon$ changes.

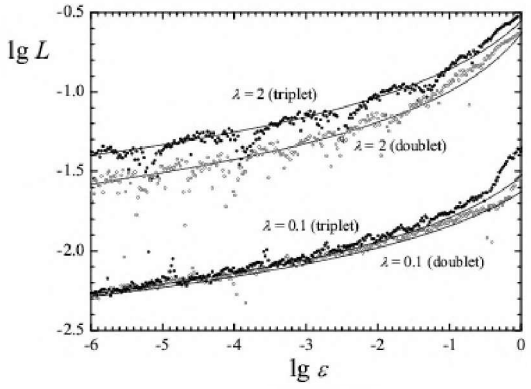


Figure 7. The maximum Lyapunov exponent of the chaotic motion of system (3) in dependence on the amplitude of perturbation: the results of direct computation and the theoretical curves (Shevchenko 2007b). Filled and white dots denote the numerical-experimental data for triplets and doublets, respectively.

Theoretical dependences are shown in Fig. 7 as solid curves. $T_{\rm L}$ are given by formulas (14, 16, 21, 22); $L = 1/T_{\rm L}$. Close correspondence is observed between the theory and experimental data for each value of $\lambda$.

### 5.7. Lyapunov exponents for the motion in supermultiplets

Now let us consider the case when the number of resonances in a multiplet is greater than 3. We assume that the guiding resonance is not at the border of the multiplet. In the case of fast chaos ($\lambda > 1/2$) one can still apply, as an approximation, the theory developed in Subsection 5.2. for the triplet case, because the influence of the "far away" resonances is exponentially small with $\lambda$. However, if chaos is slow ($\lambda < 1/2$), the triplet approximation does not work and one has to develop an approach totally different from the previous analysis. Let us consider the limit case — the case of infinitely many interacting equal-sized and equally spaced resonances.

The standard map

$$\begin{aligned} y_{i+1} &= y_i + K \sin x_i \pmod{2\pi}, \\ x_{i+1} &= x_i + y_{i+1} \pmod{2\pi} \end{aligned} \tag{23}$$

describes the motion in an infinite multiplet of equal-sized and equally spaced resonances, as it is clear from its Hamiltonian

$$H = \frac{y^2}{2} + K \sum_{n=-\infty}^{\infty} \cos(x - nt) \tag{24}$$

(Chirikov 1979). The variables $x_i$, $y_i$ of the map (23) correspond to the variables $x(t_i)$, $y(t_i)$ of the continuous system (24) taken stroboscopically at time moduli $2\pi$ (see e. g. Chirikov 1979).

A formula for the maximum Lyapunov exponent of the standard map was derived in (Chirikov 1979). By means of linearizing the map near some point (not necessarily fixed) and solving the characteristic equation one obtains the following formula for the maximum eugenvalue at $K \gg 1$:

$$l^+ \approx K|\cos x|.$$

At $K \gg 1$ the relative measure of the regular component of the phase space of the standard map is close to zero; elliptic periodic trajectories (e. g. the so-called *accelerator modes*) and small regular islands, associated to them, can exist, but the measure of these islands rapidly tends to zero with $K \to \infty$; see (Chirikov 1979; Giorgilli & Lazutkin 2000; Shevchenko 2007c). Therefore one can assume complete ergodicity of the motion at large values of $K$ and obtain the maximum Lyapunov exponent by averaging of $l^+$ over the phase plane, equivalently over $x$:

$$L_{\rm st} \approx \frac{1}{2\pi} \int_0^{2\pi} \ln K|\cos x| dx = \ln \frac{K}{2}. \tag{25}$$

As found by Chirikov in a numerical experiment, already at $K = 6$ the difference between the theoretical and observed values is less than $\approx 2\%$.

The adiabaticity parameter $\lambda \equiv \Omega/\omega_0$ is related to the standard map parameter by the formula $\lambda = 2\pi/\sqrt{K}$ (see Chirikov 1979). Therefore, for equal-sized resonance multiplets, parameterized by $\lambda$, one has

$$L_{\rm mp} \approx \ln \frac{2\pi^2}{\lambda^2}. \tag{26}$$

Since the threshold for global chaos onset is $K \approx 1$ (the condition of marginal overlap; see Chirikov 1979; Lichtenberg & Lieberman 1992), this formula can be used already at $\lambda$ less than $\approx 6$ and, with much higher precision, at $\lambda$ less than $\approx 3$. In other words, the domain of its applicability covers not only the realm of adiabatic chaos, but an actual portion of that of fast chaos as well. However, the number of resonances in the multiplet, for the model to work, should be large, because for the standard map this number is infinite.

Using formulas obtained in (Shevchenko 2004a; Shevchenko 2004b) for the dependence $L(K)$ for the standard map at $K < 1$ and $K > 4.5$, one arrives at the approximation

$$L = \Omega \cdot \begin{cases} 0.1333K, & \text{if } K \le 2, \\ \ln \frac{K}{2} + \frac{1}{K^2}, & \text{if } K > 2, \end{cases} \tag{27}$$

where $K = (2\pi/\lambda)^2$ and $\Omega = \lambda/(2\pi)$. Note that we glue two asymptotic dependences at $K = 2$, thus somewhat underestimating the actual values of $L$ in the interval $1 > K > 4.5$; see Fig. 8.

Formula (27) can be straightforwardly applied for estimating maximum Lyapunov exponents in resonance multiplets, when the number of resonances in a multiplet is large. The $\lambda$ dependences, both numerical-experimental and theoretical, of the maximum Lyapunov exponent in multiplets of equal-sized and equally spaced resonances are shown in Fig. 9. One can see that the dependence for the septet occupies an intermediate position between the dependence for the triplet and the dependence for the "infinitet", i. e. for the standard map. The numerical data for the triplet are described well by the separatrix map theory (the lower solid curve) described above in Subsections 5.2. and 5.4., notwithstanding the large amplitude of perturbation $\varepsilon = 1$.

## 6. Chaos in rotation of planetary satellites

In 1984 Wisdom, Peale, & Mignard predicted, on a basis of numerical experiments and theoretical estimates of the width of a chaotic layer, that the seventh satellite of Saturn, Hyperion, rotates chaotically with respect to its centre of masses: the orientation of the satellite and its rotation rate vary with time chaotically (Wisdom *et al.* 1984).

The cause of chaos in the rotational dynamics of satellites consists in interaction of *spin-orbit resonances*, i. e. resonances between the rotational and orbital motions (see e. g. Wisdom *et al.* 1984; Celletti & Chierchia 2000). It turned out that in the case of Hyperion this interaction is especially strong, due to the strong asymmetry of its figure and a substantial deviation of its orbit from the circular one, i. e. its appreciable eccentricity.

A primary source of observational data on the rotational dynamics of minor planetary satellites is provided by the analysis of their optical lightcurves — the series of the light flux values fixed by observers during extended intervals of time. In 1989 James Klavetter constructed accurate lightcurves of Hyperion with high

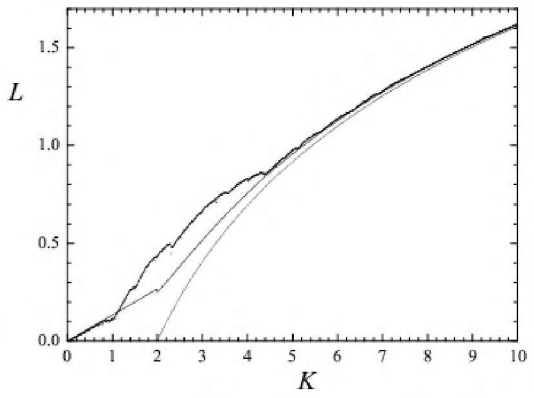


Figure 8. The dots show the numerical-experimental dependence $L(K)$ for the standard map at $0 < K < 10$ (Shevchenko 2004a; Shevchenko 2004b). The dashed curve shows the function $\ln \frac{K}{2}$. The solid curve is given by Eqs. (27) with $\Omega = 1$.

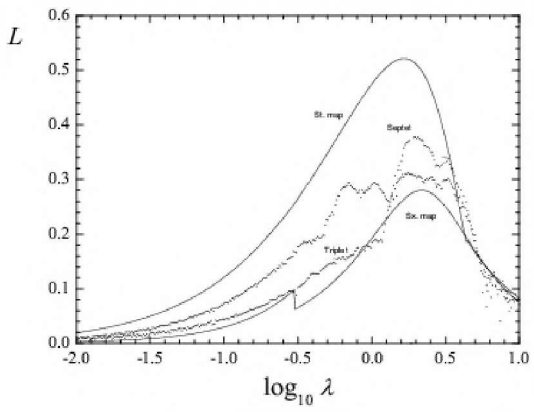


Figure 9. The $\lambda$ dependences of the maximum Lyapunov exponent in multiplets. The dots denote numerical-experimental data, and the solid curves denote theoretical dependences. The upper solid curve corresponds to the standard map theory (for the infinite multiplet), whereas the lower one corresponds to the separatrix map theory (for the triplet).

time resolution and carried out their analysis and modelling (Klavetter 1989). He concluded that this satellite was, most likely, indeed in the irregular rotation state. In 2002 Alexander Melnikov at the Pulkovo Observatory with the help of especially developed software modelled the lightcurves of Hyperion (Melnikov 2002), using both the data of Klavetter's observations and the data of Pulkovo observations by A.V.Devyatkin *et al.* By means of computation of the Lyapunov

exponents of the motion he arrived to rigorous conclusion that Hyperion was in the chaotic mode of rotation. In Fig. 10 the model lightcurves of Hyperion are presented.

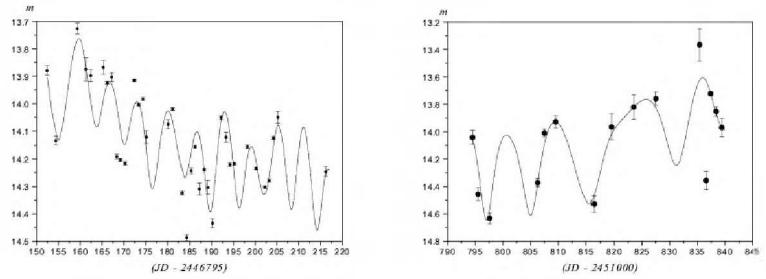


Figure 10. Model lightcurves of Hyperion: for Klavetter's observational data (on the left), and for the Pulkovo observational data obtained in 2000 (on the right). The dots denote the observational data. At the graph axes: JD is the Julian date, $m$ is the stellar magnitude.

The planar (in the orbit plane) oscillations and rotations of a satellite near *synchronous* resonance (the resonance at which the period of rotation of a satellite is equal to its orbital period, as in the case of the Moon) are described by an equation of the perturbed pendulum, where the role of the angle of the pendulum is played by the angle specifying orientation of the satellite with respect to the direction to the planet. Therefore the described above method of analytical estimating the maximum Lyapunov exponent is applicable. The resulting theoretical estimates of the Lyapunov time ($\approx$ 30 days) are in accord with the numerical-experimental ones (Shevchenko & Kouprianov 2002; Kouprianov & Shevchenko 2005).

Do there in the Solar system exist any other (besides Hyperion) chaotically tumbling satellites? For many satellites the character of rotation is unknown. The satellites for which it is established, in the majority rotate synchronously with the orbital motion. Similar to the Moon, they permanently face their planet with one side. However, in the course of the long-term dynamical evolution any satellite at some moment of time enters a state of chaotic rotation, e. g. when passing the separatrix of synchronous resonance, because otherwise it cannot be captured in synchronous rotation. Minor and large satellites have different characters of chaotic rotation. For the minor (irregular-shaped) satellites such rotation represents three-dimensional tumbling; for the majority of large satellites (close to spherical in shape) the rotational motion in the chaotic layer in the vicinity of the separatrix of synchronous resonance preserves planar character: the axis of rotation remains to be approximately orthogonal to the orbit plane (Kouprianov & Shevchenko 2005).

Our theoretical research (Kouprianov & Shevchenko 2005) showed that, besides Hyperion, in the state of chaotic rotation might also reside the 16th and 17th Saturnian satellites Prometheus and Pandora. If they rotate chaotically, the Lyapunov times of their rotation are very small: 0.5–0.6 days. The character of their rotation is not known yet. The following problem is important: whether in the course of the "chaotic tumbling" there exists a preferable orientation of

the satellite, or all orientations are equiprobable? Our calculations (Melnikov & Shevchenko 2008) testify that Prometheus and Pandora in the case of chaotic rotation do have preferable orientations of the largest axis of the figure in direction to Saturn. This complicates making conclusions about the character of rotation of these satellites from observations, because the chaotic mode is similar in some sense to the usual synchronous rotation.

So, the Lyapunov times of chaotic rotation of minor planetary satellites of the Solar system can be very small: as small as a half-day, i. e. chaos in case of its presence is quite observable. However, till now the chaotic rotation was observed only in the case of Hyperion. More than 160 satellites of planets have been discovered till now, but the state of rotation is determined from observations for only 33 of them. Amongst these 33 satellites, 25 are in synchronous resonance and 7 have the periods of rotation much less than the orbital periods.

And what about the Moon, which is of course most important to us, amongst all the planetary satellites? As it is well-known to everyone, the Moon always faces the Earth with one and the same side, i. e. it is in synchronous resonance. It is not so widely known that the Moon is subject to small oscillations with respect to the exact synchronous state. This phenomenon carries the name of *physical libration* of the Moon. How much regular and predictable are these small oscillations of our satellite, if one considers large intervals of time? Unfortunately, this problem has not been considered till now, maybe due to the complexity of the problem.

## 7. Satellite systems

If the orbital frequencies of planets in a planetary system or satellites in a satellite system are approximately commensurable, i. e. their ratio is approximately equal to the ratio of two integers, the system is either close to orbital resonance,[2] or resides in it. In the Solar system, many planetary satellites belong to resonant or close-to-resonant systems. E.g., in the satellite system of Jupiter, Io and Europa, as well as Europa and Ganymede, are in the *mean motion resonance* 2/1. In the system of Saturn, Mimas and Tethys, as well as Enceladus and Dione, are also in the mean motion resonance 2/1, Dione and Rhea are close to resonance 5/3, Titan and Hyperion are in resonance 4/3. In the system of Uranus, all resonances are approximate: Miranda and Umbriel move close to resonance 3/1, Ariel and Umbriel — 5/3, Umbriel and Titania — 2/1, Titania and Oberon — 3/2.

Captures of satellite systems in orbital resonances are natural stages of tidal evolution of these celestial-mechanical systems. Tittemore & Wisdom (1988, 1989) and Dermott, Malhotra, & Murray (1988) showed that such capture can significantly affect the further orbital dynamics of satellites and even their internal structure. In the course of capture in orbital resonance or in the course of escape from it, the system crosses the chaotic layer in the vicinity of the resonance separatrices. Inside the layer the system moves chaotically. However,

[2]The expression "close to resonance" means that the frequencies satisfy the given integer relation only approximately, and thus the corresponding resonant phase rotates, instead of libration.

the cause of chaos, as a rule, consists not in the interaction of separate mean motion resonances, but in the interaction of *subresonances* in a multiplet corresponding to a single mean motion resonance. The form and orientation of orbits in satellite systems are subject to slow variations, including monotonous (secular) precession of orbits. The splitting of orbital resonances into subresonance multiplets is caused by this precession.

The second (Miranda) and fifth (Umbriel) satellites of Uranus do not reside now in exact orbital resonance with each other, but they are rather close to resonance 3/1. According to Tittemore and Wisdom's conclusions, it is rather probable that in the past, in the course of tidal evolution of their orbits, these satellites spent some time in resonance 3/1. An example of the phase space section of the motion for one of the stages of evolution of this system is presented in Fig. 11. Numerical-experimental and analytical (obtained with the help of the separatrix map theory) estimates of the Lyapunov time were made by Melnikov & Shevchenko (2005). The Lyapunov time of the chaotic system at various stages of its evolution turned out to be equal to 50–100 yr.

Chaotic modes in the dynamics of satellite systems took place not only during the former epochs of the history of the Solar system. Champenois & Vienne (1999a, 1999b) considered the dynamics of the first and third satellites of Saturn, Mimas and Tethys, residing in orbital resonance 2/1. These dynamics might be chaotic due to the large amplitude of librations on resonance. Champenois and Vienne reduced averaged equations of the system motion to the equation of pendulum with periodic perturbations. Using this analytical representation, it is possible to obtain analytical estimates of the Lyapunov time. Such estimates were obtained by Melnikov & Shevchenko (2005), and not only analytically, but also numerically, by means of direct integration of the equations of motion. The Lyapunov times turned out to be 300–600 yr in various models. Thus, chaos in this system does not manifest itself on timescales accessible to observations.

The behavior of the Prometheus–Pandora system (the 16th and 17th Saturnian satellites — the shepherd satellites of the ring F) is much more chaotic. In 2003 Goldreich and Rappaport estimated, by means of numerical experiments, the Lyapunov time of the orbital motion of this system in the vicinity of the mean motion resonance 121/118 (Goldreich & Rappaport 2003a, 2003b). According to their results, it is equal to $\approx$ 3.3 yr. Farmer & Goldreich (2006) reduced the equations of the system motion to the equation of the pendulum with periodic perturbations. It is possible to obtain an analytical estimate of $T_{\mathrm{L}}$ (Shevchenko 2008b). It is inside the narrow range of the numerical-experimental estimates.

Therefore, the available estimates of the Lyapunov time for the Prometheus–Pandora ($\approx$ 3 yr), Miranda–Umbriel (50–100 yr) and Mimas–Tethys (300–600 yr) systems testify that the range of the times of predictable dynamics in the chaotic satellite systems is rather broad: in the known cases it extends from one year to thousand years (by the order of magnitude). In case of the Prometheus–Pandora system, orbital chaos, due to the small Lyapunov time, is potentially observable.

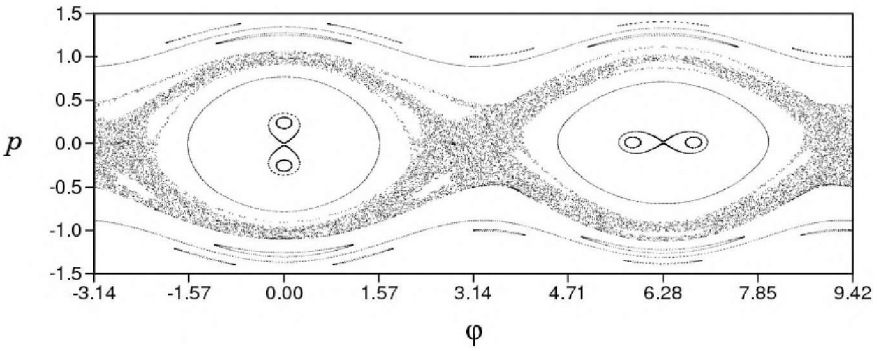


Figure 11. The phase space section for the Miranda–Umbriel system (Melnikov & Shevchenko 2005). Broad chaotic layers in the vicinity of resonance separatrices are prominent.

## 8. Orbital dynamics of asteroids and comets

The important role of resonances in the dynamics of asteroids became evident since the time of discovery of gaps (sparsely populated intervals in the distribution of mean motions) in the belt of asteroids in 1867 by Daniel Kirkwood. The most expressed minima in the distribution of the semimajor axes of the asteroidal orbits correspond to the mean motion resonances 2/1, 3/1, 4/1, 5/2 and 7/3 with Jupiter (see e. g. Murray & Dermott 1999; Morbidelli 2002). The role of chaos in "clearing" the gaps from the asteroids was found out by Wisdom in the beginning of eighties of the last century, at least for the gap corresponding to resonance 3/1 (this one and that corresponding to resonance 2/1 are the most prominent): most of the chaotic orbits close to this commensurability demonstrate sporadic jumps of the eccentricity, resulting in crossings of the orbits with the orbit of Mars. Therefore, sooner or later the asteroids are ejected from the gap area owing to close encounters with this planet. However, as was discovered later on in the works by S. Ferraz-Mello & J. C. Klafke (1991), B. Gladman *et. al.* (1997), the overlap of the 3/1 resonance with secular resonances $\nu_5$ and $\nu_6$ at moderate and high eccentricities provides much greater pumping (up to unity) of the asteroidal eccentricity and consequently a much higher rate of the population depleting, than in Wisdom's scenario, due to encounters with Earth and Venus and falling onto the Sun. The same phenomenon of overlap with secular resonances provides fast depleting of population in other important mean motion resonances in the asteroid belt, such as 2/1, 4/1, 5/2, and 7/3 (Moons & Morbidelli 1995; Gladman *et. al.* 1997; Moons, Morbidelli, & Migliorini, 1998; for a review see Morbidelli 2002).

The domain of chaos corresponding to the 3/1 resonance, computed by means of Wisdom's (1983) map in the planar elliptic restricted three-body problem Sun–Jupiter–asteroid, is shown in Fig. 12 on a representative set of initial values of the semimajor axis $a$ and eccentricity $e$ of the asteroidal orbits. The borders of this domain are determined by the calculation of the maximum Lyapunov exponent of the asteroidal motion. The initial data corresponding to the chaotic trajectories with and without jumps of the eccentricity are shown re-

spectively in black and grey. The graph reveals an evidently significant extent of the near-resonant domain of chaos.

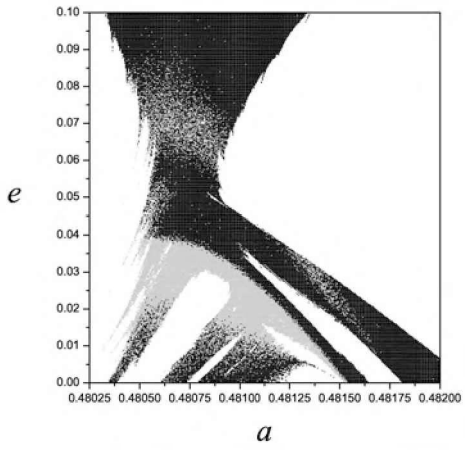


Figure 12. The region of the 3/1 mean motion resonance with Jupiter in the planar elliptic three-body problem Sun–Jupiter–asteroid (Shevchenko 2002b); $a$ is the semimajor axis, $e$ is the eccentricity of an asteroidal orbit.

The orbital resonances in the motion of asteroids subdivide into the *mean motion resonances* and *secular resonances*. The mean motion resonances represent the commensurabilities between the mean frequencies of the orbital motions of an asteroid and a planet, and the secular resonances represent the commensurabilities between the rates of precessions of the orbits of an asteroid and a planet. Besides the usual (two-body) mean motion resonances, an essential role in the dynamics of asteroids is played by the so-called *three-body mean motion resonances* (Murray, Holman, & Potter 1998; Nesvorný & Morbidelli 1998, 1999; Morbidelli 2002; Cachucho, Cincotta, & Ferraz-Mello 2010). In this case the resonant phase represents a combination of the angular coordinates of an asteroid and two planets (e. g. an asteroid, Jupiter and Saturn). Both for the two-body and three-body resonances, the equations of motion in typical cases are reduced approximately to the equation of pendulum with periodic perturbations; therefore, it is possible to estimate analytically the Lyapunov times of the motion (Shevchenko 2007b).

As follows from the orbital data placed at the AstDyS web service[3] (Knežević & Milani 2000) for hundreds of thousands asteroids, the Lyapunov times of the asteroidal motion in the main belt that are less than 400 yr are absent among the numerical-experimental estimates. It is usually not known what is the resonant multiplet that serves as the cause of the chaotic behavior of a particular asteroid, because it is rather difficult to identify three-body resonances (which are most numerous among the important resonances).

[3]http://hamilton.dm.unipi.it/cgi-bin/astdys/

In studies of chaotic dynamics of celestial bodies, the motion in perturbed highly-eccentric orbits is of special interest, because it is often strongly chaotic. The asteroids and comets crossing the orbits of planets are by all means amongst the most chaotic objects of the Solar system. Giovanni Valsecchi (2007) showed convincingly that the history of researches of chaotic dynamics of the minor bodies of the Solar system counts from the works on the dynamics of the Lexell comet accomplished by Leonhard Euler's pupil Andrej Lexell in the seventies of the 18th century and by U. Le Verrier in the forties and fifties of the 19th century. The works of Lexell were published in the proceedings of Academia Petropolitana — the Russian Academy of Sciences. In Valsecchi's saying, "It can be said that the work of Lexell started the modern understanding of the dynamics of small solar system bodies" (Valsecchi 2007). This modern understanding consists in taking into account the paramount role of resonances and close encounters with planets. In celestial-mechanical calculations of Le Verrier, the concept of essential dependence on initial conditions appeared for the first time: very small model variations (about several metres per second) of the velocity of the Lexell comet in its orbit perihelion resulted in *qualitative* changes of the orbit.

In 1770, the Lexell comet passed at an unprecedentedly close distance from the Earth. Nine years later, it was most probably ejected from the Solar system as a result of its close encounter with Jupiter. In the history of astronomical observations it became the first outstanding example of an object closely encountering the Earth. This example is enough to make it clear that the problem of assessment of the degree of predictability of the motion of planet-crossing bodies has not only theoretical but also important practical significance.

Such an assessment for the orbital motion of the asteroids passing close to the Earth (near-Earth asteroids — NEAs), and other objects potentially hazardous to the Earth, is one of the most complicated aspects of the problem of asteroid-cometary hazard. Some asteroids move in virtually predictable orbits, others do not. In A. Whipple's (1995) saying, "The existence of a significant population of extremely chaotic Earth-crossing asteroids must be factored into the thinking about the potential hazard posed by these objects. An asteroid with a Lyapunov time of 20 years may be considered as an example. If the initial error in its position is 100 km (a very optimistic assumption) then that error will grow to one Earth radius in 83 years and to an Earth–Moon distance in 165 years. Assessments of the threat from specific objects like this can be made for only short spans of time."

The Lyapunov exponents for this kind of objects can be estimated by means of analysis of the so-called *Kepler map*. The equations of this map were deduced by T. Petrosky (1986), and independently by Chirikov & Vecheslavov (1986, 1989), in connection with apparition of the Halley comet. The model of the comet motion consisted in the assumption that the main perturbing effect of Jupiter is concentrated on a rather short interval of time when the comet is close to its orbit perihelion. This effect is determined by the phase of Jupiter at this short time span. The deduction of the formula for the Lyapunov time of the motion of comets and other planet-crossing objects (such as NEAs) is based on the Kepler map concept as a generalized separatrix map, the non-perturbed parabolic orbit playing the role of the non-perturbed separatrix (Shevchenko 2007b). This formula gives the lower bound for the possible values of the Lya-

punov time. As far as it is known, at present none of the observed objects breaks this border.

As a whole, the planet-crossing asteroids (NEAs among them) and comets are amongst the most chaotic objects of the Solar system. The Lyapunov times of these objects in comparison with the Lyapunov times of the majority of usual asteroids can be very small — as low as several years.

### 8.1. Two-body mean motion resonances

Let us consider analytical estimating the maximum Lyapunov exponents of the orbital motion of asteroids in two-body mean motion resonances, following Shevchenko (2007b).

The Hamiltonian of the motion of a zero-mass test particle in the gravitational field of the Sun and Jupiter, in the plane of Jupiter's orbit, in the vicinity of a mean motion resonance with Jupiter can be represented in some approximation in the following form (Holman & Murray 1996; Murray & Holman 1997):

$$H = \frac{1}{2}\beta\Lambda^2 - \sum_{p=0}^{q} \varphi_{k+q,k+p,k} \cos(\psi - p\omega), \tag{28}$$

where $\beta = 3k^2/a^2$, $\Lambda = \Psi - \Psi_{\rm res}$, $\Psi = (\mu_1 a)^{1/2}/k$, $\Psi_{\rm res} = (\mu_1^2/(k^2(k+q)n_{\rm J}))^{1/3}$, $\mu_1 = 1-\mu$, $\omega \equiv -\varpi$ (i. e. $\omega$ is minus the longitude of asteroid's perihelion; its time derivative is assumed to be constant); $a$ and $e$ are asteroid's semimajor axis and eccentricity. The integer non-negative numbers $k$ and $q$ define the resonance: the ratio $(k+q)/k$ equals the resonant ratio of the orbital frequencies of the particle and Jupiter. The phase $\psi \equiv kl - (k+q)l_{\rm J}$, where $l$ and $l_{\rm J}$ are the mean longitudes of an asteroid and Jupiter.

Here the units are chosen in such a way that the total mass (Sun plus Jupiter), the gravitational constant, Jupiter's semimajor axis $a_{\rm J}$ are all equal to one; $\mu = 1/1047.355$, $\mu_1 = 1-\mu$. Jupiter's mean longitude $l_{\rm J} = n_{\rm J}t$, eccentricity $e_{\rm J} = 0.048$. Jupiter's mean motion $n_{\rm J} = 1$, i. e. the time unit equals $\frac{1}{2\pi}$th part of Jupiter's orbital period.

According to Eq. (28), the resonance $(k+q)/k$ splits in a cluster of $q+1$ subresonances $p = 0, 1, \ldots, q$. The coefficients of the resonant terms are

$$|\varphi_{k+q,k+p,k}| \approx \frac{\mu}{q\pi a_{\rm J}} \binom{q}{p} \left(\frac{\varepsilon}{2}\right)^p \left(\frac{\varepsilon_{\rm J}}{2}\right)^{q-p}, \tag{29}$$

where $\varepsilon = ea_{\rm J}/|a-a_{\rm J}|$, $\varepsilon_{\rm J} = e_{\rm J}a_{\rm J}/|a-a_{\rm J}|$ (Holman & Murray 1996; Murray & Holman 1997). The approximation (29) is good, if $\varepsilon q < 1$ (Holman & Murray 1996). Besides, the model is restricted to the resonances of relatively high order, $q \geq 2$.

The signs of the coefficients $\varphi_{k+q,k+p,k}$ alternate with changing $p$; therefore, the coefficients with numbers $p$ and $p+2$ are always of the same sign. This means that whatever is the choice of the guiding resonance in the multiplet, its closest neighbors have coefficients of equal signs, and $\eta$ is always non-negative.

The frequency of small-amplitude oscillations on subresonance $p$ is

$$\omega_0 = (\beta|\varphi_{k+q,k+p,k}|)^{1/2} \approx \frac{a_\mathrm{J}}{|a-a_\mathrm{J}|} n_\mathrm{J} \left( \mu_1 \mu \frac{4q}{3\pi} \binom{q}{p} \left(\frac{a}{a_\mathrm{J}}\right) \left(\frac{\varepsilon}{2}\right)^p \left(\frac{\varepsilon_\mathrm{J}}{2}\right)^{q-p} \right)^{1/2}, \tag{30}$$

and the perturbation frequency is

$$\Omega = \dot{\omega} \approx \frac{\mu_1 \mu}{2\pi} n_\mathrm{J} \left(\frac{a}{a_\mathrm{J}}\right)^{1/2} \left(\frac{a_\mathrm{J}}{a-a_\mathrm{J}}\right)^2 \tag{31}$$

(cf. Holman & Murray 1996; Murray & Holman 1997). The ratio of $\Omega$ and $\omega_0$ gives the value of $\lambda$.

Table 1. Numerical and analytical estimates of Lyapunov times for two-body mean motion resonances, $T_\mathrm{L}$ is in years, $e = 0.1$ (Shevchenko 2007b)

| $\frac{k+q}{k}$ | $\lambda$ | $\log_{10} T_\mathrm{L}^\mathrm{num}$ † | $\log_{10} T_\mathrm{L}^\mathrm{theor}$ | Res. type |
|---|---|---|---|---|
| 3/1 | 0.093 | 3.8–4.3 | 4.3 | sd |
| 5/2 | 0.192 | 3.5–3.8 | 4.1 | st |
| 7/3 | 0.415 | 3.8–4.2 | 4.0 | st |
| 9/4 | 0.932 | 3.9–4.3 | 4.2 | fd* |
| 11/5 | 1.970 | 3.9–4.3 | 4.3 | ft* |
| 9/5 | 0.323 | 3.6–3.8 | 3.7 | st* |
| 7/4 | 0.166 | 3.2–3.3 | 3.7 | st |
| 12/7 | 0.594 | 3.6–4.0 | 3.9 | fd* |
| 5/3 | 0.101 | 2.5–3.3 | 3.7 | sd |
| 8/5 | 0.156 | 2.5–3.3 | 3.6 | st* |
| 11/7 | 0.264 | 3.3–3.6 | 3.5 | st* |

† (Morbidelli & Nesvorný 1999; Holman & Murray 1996)

Now we are able to apply the theory developed in Section 5. For comparison, we take the data on the numerical values (based on integrations) of $T_\mathrm{L}$ for the motion near the mean motion resonances presented in Fig. 1 in (Morbidelli & Nesvorný 1999) and Fig. 6 in (Holman & Murray 1996). The theoretical estimates are made by means of formulas (14, 16, 21, 22). Before they are used, the guiding resonance in the multiplet is identified (it has the maximum value of $|\varphi_{k+q,k+p,k}|$), and its two closest neighbors are considered as the perturbing resonances. Then the formula is chosen in accord with the resonance type (fastly chaotic triad "ft", fastly chaotic duad "fd", slowly chaotic triad "st", or slowly chaotic duad "sd"). If the amplitudes of the neighbors differ from each other less than twice, the model resonance is considered to be a triad, otherwise a duad. Those resonances which have $\varepsilon q > 1$ are marked in Table 1 by an asterisk.

The analytical estimates of the maximum Lyapunov exponent are generally in agreement with the numerical ones. However, some differences can be clearly seen, especially in the domain of slow chaos. This should be attributed to the imperfectness of model (28), and mainly to the fact that the coefficients

$\varphi_{k+q,k+p,k}$ are treated as constants. They fix the frequencies $\omega_0$ of small amplitude oscillations on subresonances, and if the period of perturbation is large in relation to the period of these oscillations, the variations of $\omega_0$ can have greater dynamical effect.

Also another effect can be important. The differential distribution, built by Shevchenko, Kouprianov, & Melnikov (2003) for a representative set of starting values (the trajectories were computed by means of Wisdom's (1983) map in the planar elliptic restricted three-body problem), demonstrates that the maximum Lyapunov exponent of the trajectories near the 3/1 mean-motion resonance with Jupiter has two preferable numerical values: the distribution has a bimodal peak structure. This signifies that there are two distinct domains of chaos in phase space; thus the perturbed pendulum model as applied to this low-order mean motion resonance turns out to be too approximate. This example shows also that generally a closer look at the numerical data may be necessary when comparing it with a theory.

Now consider an example of an estimate for an actual asteroid, namely 522 Helga. This object is famous to be the first example of "stable chaos" among asteroids (Milani & Nobili 1993): while its Lyapunov time is relatively small (6900 yr), its orbit does not exhibit any gross changes on cosmogonic time scales, according to numerical experiments. It is known to be in the 12/7 mean motion resonance.

Let us apply our method. The necessary data on $a$, $e$, and the perihelion frequency $g = \dot{\varpi}$ are taken from the "numb.syn" catalogue of the AstDyS web service. $T_{\rm pert}$ is defined by the value of $g$. One finds that the guiding subresonance in the resonance sextet is the third one ($p = 2$), consequently the perturbing neighbors in our model correspond to $p = 1$ and 3. The quantity $\varepsilon q = 0.624 < 1$, therefore the potential model is valid. The derived separatrix map parameters are: $\lambda = 2.325$, $\eta = 0.812$, consequently the model resonance type is the fastly chaotic triplet "ft". Applying formula (14), one has $T_{\rm L} = 9700$ yr. The agreement with the values, obtained in integrations in the full problem, (6900 yr, according to Milani & Nobili (1993), and 6860 yr, according to the AstDyS data) are quite satisfactory.

### 8.2. Three-body mean motion resonances

Let us consider analytical estimating the maximum Lyapunov exponents of the orbital motion of asteroids in three-body mean motion resonances, following Shevchenko (2007b).

The three-body resonances can be described by the perturbed pendulum model (Murray, Holman, & Potter 1998; Nesvorný & Morbidelli 1998; Nesvorný & Morbidelli 1999). The Hamiltonian of the motion of a zero-mass test particle near a three-body resonance $\{m_{\rm J} m_{\rm S} m\}$ with Jupiter and Saturn in the planar-elliptic problem can be expressed, in some approximation, in the following form (Nesvorný & Morbidelli 1999):

$$H = \alpha S^2 + \sum_{p_{\rm J},p_{\rm S},p} \beta_{p_{\rm J}p_{\rm S}p} \cos \sigma_{p_{\rm J}p_{\rm S}p} \,, \tag{32}$$

where the resonant argument conjugated to $S$ is $\sigma_{p_{\rm J}p_{\rm S}p} = m_{\rm J} l_{\rm J} + m_{\rm S} l_{\rm S} + ml + p_{\rm J}\varpi_{\rm J} + p_{\rm S}\varpi_{\rm S} + p\varpi$. It is assumed that the time derivatives of $l_{\rm J}$, $l_{\rm S}$, $\varpi_{\rm J}$, $\varpi_{\rm S}$ are

constants; $\alpha = -(3/2)n^2 a_{\rm res}^{-2}$. Analytical expressions $\beta_{p_{\rm J}p_{\rm S}p}(e)$ for some important three-body resonances are given in Tables 3–6 in (Nesvorný & Morbidelli 1999). It is clear from Eq. (32) that the three-body resonance $\{m_{\rm J}m_{\rm S}m\}$ splits in a cluster of subresonances with various $\{p_{\rm J}p_{\rm S}p\}$ combinations.

The frequency of small-amplitude oscillations on subresonance $\{p_{\rm J}p_{\rm S}p\}$ is (Nesvorný & Morbidelli 1999):

$$\omega_0 = 2\pi n(3\beta_{p_{\rm J}p_{\rm S}p})^{1/2} a_{\rm res}^{-1}. \tag{33}$$

The perturbation frequency $\Omega$ is generally an algebraic combination of perihelion frequencies of Jupiter, Saturn and the asteroid. The formula for this combination is defined by the choice of the guiding subresonance (see below). The ratio of $\Omega$ and $\omega_0$ gives the value of $\lambda$.

Table 2. Numerical and analytical estimates of Lyapunov times for asteroids in three-body mean motion resonances (Shevchenko 2007b)

| Asteroid | Resonance $\{m_{\rm J}m_{\rm S}m\}$ | $\lambda$ | $T_{\rm L}^{\rm num}$, yr † | $T_{\rm L}^{\rm num}$, yr ‡ | $T_{\rm L}^{\rm theor}$, yr | Res. type |
|---|---|---|---|---|---|---|
| 258 Tyche | $2+2-1$ | 0.536 | 35900 | – | 43100 | ft |
| 485 Genua | $3-1-1$ | 0.376 | 6550 | 6500 | 35700 | sd |
| 1642 Hill | $3-1-1$ | 0.643 | 36100 | – | 43300 | fd |
| 936 Kunigunde | $6+1-3$ | 0.624 | 22200 | – | 54600 | fd |
| 490 Veritas | $5-2-2$ | 0.546 | 10200 | 8500 | 9100 | fd |
| 2039 Paine-Gaposchkin | $5-2-2$ | 0.449 | 22000 | – | 6020 | sd |
| 3460 Ashkova | $5-2-2$ | 0.433 | 65100 | 8300 | 5940 | sd |

† AstDyS
‡ (Nesvorný & Morbidelli 1998; Nesvorný & Morbidelli 1999; Milani, Nobili, & Knežević 1997)

We consider the asteroids residing close to the three-body resonances studied in (Nesvorný & Morbidelli 1999) and utilize the analytical data in Tables 3–6 in (Nesvorný & Morbidelli 1999) on the coefficients of the resonant terms. The theoretical estimates of Lyapunov times are made by means of formulas (14, 16, 21, 22). Before they are used, the guiding resonance in the multiplet is identified (it has the maximum value of $|\beta_{p_{\rm J}p_{\rm S}p}|$), and its two closest neighbors are considered as the perturbing resonances. Then, the formula is chosen in accord with the resonance type ("ft", "fd", "st", "sd").

The identification of the guiding resonances in the multiplets reveals that the three-body resonances under study subdivide in two distinct classes: those for which the perturbation frequency $\Omega$ in model (3) is equal to $\dot{\varpi} - \dot{\varpi}_{\rm J}$, and those for which it is equal to $\dot{\varpi}_{\rm S} - \dot{\varpi}_{\rm J}$. The resonances $5-2-2$ and $3-1-1$ belong to the first class, while $2+2-1$ and $6+1-3$ to the second. We use $\dot{\varpi}_{\rm J} = 4.257''/{\rm yr}$ and $\dot{\varpi}_{\rm S} = 28.243''/{\rm yr}$ (Bretagnon 1990). The data on $\dot{\varpi}$, $a$, and $e$ are taken from the "numb.syn" catalogue of the AstDyS web service.

Theoretical estimates obtained in this way are presented in Table 2. Some of them are in accord with the numerical ones (in particular, in the case of Veritas), others are not. From the fact of disagreement in the cases of Genua and Paine-

Gaposchkin, one can judge that these objects do not, most probably, reside in the chaotic layers of the prescribed resonance multiplets. Thus analytical estimating the Lyapunov times represents a promising tool for discerning between possible models of chaos in the motion of actual asteroids, and, generally, celestial bodies.

## 9. Multiple asteroids

In the beginning of the nineties of the 20th century, B. Chauvineau, P. Farinella, and F. Mignard found out that the dynamics of a small satellite of a rotating irregular-shaped asteroid can be highly chaotic (Chauvineau *et al.* 1993). In 1997, J.-M. Petit and co-workers made a unique (up to present time) estimation of the Lyapunov time of the possible chaotic dynamics of a double asteroid: by means of a numerical experiment they established that the orbital motion of Dactyl, a satellite of the asteroid 243 Ida, can be chaotic with the Lyapunov time ranging from 9 d to 4 yr (Petit *et al.* 1997). In highly chaotic modes of the motion, the satellite either falls on the surface of Ida or escapes very soon.

The population of double asteroids includes, among others, the asteroid 762 Pulcova. Although more than 160 multiple asteroids have been discovered already, and new and new are discovered, one should not expect, apparently, that chaos in such systems can be often observed. The systems for which the type of dynamics is already known are regular, and, moreover, as a rule they possess double synchronization: the periods of rotation of both components about their centres of masses are equal to the orbital period, thus both components are always facing each other by the same sides, i. e. one deals with the systems that have reached the final stage of tidal evolution.

## 10. In brief about planets

And at the highest level of the structural hierarchy of the Solar system there are resonances and chaos. The approximate orbital commensurabilities Jupiter–Saturn (the ratio of mean motions $\approx 5/2$), Saturn-Uranus ($\approx 3/1$), Uranus–Neptune ($\approx 2/1$), and the Neptune–Pluto resonance (3/2) are well known. At the end of the eighties G. Sussman and J. Wisdom and, independently, J. Laskar in complicated numerical experiments obtained the first estimates of the Lyapunov time of the Solar system (Sussman & Wisdom 1988; Laskar 1989). It turned out that it is not at all infinite, i. e. the motion of the Solar system is not regular. Moreover, the Lyapunov time is rather small: it is three orders of magnitude less than the age of the Solar system. According to Sussman and Wisdom's calculations, the Lyapunov time of the outer Solar system (that from Jupiter to Pluto) is equal to $\approx 10$ million years. And for the system of all planets, either with Pluto or without it, $T_{\mathrm{L}} \approx 5$ million years.

At a first glance it might seem that the basic contribution to chaos must be brought by the planets with relatively small masses, i. e. the planets of terrestrial group, as well as Pluto ranked as a planet until recently. However, if the calculations were limited only to four planets-giants, then, as it was found out in 1992 by Sussman and Wisdom and confirmed in 1999 by N. Murray and M. Holman, chaos remains and, moreover, the Lyapunov time practically does not change: $T_{\mathrm{L}} \approx 5$–7 million years (Sussman & Wisdom 1992; Murray & Holman 1999).

Murray & Holman (1999) found that chaos could be due to a multiplet of subresonances associated to a particular three-body resonance Jupiter–Saturn–Uranus. This conclusion, however, has a preliminary character, because no full agreement of the analytical model with the numerical experiments has been achieved up to now. If it is true, the degree of chaoticity of the Solar system has, in some sense, an arbitrary character: if the semimajor axis of the orbit of Uranus differed from the present value by only several diameters of Uranus, the chaoticity would sharply decrease, if not at all practically disappear. Recent numerical-experimental studies by W. Hayes and co-workers (Hayes 2007, 2008; Hayes *et al.* 2010) show that the spatial interplay of chaos and order may be present even on much finer scales. In the future researches, if the guiding three-body resonance is accurately identified, then analytical estimating the Lyapunov time $T_{\mathrm{L}}$ will be possible to accomplish using the separatrix map theory.

## In conclusion

The idea of full predefiniteness of the motion of both large and small bodies of the Solar system was fundamental in celestial mechanics until the eighties of the 20th century, but now we know that the time of predictable motion of any body in the Solar system is limited, by the order of magnitude, by the value of its Lyapunov time. For some bodies this time is large and the motion is practically regular; for others, on the contrary, dynamical chaos manifests itself on small timescales and, therefore, it can be observed directly.

Direct astronomical observations of chaos in case of its presence are possible in investigations of the rotational dynamics of minor planetary satellites, where the Lyapunov times are small enough for the chaotic character of dynamics to manifest itself on the intervals of time accessible to observations. The presence of chaos can be as well identified in the dynamics of comets, if, as in case of the Halley comet, historical chronicles about the dates of comet's returnings are available. As a rule, highly chaotic is the motion of the planet-crossing asteroids and comets. Judging by the Lyapunov times, chaos (if it is present) is also observable in the motion of some satellite systems and in the inner orbital dynamics of multiple asteroids.

In the dynamics of many other bodies of the Solar system, chaos is obscure, due to either too large Lyapunov times at nowadays or to remoteness of the time epochs when the motion was strongly chaotic, but it is important to remember that in the long-term evolution its role is great and it imprints and sculptures the present appearance and structure of the Solar system, e. g. in the form of Kirkwood gaps in the main belt of asteroids.

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