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Theoretical Description and Basic Physics of Stellar Pulsations

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Abstract.

As an introduction to the subject basic properties of stellar pulsations are derived using simple intuitive estimates. With respect to a theoretical description of pulsating stars the physical principles governing stellar structure and dynamics are discussed. The associated equations are simplified by the assumption of spherical symmetry thus providing the basis for the study of radial pulsations.

Key words: asteroseismology — hydrodynamics — radiative transfer — stars: oscillations

1. Preliminary Considerations

Variability of stars, observed either by photometric or spectral methods, can originate from various effects. It could be caused by eclipses in binaries, by disks in cataclysmic variables, by nuclear explosions in Novae and Supernovae, by star spots associated with magnetic fields, or by oscillations of the star around its equilibrium, i.e., by stellar pulsations, which are the subject of the current series of lectures. In order to distinguish them from other sources of variability we define them in a first attempt as an intrinsic property of a single, isolated star exhibiting (possibly multiple) periodic variability of its effective temperature, radius and luminosity.

Pulsating stars are of fundamental importance for astrophysics, since the properties of the pulsations allow for reliable estimates of stellar parameters, and certain classes of pulsating stars (e.g., Cepheids) can be used for distance determinations. In asteroseismology direct information on stellar structure and interiors is obtained from the spectrum of observed oscillation frequencies. Historically, the hypothesis, that stellar pulsations or oscillations may be responsible for observed stellar variability was first raised by Shapley in 1914 and considered theoretically by Eddington in 1918 (see Cox, 1980). For further reading we recommend the article by Ledoux & Walraven (1958) and the textbooks by Cox (1980) and Unno et al. (1989).

In order to identify the stellar parameters governing the observed timescale of pulsation-induced variability we consider the various timescales occurring in stellar physics.

The mechanical or dynamical timescale is determined by the acceleration of a mass element under the action of gravity. Denoting the radial position of a mass element with r, the time with t, the mass of the star with M, and the gravitational constant with G, we estimate the acceleration as

$$\frac{d^2r}{dt^2} \propto -\frac{GM}{r^2} \tag{1}$$

Using the stellar radius R as an estimate for r and the dynamical timescale τ_{Dyn} as an estimate for t we are left with

$$\frac{R}{\tau_{Dyn}^2} \propto \frac{GM}{R^2} \tag{2}$$

Solving for τ_{Dyn} we obtain

$$\tau_{Dyn} \propto (G\rho)^{-1/2} \tag{3}$$

where ρ denotes the mean density of the star. Thus the dynamical timescale of a star is entirely determined by its mean density and varies between milliseconds for compact neutron stars and some 100 days for giants.

The thermal (Kelvin - Helmholtz) timescale τ_{KH} of a star may be defined by the time needed to radiate its thermal energy content $E_{thermal}$ at its current luminosity L:

$$\tau_{KH} \propto \frac{E_{thermal}}{L} \tag{4}$$

Due to the virial theorem the thermal and gravitational potential energy E_{Grav} of a star are of the same order of magnitude ($E_{thermal} \propto E_{Grav} \propto \frac{GM^2}{R}$) and we obtain

$$\tau_{KH} \propto \frac{GM^2}{LR} \propto 10^7 years \frac{(M/M_\odot)^2}{(L/L_\odot)(R/R_\odot)} \tag{5}$$

Similar to the thermal timescale the nuclear timescale τ_{nuc} of a star may be defined by the time needed to radiate its nuclear energy content E_{nuc} at its current luminosity L. Since the nuclear energy content of a star is proportional to its mass we are left with

$$\tau_{nuc} \propto \frac{E_{nuc}}{L} \propto \frac{M}{L} \propto 10^{10} years \frac{M/M_\odot}{L/L_\odot} \tag{6}$$

Comparing the nuclear, thermal and mechanical timescales of a star with the observed timescale of stellar pulsations of at most a few hundred days we conclude that the mechanical timescale is relevant for stellar pulsations. Moreover, the physics governing pulsations should be dominated by the mechanics of the system. The pulsation - induced variability of stellar parameters is usually small compared to their mean time independent values. Thus pulsations may be regarded as oscillations around the mechanical (hydrostatic) equilibrium, where the perturbed equilibrium is readjusted on the dynamical timescale.

Oscillations require a restoring force. In a star, two types of restoring forces are available: Stellar matter is compressible and the perturbation of the density of a mass element will be associated with a perturbation of its pressure implying forces which counteract the density perturbation and tend to restore the unperturbed configuration. In a continuous medium this restoring force gives rise to the existence of sound waves. Since pressure is the restoring force, standing sound waves in a star are denoted as p - modes. Buoyancy is the origin of a second restoring force. For its action it requires a non vanishing acceleration (the gravity g in a star) and a finite density gradient $\frac{d\rho}{dr}$. An aspherical displacement of a mass element will then induce a restoring buoyancy force proportional to the gravity and the density gradient ($\propto g \cdot \frac{d\rho}{dr}$). In a continuous medium it leads to the existence of gravity waves. Since gravity is an essential ingredient in buoyancy, standing gravity waves in a star are denoted as g - modes.

With respect to the geometry we distinguish radial from nonradial pulsations. For radial pulsations the perturbations preserve the spherical geometry of the hydrostatic star, whereas nonradial pulsations allow for non - spherically deformed perturbations. Since buoyancy cannot act in spherical geometry, radial g - modes do not exist and radial pulsations do consist of p - modes only. For the same reason pure gravity modes - should they exist - have to be nonradial. Nonradial pulsations contain both g - and p - modes, where the strict classification of a given mode as g - or p - mode is not always meaningful, since there are modes with a mixed character, where both restoring forces act simultaneously.

On the basis of the hypothesis that stellar pulsations may be regarded as standing waves in a star we would like to provide a simple intuitive estimate of their pulsation periods, restricting ourselves to considering radial acoustic p - modes. As a guidance the analogue of an organ pipe as an acoustic resonator turns out to be helpful. The acoustic frequency spectrum of an organ pipe is obtained by considering the wavelengths λ of standing waves which a pipe with length L and rigid boundaries at the top and at the bottom (corresponding to nodes of standing acoustic waves) allows for. If $n-1$ denotes the number of nodes within the pipe of the standing sound wave, $\lambda/2$ can take the infinite number of discrete values L/n. Assuming now that a star can be regarded as an acoustic resonator similar to an organ pipe with nodes of standing waves at the center ($r=0$) and the surface ($r=R$) we identify the length L of the organ pipe with the stellar radius R and obtain from $\lambda \propto L/n$ by analogy as an order of magnitude estimate for the wavelengths of standing sound waves in a star $\lambda \propto R/n$. Wavelengths and associated frequencies ν are in both cases related by

$$\nu\lambda = c_{Sound} \tag{7}$$

where the sound speed c_{Sound} is given by

$$c^2{}_{Sound} = \gamma p/\rho \propto p/\rho \tag{8}$$

p, ρ and γ denote pressure, density and the adiabatic exponent, respectively. Thus the spectrum of acoustic frequencies of an organ pipe is estimated as

$$\nu = c_{Sound}/\lambda \propto n\frac{\sqrt{p/\rho}}{L} \tag{9}$$

For the (radial) acoustic spectrum of a star we obtain the estimate

$$\nu = c_{Sound}/\lambda \propto n\frac{\sqrt{p/\rho}}{R} \tag{10}$$

For a star, the ratio p/ρ can be estimated from the condition of hydrostatic equilibrium:

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{GM_r}{r^2} \tag{11}$$

where M_r denotes the mass within a sphere of radius r. Using $\frac{p}{R}$ as an estimate for $\frac{\partial p}{\partial r}$, M as an estimate for M_r and R as an estimate for r we obtain

$$\frac{1}{\rho}\frac{p}{R} \propto \frac{GM}{R^2} \tag{12}$$

Thus the ratio p/ρ is given by

$$\frac{p}{\rho} \propto \frac{GM}{R} \tag{13}$$

and the radial acoustic spectrum of a star (see equation 10) is estimated as

$$\nu \propto n\sqrt{\frac{GM}{R^3}} \propto n\sqrt{G\rho} \tag{14}$$

Replacing the frequency by the pulsation period $\Pi = 1/\nu$ we obtain for the radial fundamental mode ($n = 1$):

$$\Pi\sqrt{\rho} = constant \tag{15}$$

Equation 15 represents the period - density - relation for the radial fundamental mode of stellar pulsations. Note that according to our estimates the density occurring in equation 15 has to be regarded as the mean density of the star. A familiar form of the period - density - relation (see, e.g., Cox, 1980) reads:

$$\Pi(\rho/\rho_{\odot})^{1/2} = Q \quad with \quad 0.03d \lesssim Q \lesssim 0.12d \tag{16}$$

The variation of Q is caused by the influence on the pulsation period of different stellar structures, which was not accounted for by our simple estimates. Note that the period - density relation is consistent with our initial findings that the timescale of pulsations is given by the dynamical timescale (equation 3).

2. Physics of Stellar Structure and Dynamics

For continuous systems like stars two kinds of descriptions are common. In the Eulerian framework fixed positions in space are considered, position vectors $\vec{r}$ and time t are used as independent variables. Accordingly, the Eulerian time derivative $\frac{\partial}{\partial t}|_{\vec{r}}$ is defined at constant position vector $\vec{r}$. In the Lagrangean framework fixed mass elements are considered, the initial position vector $\vec{r_0}$ of a mass element and the time t are used as independent variables. Accordingly, the Lagrangean time derivative $\frac{d}{dt}|_{\vec{r_0}}$ is defined at constant initial position vector $\vec{r_0}$ of the mass element considered. Note that in the Lagrangean description the actual position vector $\vec{r} = \vec{r}(\vec{r_0}, t)$ is a (time dependent) dependent variable. For the definition of the velocity $\vec{v}$ the Lagrangean description is adopted:

$$\vec{v} = \frac{d\vec{r}}{dt} \tag{17}$$

Using the relation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}\nabla) \tag{18}$$

between the Lagrangean and the Eulerian time derivatives the acceleration $\frac{d\vec{v}}{dt}$ can be written as

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} \tag{19}$$

Depending on which of the equivalent descriptions is more convenient for the particular situation studied, either the Eulerian or the Lagrangean approach (or even a combination of them) is used.

The physical principles governing stellar structure and dynamics comprise the conservation laws for mass, momentum and energy together with Poisson's equation for the gravity and a prescription for the energy transport. In its differential form mass conservation is described by the continuity equation

$$\frac{d\rho}{dt} + \rho\nabla\vec{v} = 0 \tag{20}$$

Alternatively, the continuity equation in the Eulerian approach may be written as

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{v}) = 0 \tag{21}$$

By definition, an incompressible motion is characterized by a vanishing Lagrangean time derivative of the density ($\frac{d\rho}{dt} = 0$). According to equation 20 this condition is equivalent to $\nabla\vec{v} = 0$, i.e., to a vanishing divergence of the velocity field. Incompressibility and homogeneity, which would correspond to a vanishing gradient of the density ($\nabla\rho = 0$), must not be confused.

In the absence of viscosity and magnetic fields, momentum conservation is described by Euler's equation:

$$\rho\frac{d\vec{v}}{dt} = \rho(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v}) = -\nabla p - \rho\nabla\phi \tag{22}$$

The left hand side of equation 22 describes the inertial forces in either the Lagrangean or the Eulerian framework, the first term on the right hand side corresponds to forces induced by pressure gradients, the second refers to the gravitational force where ϕ is the gravitational potential. It is determined by Poisson's equation:

$$\triangle\,\phi = 4\pi G\rho \tag{23}$$

The solution of Poisson's equation 23 may be represented as:

$$\phi(\vec{r},t) = -G \int \frac{\rho(\vec{x},t)}{|\vec{x}-\vec{r}|} d^3x \tag{24}$$

Based on the first law of thermodynamics energy conservation may be expressed as

$$\rho \frac{du}{dt} = -p\nabla\vec{v} + \rho\varepsilon - \nabla\vec{F} \tag{25}$$

where u, $\vec{F}$ and ε denote the specific internal energy, the heat flux and the specific energy generation rate, respectively. The variation with time of the internal energy of a mass element is given by the mechanical work done by the element (first term on the r.h.s. of equation 25), the (nuclear) energy generation within the element (second term) and the heat deposited in it, expressed in terms of the divergence of the heat flux (third term). With $V = 1/\rho$, the continuity equation 20 and some basic thermodynamics two terms of equation 25 may be rearranged to yield:

$$\rho \frac{du}{dt} + p\nabla\vec{v} = \rho(\frac{du}{dt} + p\frac{dV}{dt}) = \rho T \frac{ds}{dt} \tag{26}$$

where T and s denote temperature and specific entropy, respectively. Thus an alternative form of energy conservation (equation 25) is given by

$$\rho T \frac{ds}{dt} = \rho\varepsilon - \nabla\vec{F} \tag{27}$$

In stellar interiors energy transport is usually approximated by a diffusion type process, where the heat flux is proportional and opposite to the temperature gradient:

$$\vec{F} = -D\nabla T \tag{28}$$

The particular transport process enters through the diffusion coefficient D. In the optically thick regime (e.g., in stellar interiors) radiation transport can be treated in the diffusion approximation with D given by:

$$D = \frac{4ac}{3\kappa\rho} T^3 \tag{29}$$

where a, c and κ are the radiation constant, the speed of light and the Rosseland mean of the opacity, respectively.

If nuclear processes are of interest, the system of equations has to be complemented by the variation with time of the chemical composition (X_i denotes the mass fraction of nucleus i) induced by nuclear reactions:

$$\frac{dX_i}{dt} = \frac{dX_i}{dt}(X_j, p, T) \tag{30}$$

The specific dependence on chemical composition, pressure and temperature of the reaction rate entering equation 30 is provided by nuclear physics.

A closure of the system of equations given above is accomplished by the prescription of a thermal and a caloric equation of state (EOS) provided by thermodynamics and atomic physics. Depending on the thermodynamic basis adopted it may formally be written as, e.g.,

$$p = p(\rho, T) \quad or \quad \rho = \rho(p, T) \quad or \quad s = s(p, T) \tag{31}$$

Moreover, the Rosseland mean of the opacity $\kappa = \kappa(p, T)$ and the nuclear energy generation rate $\varepsilon = \varepsilon(p, T)$ have to be provided by atomic and nuclear physics either in parametrized or in tabular form.

The problem posed by the system of equations introduced here consists of their application to stellar structure and dynamics and their mathematical solution. Concerning the latter, a numerical treatment of the equations with subsequent numerical simulations seems to be an appropriate strategy. However, concerning stellar pulsations reliable nonlinear 3D simulations satisfying the necessary accuracy requirements are still not yet feasible. Therefore the theoretical study of stellar pulsations still relies on simplifications and approximations.

3. Radial Pulsations

As an attempt to reduce the mathematical complexity of the problem we simplify its geometry by assuming spherical symmetry, i.e., we restrict our studies to radial pulsations. However, according to the preliminary considerations in section 1 this assumption does not only simplify the mathematics, it also leads to a loss of physical effects, such as buoyancy. As a consequence, e.g., g - modes are excluded in this approach. Therefore the interpretation and generalisation of results based on a radial analysis has to be dealt with caution. For convenience, we introduce spherical polar coordinates (r, θ, φ) and adopt the Lagrangean description. Then the mass M_r contained within a sphere of radius r is given by (subscripts 0 refer to initial quantities in the Lagrangean sense):

$$M_r = \int_0^{r(r_0,t)} \rho(r', t) 4\pi r'^2 dr' = \int_0^{r_0} \rho(r_0{}') 4\pi r_0{}'^2 dr_0{}' = M_{r_0} \tag{32}$$

With this definition the conservation of mass is expressed as

$$M_r = M_{r_0} \quad ; \quad \frac{dM_r}{dt} = 0 \tag{33}$$

and $M_r = M_{r_0}$ is chosen as a new Lagrangean variable replacing r_0. Thus M_r (and t) have become independent Lagrangean variables, whereas $r(M_r, t)$ is a dependent variable. The relation between r and M_r is obtained by differentiation of the definition of M_r (equation 32):

$$\frac{\partial M_r}{\partial r} = 4\pi r^2 \rho \quad or \quad \frac{\partial r}{\partial M_r} = \frac{1}{4\pi r^2 \rho} \tag{34}$$

Note that in equation 34 the derivatives have to be interpreted in the Lagrangean sense. In spherical symmetry the gravitational force occurs in Euler's equation 22 in terms of the gradient $\frac{\partial \phi}{\partial r}$ of the potential ϕ. It is determined by Poisson's equation 23 which in spherical symmetry is given by:

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\phi}{\partial r}) = 4\pi G\rho \tag{35}$$

By multiplication of equation 35 with r^2 and integration we obtain:

$$r^2\frac{\partial\phi}{\partial r} = \int 4\pi G\rho r^2 dr = GM_r \tag{36}$$

For the gravity we are thus left with

$$\frac{\partial\phi}{\partial r} = \frac{GM_r}{r^2} \tag{37}$$

Note that the particular choice of the Lagrangean variables allows for an algebraic representation of the gravitational force. No further integration is required.

To present the equations governing radial pulsations in their conventional form, some transformations and definitions have to be introduced: The radial component F_r of the heat flux is replaced by the luminosity $L(r)$ through

$$L(r) = 4\pi r^2 F_r \tag{38}$$

Choosing pressure p and temperature T as the thermodynamic basis we write the differential of the specific entropy $s = s(p, T)$ as

$$Tds = c_p dT - \frac{\delta}{\rho}dp \tag{39}$$

where c_p denotes the specific heat at constant pressure and the coefficients α and δ of the differential form of the equation of state $\rho = \rho(p, T)$ are defined as

$$\alpha = \left.\frac{\partial \log\rho}{\partial \log p}\right|_T \quad ; \quad \delta = -\left.\frac{\partial \log\rho}{\partial \log T}\right|_p \tag{40}$$

The transformation from r to M_r as an independent variable is accomplished by using equation 34 in the form

$$\frac{\partial}{\partial r} = 4\pi r^2\rho\frac{\partial}{\partial M_r} \tag{41}$$

We are thus left with the following system of equations describing the spherically symmetric structure and dynamics of a star ($\frac{\partial}{\partial t}$ refers to the Lagrangean time derivative):

Mass conservation

$$\frac{\partial r}{\partial M_r} = \frac{1}{4\pi r^2\rho} \tag{42}$$

Momentum conservation

$$\frac{\partial p}{\partial M_r} = -\frac{GM_r}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} \tag{43}$$

Energy conservation

$$\frac{\partial L}{\partial M_r} = \varepsilon - c_p\frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial p}{\partial t} \tag{44}$$

Energy transport

$$\frac{\partial T}{\partial M_r} = -\frac{3\kappa L}{64\pi^2 acr^4 T^3} \tag{45}$$

Change of chemical composition by nuclear processes

$$\frac{\partial X_i}{\partial t} = \frac{\partial X_i}{\partial t}(X_j, p, T) \tag{46}$$

This system of five partial differential equations needs to be closed by the prescription of an equation of state and by specifying the nuclear energy generation rate ε and the opacity κ. We note that energy transport processes other than radiation diffusion are not taken into account in equation 45. In particular, energy transport by convection is disregarded.

Three terms involving a time derivative occur in equations 42 — 46, each of them being related to one of the characteristic stellar timescales discussed in section 1: The acceleration term in equation 43 is associated with the dynamical timescale, the time derivative of the entropy in equation 44 (expressed by the time derivatives of temperature and pressure, respectively) with the thermal timescale and the time derivatives of the mass fractions in equation 46 with the nuclear timescale.

Stellar evolution relies on hydrostatic equilibrium ($\frac{\partial^2 r}{\partial t^2} = 0$) and is governed by the nuclear and the thermal timescales. Thus the description of standard stellar evolution is included in equations 42 — 46 as the special case of vanishing acceleration.

On the other hand, the study of pulsations requires deviations from hydrostatic equilibrium ($\frac{\partial^2 r}{\partial t^2} \neq 0$), whereas on the timescale of pulsations the nuclear changes of the chemical composition may be ignored. Thus nuclear processes, i.e., equations 46 are usually ignored and the chemical composition in terms of the mass fractions X_i is assumed to be constant on the dynamical timescale of pulsations. Thus pulsations are governed by the dynamical and thermal timescales. Under certain conditions for pulsations even the change of the entropy (i.e., its time derivative in equation 44) may be neglected. Then the energy equations can be disregarded altogether and we are left with a mechanical system, where pulsations are completely determined by the dynamical timescale.

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References

Cox J. P., 1980, *Theory of stellar pulsation*

Ledoux P., Walraven T., 1958, *Handbuch der Physik*, **51**, 353

Unno W., Osaki Y., Ando H., Saio H., Shibahashi H., 1989, *Nonradial oscillations of stars*