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EFFICIENCY OF TESTS BASED ON WEIGHTED RANKINGS FOR RANDOMIZED BLOCKS WHEN THE ALTERNATIVES HAVE AN A PRIORI ORDERING*

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ABSTRACT

Salama and Quade (1981) proposed a family of nonparametric tests based on a method of weighted within-block rankings, for testing the hypothesis of no treatment effects against a postulated ranking of the treatments in a complete randomized blocks layout.

These tests and others are compared with respect to asymptotic efficiency.

1. INTRODUCTION

Let Z_{ij}, $1\leq i\leq m$, $1\leq j\leq n$, be observations corresponding to a complete randomized block layout with m treatments and n blocks. We assume that the Z_{ij}'s are independent with distribution $G(z-\theta_i-\mu_j)$, where G is continuous.

We are interested testing the null hypothesis of no

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treatment effects

$$H: \theta_1 = \qquad = \theta_m$$

against

$$K: \theta_1 \le \theta_2 \le \qquad \le \theta_m$$

where at least one of the inequalities is strict.

Classical normal theory statistics for this problem have been considered by Bartholomew (1961), Nüesch (1966), Hogg (1965) and others. These statistics are based on $\{Z_{i.} - Z_{j.} \quad i<j\}$,where $Z_{i.} = \sum_{j=1}^{n} Z_{ij}/n$, $i = 1, \ldots, m$. For instance, the statistic

$$T = \sum_{i<j} (Z_{i.} - Z_{j.})/\hat{\sigma}_X,$$

where $\hat{\sigma}_X$ is the appropriate estimate of the standard deviation of $\sum_{i<j} (Z_{i.} - Z_{j.})$, has been considered by Hollander (1967) for $Z_{ij} = b_j + (i-1)\theta + \epsilon_{ij}$, where the ϵ_{ij} are independent and identically distributed according to $N(0, \sigma^2)$.

A nonparametric test for the null hypothesis H was proposed by Jonckheere (1954). Jonckheere's procedure is to reject H for large values of

$$\sum_{j=1}^{n} \tau_j$$

where τ_j is Kendall's rank correlation coefficient between postulated order and observation order in the j-th block. A similar test was given by Page (1963). This test is based on

$$\rho = \sum_{j=1}^{n} \rho_j$$

where ρ_j is Spearman's rank correlation coefficient between postulated order and observation order in the j-th block.

The Page's statistic and the Jonckheere's statistic are special cases of the average rank correlation

$$W^* = \sum_{j=1}^{n} C_j/n,$$

where C_j, $j = 1,$ n are the values of some index of rank correlation between $(Z_{1j}, \ldots, Z_{mj})$ and the predicted ranking $(1, \ldots, m)$.

For the case $m = 2$, a nonparametric test for the null hypothesis H is the sign test. This test has an asymptotic relative efficiency (A.R.E.) of $2/\pi \cong 0.637$ with respect to the classical t-test, under normality. The loss efficiency of the sign test can be attributed to the absence of between-block comparisons.But for this special case there is an efficient nonparametric test, which utilizes the information on interblock comparisons: the Wilcoxon signed test (1945). The A.R.E. of this test with respect to the t -test is $3/\pi \cong 0.955$, under normality.

Another procedure for testing the hypothesis H was proposed by Hollander (1967), which generalizes the signed-rank test. Let $R_{uv}^{(j)}$ be the rank of $|Z_{vj}-Z_{uj}|$ among the n absolute differences within blocks between treatments u and v and let

$$T_{uv} = \sum_{j=1}^{n} R_{uv}^{(j)} \psi_{uv}^{(j)}$$

where

$$\psi_{uv}^{(j)} = \begin{cases} 1 & \text{if } Z_{uj} < Z_{vj} \\ 0 & \text{otherwise} \end{cases} \qquad (1.1)$$

Hollander's test is based on

$$Y = \sum_{u<v}$$

Under Y is neither distribution-free for finite n, nor asymptotically distribution-free. However, it can be made asymptotically distribution-free by dividing it by a consistent estimate of the null standard deviation of Y.

Doksum (1967) has proposed a test that is very similar to the Y test. Doksum uses the random variables

$$U_{uv} = T_{uv} - \sum_{j=1}^{n} \psi_{uv}^{(j)}$$

and considers the statistic

$$U = \left\{ \frac{18n(n-1)}{(m^2-1)\{2n-1+(m-2)[24(n-2)\hat{\lambda}(G)+13-6n]\}} \right\}^{1/2}$$

$$\times \sum_{u<v} \left\{ \sum_{k=1}^{m} U_{uk}/m - \sum_{k=1}^{m} U_{vk}/m \right\}$$

where $\hat{\lambda}(G)$ is a consistent estimate of

$$\lambda(G) = P(X_1 < X_2 + X_3 - X_4 \, , \, X_1 < X_5 + X_6 - X_7) \quad (1.2)$$

for independent observations $X_1, \ldots, X_7$ from G. This test is only asymptotically distribution free, under H.

A method of weighted within-block rankings, which generalizes the standard method based on unweighted rankings, was proposed by Quade (1972, 1979).

Let us associate to each block a measure of variability $D_j = \psi(Z_{1j}, \ldots, Z_{mj})$, j 1,..., n, such that is symmetric in its m arguments and satisfies

$$\psi(z_1+c, \ldots, z_m+c) = \psi(z_1, \quad z_m)$$

for all c.

For simplicity we assume that with probability 1 there will be no ties among the D_j's.

Let Q_j be the rank of D_j and b_{Q_j} the relative weight given to the j-th block, for $j = 1, \ldots, n$, where $b_1, \ldots, b_n$ are constants such that $0 \le b_1 \le \ldots \le b_n$.

Then the weighted average rank correlation becomes

$$W = \sum_{j=1}^{n} b_{Q_j} C_j / \sum_{j=1}^{n} b_j$$

where C_j is any measure of rank correlation. The signed-rank statistic for $m = 2$ is a special case, in which $C = \psi_{12}^{(j)}$ and $b_j = j$.

Monte Carlo work by Silva (1977) suggests that the choice of measure of variability D_j may not be crucial.

The test statistic W is distribution-free, under H. Moreover if the block weights $b_1, \ldots, b_n$ are so chosen that

$$\max_{1 \le j \le n} b_j^2 / \sum_{j=1}^{n} b_j^2 \to 0 \qquad (1.3)$$

as $n \to \infty$, and if $0 < V(C_j) < \infty$, then under H, W is asymptotically normally distributed (see Salama and Quade (1981)).

Tests using Spearman and Kendall correlation are studied by Salama and Quade (1981) and their distributions are obtained for both small and large experiments. These tests and others competitors are compared in small experiments with respect to expected significance level.

In Section 2 we obtain a general expression for the A.R.E.

of the W-test with respect to the T-test, for different weights b_j, $C_j = \tau_j$ and $D_j = R_j$, where

$$R_j = \max_{1\le i\le m} Z_{ij} - \min_{1\le i\le m} Z_{ij}.$$

In section 3 we deduce a computable expression for this efficiency in the case $m = 3$ and an underlying normal or Cauchy distribution. In Section 4 we present similar results for the A.R.E. in the case $m = 4$ and a normal underlying distribution.

2. ASYMPTOTIC RELATIVE EFFICIENCY OF THE WEIGHTED AVERAGE RANK CORRELATION

We shall restrict attention to Kendall's rank correlation.

Let W_τ denote the weighted average Kendall's rank correlation

$$W_\tau = \sum_{j=1}^{n} b_{Q_j} \tau_j / \sum_{j=1}^{n} b_j$$

where τ_j is the Kendall rank correlation

$$\tau_j = \frac{4}{m(m-1)} \sum_{u<v} \psi_{uv}^{(j)}$$

and $\psi_{uv}^{(j)}$ is given by (1.1).

Suppose that the sequence of weights b_j for $j = 1$, satisfies the condition (1.3) as $n \to \infty$.

Let us consider the sequence of alternative hypotheses

$$K_n\colon G_{ij}(z) = G_i(z-\mu_j)$$
$$= G(z-(i-1)\theta-\mu_j)$$

where $\theta = cn^{-1/2}$, $i = 1 \quad \ldots, m$, $j = 1, \ldots, n$ and $c>0$.

The following theorems are consequences of the special central limit theorem of Hájek and Šidák (1967).

Theorem 2.1. If $0 < \int G_i \, dG_k < 1$ at least (i,k) pair, then W_τ, suitably standardized, has an asymptotic normal distribution, under K_n.

Assume now that we have two tests based on the statistics T_{1n} and T_{2n} (from the group of T, τ, ρ, Y, U, W_τ and W_ρ). From Noether (1955) it follows that the Pitman asymptotic relative efficiency of T_{1n} with respect to T_{2n} (hereafter denoted by $e_{T_1,T_2}(G)$) is given by

$$\lim_{n\to\infty} \{[dE_\theta(T_{1n})/d\theta]_{\theta=0}[dE_\theta(T_{2n})/d\theta]^{-1}_{\theta=0}\}^2 \sigma_0^2(T_{2n})/\sigma_0^2(T_{1n})$$

where E_θ indicates that the mean is taken under K_n and σ_0^2 is the null-hypothesis variance.

Theorem 2.2 gives an expression for $[dE_\theta(W_\tau)/d\theta]_{\theta=0}$ for different weights b_j, j = 1, n. Before stating this theorem, we consider the following lemma.

Lemma 2.1. Let M_{jk}, j = 1,..., n, k = 1, ..., n be the indicator function of the event $D_j > D_k$, and let $\psi_{uv}^{(j)}$ 1, v = 1, ..., m be given by (1.1). Then

$$\lim_{n\to\infty} [dE_\theta(\psi_{uv}^{(j)}(\sum_{k=1}^{n} M_{jk})^r)/d\theta]_{\theta=0} = [dE_\theta(\psi_{uv}^{(j)} \prod_{i=2}^{r+1} M_{ji})/d\theta]_{\theta=0},$$

r = 1,2,....

We omit the proof of this lemma, since it is very similar to that of Lemma 2.1 in Ferretti and Yohai (1986).

Put now

$$d_{uv,r}(G) = [dE_\theta(\psi_{uv}^{(1)} \prod_{k=2}^{r+1} M_{1k})/d\theta]_{\theta=0}, \tag{2.1}$$

u = 1,..., m, v = 1, ..., m, r = 1, 2,

Then (2.2) holds.

From Theorem 2.2 we obtain $e_{W_\tau,T}(G)$, $e_{W_\tau,\tau}(G)$, $e_{W_\tau,\rho}(G)$, $e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$.

<u>Corollary 2.2.1.</u> Let $\sigma^2(G)$ be the variance of G. Under the K_n alternatives and if $b_j = f(j/n)$, where f satisfies the conditions of Theorem 2.2, the A.R.E. of the W_τ-test with respect to the T-test is given by

$$e_{W_\tau,T}(G) = \frac{864\,[\sum_{u<v}\sum_{i=0}^{\infty} c_i\, d_{uv,i}(G)]^2\,\sigma^2(G)}{m^2(m-1)^2(m+1)(2m+5)\int_0^1 f^2(x)\,dx} \tag{2.3}$$

<u>Proof.</u> Under H Kendall's rank correlation coefficient has variance $2(2m+5)/9m(m-1)$, so

$$\sigma_0^2(W_\tau) = \frac{2(2m+5)\sum_{j=1}^{n}(f(j/n))^2/n}{9nm(m-1)[\sum_{j=1}^{n} f(j/n)/n]^2} \tag{2.4}$$

Also

$$[dE_\theta(T)/\,d\theta]_{\theta=0} \sim [12\sigma^2(G)/nm(m-1)(m+1)]^{-1/2}$$

and

$$\sigma_0^2(T) \sim 1$$

From (2.2) the expression for $e_{W_\tau,T}(G)$ follows.

<u>Corollary 2.2.2.</u> Under the K_n alternatives and if $b_j = f(j/n)$ where f satisfies the condition of Theorem 2.2,

$$e_{W_\tau,\tau}(G) = \frac{36\,[\sum_{u<v}\sum_{i=0}^{\infty} c_i\, d_{uv,i}(G)]^2}{m^2(m-1)^2(m+1)^2\,[\int_{-\infty}^{\infty} g^2(x)\,dx]^2\int_0^1 f^2(x)\,dx} \tag{2.5}$$

<u>Proof.</u> Hollander (1967) shows that

$$[dE_\theta(\tau)/d\theta]_{\theta=0} = 2n(m+1)\int_{-\infty}^{\infty} g^2(x)\,dx/3$$

and

$$\sigma_0^2(\tau) = \frac{2n(2m+5)}{9m(m-1)}$$

Then (2.5) follows from (2.2) and (2.4).

Corollary 2.2.3. Under the K_n alternatives and if $b_j = f(j/n)$, where f satisfies the conditions of Theorem 2.2,

$$e_{W_\tau,\rho}(G) = \frac{72[\sum_{u<v}\sum_{i=0}^{\infty} c_i\, d_{uv,i}(G)]^2}{m^3(m-1)^2(2m+5)\int_0^1 f^2(x)\,dx\,[\int_{-\infty}^{\infty} g^2(x)\,dx]^2} \quad (2.6)$$

Proof. Hollander (1967) shows that

$$[dE_\theta(\rho)/d\theta]_{\theta=0} = nm\int_{-\infty}^{\infty} g^2(x)\,dx$$

and

$$\sigma_0^2(\rho) = m/(m-1).$$

Then (2.6) follows from (2.2) and (2.4).

Corollary 2.2.4. Under the K_n alternatives and if $b_j = f(j/n)$, where f satisfies the condition of Theorem 2.2,

$$e_{W_\tau,Y}(G) = \frac{36(3+2(m-2)(12\lambda(G)-3))[\sum_{u<v}\sum_{i=0}^{\infty} c_i\, d_{uv,i}(G)]^2}{m^2(m-1)^2(m+1)^2[\int_{-\infty}^{\infty} g_1^2(x)\,dx]^2\int_0^1 f^2(x)\,dx} \quad (2.7)$$

where $\lambda(G)$ is given by (1.2), and G_1 is the distribution function of $X_1 - X_2$, with corresponding density g_1, when X_1, X_2 are independent and identically distributed according to G.

Proof. Hollander (1967) has shown that

$$\lim_{n \to \infty} [dE_\theta(Y)/d\theta]_{\theta=0}/n(n-1) = m(m-1)(m+1)\int_{-\infty}^{\infty} g_1^2(x)\, dx\ /6$$

and

$$\sigma_0^2(Y) = n(n+1)(2n+1)m(m-1)(3 + 2(m-2)\rho_0^n(G))/144,$$

$$\rho_0^n(G) = \frac{(24\lambda(G)-6)n^2 + (48\mu(G)-72\lambda(G)+7)n + 48\lambda(G)-48\mu(G)+1}{(n+1)(2n+1)},$$

and

$$\mu(G) = P(X_1 < X_2 \quad X_1 < X_5 + X_6 - X_7)$$

when $X_1, X_2, \ldots, X_7$ are independent and identically distributed according to G.

Also, Hollander (1966) has proved that

$$\rho^*(G) = \lim_{n \to \infty} \rho_0^n(G) = 12\lambda(G) - 3.$$

Then (2.7) follows from (2.2) and (2.4).

Corollary 2.2.5. Under the K_n alternatives and if $b_j = f(j/n)$, where f satisfies the condition of Theorem 2.2,

$$e_{W_\tau,U}(G) = \frac{36(2r+1)(2+6(m-2)(4\lambda(G)-1)[\sum_{u<v}\sum_{i=0}^{\infty} c_i\, d_{uv,i}(G)]^2}{m^3(m-1)^2(m+1)(2m+5)\int_0^1 f^2(x)\, dx[\int_{-\infty}^{\infty} g_1^2(x)\, dx]^2} \quad (2.8)$$

Proof. Doksum (1967) shows that

$$e_{U,T}(G) = \frac{24m\sigma^2(G)[\int_{-\infty}^{\infty} g_1^2(t)\, dt]^2}{2 + 6(m-2)(4\lambda(G) - 1)}$$

Then (2.8) follows from (2.3).

3. ASYMPTOTIC EFFICIENCY OF W_r-TEST FOR THREE TREATMENTS

3.1. Calculation of $d_{uv,0}(G)$ and $d_{uv,i}(G)$, $i = 1, 2, \ldots$

In this section we will suppose $D_j = R_j$, $j = 1, \ldots, n$.

Theorem 3.1 gives a computable expression for $d_{uv,0}(G)$ and $d_{uv,i}(G)$, $i = 1, 2,$ for general G.

Theorem 3.1. Let Z_1, Z_2, Z_3 be i.i.d. random variables with distribution G. Define $U = Z_3 - Z_2$, $V = Z_2 - Z_1$ and let $h(u,v)$ be the joint density of U and V. Then

$$d_{uv,0}(G) = (v - u) \int_{-\infty}^{\infty} g^2(x)\, dx \tag{3.1}$$

and

$$d_{uv,i}(G) = 6^i (v - u) i \{K_{1i} + K_{2i} + 2K_{3i}\}, \quad i = 1, 2, \tag{3.2}$$

where

$$K_{1i} = \int_{-\infty}^{0} \int_{-\infty}^{0} [H(x+y)]^{i-1} H_1(x+y)\, h(x,y)\, dx\, dy \tag{3.3}$$

$$K_{2i} = \int_{-\infty}^{0} \int_{-\infty}^{0} [H(x+y)]^{i-1} H_2(x+y)\, h(x,y)\, dx\, dy \tag{3.4}$$

$$K_{3i} = \int_{-\infty}^{0} \int_{-\infty}^{0} \int_{u+v}^{0} [H(u+v)]^{i-1}\, h(u+v-w,w)\, h(u,v)\, dw\, dv\, du \tag{3.5}$$

and

$$H(z) = \int_{z}^{0} \int_{z-y}^{0} h(x,y)\, dx\, dy \tag{3.6}$$

$$H_1(z) = \int_{-\infty}^{z} h(0,y)\, dy \tag{3.7}$$

$$H_2(z) = \int_{-\infty}^{z} h(x,0)\, dx. \tag{3.8}$$

We omit the proof of this theorem, since it is very similar to that of Theorem 3.1 in Ferretti and Yohai (1986).

The following corollaries are consequences of Theorem 3.1.

Corollary 3.1.1. If G is normal with variance 1 and Φ, ϕ are respectively the standardized normal distribution function and the standardized normal density function, we have

$$d_{uv,0}(G) = (v - u)/(2\pi^{1/2}) \tag{3.9}$$

$$K_{1i} = K_{2i} = \int_{-\infty}^{0} H_1(2^{1/2}w)[H(2^{1/2}w)]^{i-1}(2\Phi(-w/3^{1/2}) - 1)\phi(w) \, dw \tag{3.10}$$

and

$$K_{3i} = (8\pi)^{-1/2}\int_{-\infty}^{0} [H(w)]^{i-1}(2\Phi(-w/6^{1/2}) - 1)^2\phi(w)\, dw \tag{3.11}$$

where

$$H(z) = \int_{z/2^{1/2}}^{0} (2\Phi(-u/3^{1/2}) - 1)\phi(u)\, du \tag{3.12}$$

$$H_1(z) = 2^{-1}\pi^{-1/2}\Phi(z(2/3)^{1/2}). \tag{3.13}$$

Corollary 3.1.2. If G is Cauchy with scale parameter 1, we have

$$d_{uv,0}(G) = (v - u)/(2\pi), \tag{3.14}$$

$$K_{1i} = K_{2i} = (-1/\pi)\int_{-\infty}^{0} H_1(2w)[H(2w)]^{i-1}[\ln(w^2 + 1)/2w + \arctan(w)]\gamma_1(w)\, dw \tag{3.15}$$

and

$$K_{3i} = (1/(2\pi^2))\int_{-\infty}^{0} [\ln(w^2 + 1)/2w + \arctan(w)]^2[H(2w)]^{i-1} \times[\gamma_1(w)]^2\, dw \tag{3.16}$$

where γ_1 is the Cauchy density function with location parameter

0 and scale parameter 1,

$$H(z) = \{[\text{arc tan}(z/2)]^2 - [\ln(z^2/4 + 1)]^2/4 - \int_{z/2}^{0} \ln(x^2 + 1)/x \, dx\}/2\pi^2 \tag{3.17}$$

and

$$H_1(z) = (2\pi^2)^{-1}(\text{arc tan}(z/2) + \pi/2 + z/(4 + z^2)) \tag{3.18}$$

3.2. Some Results and Conclusions

(a) Let $b_j = (j/n)^r$ for $r = 1, 2,$ and $D_j = R_j$.

From Corollaries 2.2.1, 2.2.2, 2.2.3, 2.2.4 and 2.2.5 with $f(x) = x^r$ we see that

$$e_{W_\tau,T}(G) = 96(36)^r r^2 \{K_{1r} + K_{2r} + 2K_{3r}\}^2 \sigma^2(G)(2r + 1)/11,$$

$$e_{W_\tau,\tau}(G) = (2r + 1)r^2(36)^r\{K_{1r} + K_{2r} + 2K_{3r}\}^2/[\int_{-\infty}^{\infty} g^2(x)\, dx]^2,$$

$$e_{W_\tau,\rho}(G) = 32(2r + 1)r^2(36)^r\{K_{1r} + K_{2r} + 2K_{3r}\}^2/33[\int_{-\infty}^{\infty} g^2(x)\, dx]^2$$

$$e_{W_\tau,Y}(G) = \frac{(2r + 1)r^2(36)^r(3 + 2(12\lambda(G) - 3))\{K_{1r} + K_{2r} + 2K_{3r}\}^2}{11[\int_{-\infty}^{\infty} g_1^2(x)\, dx]^2}$$

$$e_{W_\tau,U}(G) = \frac{4(2r + 1)(2 + 6(4\lambda(G) - 1))r^2(36)^r\{K_{1r} + K_{2r} + 2K_{3r}\}^2}{33[\int_{-\infty}^{\infty} g_1^2(x)\, dx]^2}$$

where K_{1r}, K_{2r} and K_{3r} are given by (3.3), (3.4) and (3.5) respectively with $i = r$.

Some values of $e_{W_\tau,T}(G)$, $e_{W_\tau,\tau}(G)$, $e_{W_\tau,\rho}(G)$, $e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$ when G is normal are shown in Table I. In Table II we list some values of $e_{W_\tau,\tau}(G)$, $e_{W_\tau,\rho}(G)$, $e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$ when G is Cauchy.

For each distribution five values of r were considered: 1, 2, 3, 4, 14.

TABLE I

Asymptotic Efficiencies $e_{W_\tau,T}(G)$, $e_{W_\tau,\tau_r}(G)$, $e_{W_\tau,\rho}(G)$, $e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$, when $b_j = (j/n)^r$, $m = 3$ and G is normal

r	$e_{W_\tau,T}(G)$	$e_{W_\tau,\tau}(G)$	$e_{W_\tau,\rho}(G)$	$e_{W_\tau,Y}(G)$	$e_{W_\tau,U}(G)$
1	0.872	1.255	1.217	0.905	0.902
2	0.820	1.181	1.145	0.851	0.848
3	0.748	1.077	1.045	0.776	0.773
4	0.683	0.984	0.954	0.709	0.706
14	0.368	0.530	0.514	0.382	0.380

TABLE II

Asymptotic Efficiencies $e_{W_\tau,\tau_r}(G)$, $e_{W_\tau,\rho}(G)$, $e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$, when $b_j = (j/n)^r$, $m = 3$ and G is Cauchy

r	$e_{W_\tau,\tau}(G)$	$e_{W_\tau,\rho}(G)$	$e_{W_\tau,Y}(G)$	$e_{W_\tau,U}(G)$
1	0.595	0.577	0.846	0.839
2	0.340	0.330	0.484	0.480
3	0.217	0.210	0.308	0.306
4	0.150	0.146	0.214	0.212
14	0.026	0.025	0.037	0.037

We note that for each considered G the values $e_{W_\tau,T}(G)$, $e_{W_\tau,\tau}(G)$, $e_{W_\tau,\rho}(G)$, $e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$ are decreasing functions of r. In each case the maximum occurs at $r = 1$. For $r = 1$ and G normal, $e_{W_\tau,\tau}(G) = 1.255$, $e_{W_\tau,\rho}(G) = 1.217$, $e_{W_\tau,Y}(G) = 0.905$ and $e_{W_\tau,U}(G) = 0.902$. When $r = 1$ and G is Cauchy, $e_{W_\tau,\tau}(G) = 0.595$, $e_{W_\tau,\rho}(G) = 0.577$, $e_{W_\tau,Y}(G) = 0.846$ and $e_{W_\tau,U}(G) \doteq 0.839$.

(b) Let $b_j = (1 + j/n)^r$ for $r = 1, 2,$ and $D_j = R_j$.

Since b_j can be expressed as $b_j = \sum_{i=0}^{\infty} \binom{r}{i} (j/n)^r$, we conclude from corollaries 2.4.1, 2.4.2, 2.4.3, 2.4.4 and 2.4.5

$$e_{W_\tau,T}(G) = \frac{96[\int_{-\infty}^{\infty} g^2(x)\,dx + \sum_{i=1}^{\infty} \binom{r}{i} 6^i i\{K_{1i} + K_{2i} + 2K_{3i}\}]^2 \sigma^2(G)}{11\int_0^1 f^2(x)\,dx}$$

$$e_{W_\tau,\tau}(G) = \frac{[\int_{-\infty}^{\infty} g^2(x)\,dx + \sum_{i=1}^{\infty} \binom{r}{i} 6^i i\{K_{1i} + K_{2i} + 2K_{3i}\}]^2}{[\int_{-\infty}^{\infty} g^2(x)\,dx]^2 \int_0^1 f^2(x)\,dx}$$

$$e_{W_\tau,\rho}(G) = \frac{32[\int_{-\infty}^{\infty} g^2(x)\,dx + \sum_{i=1}^{\infty} \binom{r}{i} 6^i i\{K_{1i} + K_{2i} + 2K_{3i}\}]^2}{33[\int_{-\infty}^{\infty} g^2(x)\,dx]^2 \int_0^1 f^2(x)\,dx}$$

$$e_{W_\tau,Y}(G) = \frac{[3 + 2(12\lambda(G) - 3)][\int_{-\infty}^{\infty} g^2(x)\,dx + \sum_{i=1}^{\infty} \binom{r}{i} 6^i i\{K_{1i} + K_{2i} + 2K_{3i}\}]^2}{11[\int_{-\infty}^{\infty} g_1^2(x)\,dx]^2 \int_0^1 f^2(x)\,dx}$$

$$e_{W_\tau,U}(G) = \frac{4[2 + 6(4\lambda(G) - 1)][\int_{-\infty}^{\infty} g^2(x)\,dx + \sum_{i=1}^{\infty} \binom{r}{i} 6^i i\{K_{1i} + K_{2i} + 2K_{3i}\}]^2}{33[\int_{-\infty}^{\infty} g_1^2(x)\,dx]^2 \int_0^1 f^2(x)\,dx}$$

where K_{1i}, K_{2i} and K_{3i} are given by (3.3), (3.4) and (3.5) respectively.

Some values of $e_{W_\tau,T}(G)$, $e_{W_\tau,\tau}(G)$, $e_{W_\tau,\rho}(G)$, $e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$ when G is normal are shown in Table III. When G is Cauchy,

TABLE IV

Asymptotic Efficiencies $e_{W_\tau,\tau}(G)$, $e_{W_\tau,\rho}(G)$, $e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$, when $b_j = (1 + j/n)^r$, $m = 3$ and G is Cauchy

r	$e_{W_\tau,\tau}(G)$	$e_{W_\tau,\rho}(G)$	$e_{W_\tau,Y}(G)$	$e_{W_\tau,U}(G)$
1/12	0.994	0.963	1.412	1.402
1/8	0.990	0.960	1.408	1.397
1/6	0.987	0.957	1.404	1.393
1/4	0.980	0.951	1.394	1.383
1/3	0.972	0.943	1.382	1.372
1/2	0.956	0.927	1.359	1.348
1	0.895	0.868	1.273	1.263
2	0.747	0.724	1.061	1.053
3	0.598	0.580	0.850	0.844
4	0.473	0.458	0.672	0.667
6	0.299	0.290	0.425	0.422
8	0.200	0.194	0.284	0.282
10	0.142	0.138	0.202	0.201
12	0.107	0.103	0.151	0.150
14	0.083	0.081	0.118	0.117

4. ASYMPTOTIC EFFICIENCY OF THE W_τ-TEST FOR FOUR TREATMENTS

In this section we will suppose $D_j = R_j$, $j = 1, 2$,

Theorem 4.1 gives a computable expression for $d_{uv,0}(G)$ and $d_{uv,i}(G)$, $i = 1, 2$, for general G.

Theorem 4.1. Let Z_1, Z_2, Z_3, Z_4 be i.i.d. random variables with distribution G. Define $U = Z_2 - Z_1$, $V = Z_3 - Z_2$, $W = Z_4 - Z_3$ and let $h^*(u,v,w)$ be the joint density of U, V and W. Then

$$d_{uv,0}(G) = 2(v - u)\int_{-\infty}^{\infty} g^2(x)\, dx \tag{4.1}$$

and

$$d_{uv,i}(G) = 4(24)^i (v-u) i \{K^*_{1i} + K^*_{2i} + K^*_{3i} + 3K^*_{4i}\}, \; i = 1, 2, \tag{4.2}$$

where

$$\begin{aligned} K^*_{si} = \{&\int_{-\infty}^{0} \int_{-\infty}^{0} \int_{-\infty}^{0} H^*_{2s-1}(u+v+w)[H^*(u+v+w)]^{i-1} h^*(u,v,w)\, du\, dv\, dw \\ &+ \int_{-\infty}^{0} \int_{-\infty}^{0} \int_{-\infty}^{0} H^*_{2s}(u+v+w)[H^*(u+v+w)]^{i-1} h^*(u,v,w)\, du\, dv\, dw \end{aligned} \tag{4.3}$$

for s = 1, 2, 3,

$$K^*_{4i} = \int_{-\infty}^{0} \int_{-\infty}^{0} \int_{-\infty}^{0} H^*_7(u+v+w)[H^*(u+v+w)]^{i-1} h^*(u,v.w)\, du\, dv\, dw \tag{4.4}$$

where

$$H^*(z) = \int_z^0 \int_{z-u}^0 \int_{z-u-v}^0 h^*(u,v,w)\, du\, dv\, dw \tag{4.5}$$

$$H^*_1(z) = \int_{-\infty}^{z} \int_{-\infty}^{0} h^*(0,v,w)\, dv\, dw \tag{4.6}$$

$$H^*_2(z) = \int_z^0 \int_{-\infty}^{z-w} h^*(0,v,w)\, dv\, dw \tag{4.7}$$

$$H^*_3(z) = \int_{-\infty}^{z} \int_{-\infty}^{0} h^*(u,0,w)\, du\, dw \tag{4.8}$$

$$H^*_4(z) = \int_z^0 \int_{-\infty}^{z-w} h^*(u,0,w)\, du\, dw \tag{4.9}$$

$$H^*_5(z) = \int_{-\infty}^{z} \int_{-\infty}^{0} h^*(u,v.0)\, du\, dv \tag{4.10}$$

$$H_6^*(z) = \int_z^0 \int_{-\infty}^{z-v} h^*(u,v,0)\, du\, dv \tag{4.11}$$

and

$$H_7^*(z) = \int_z^0 \int_{z-v}^0 h^*(z-v-w,v,w)\, dv\, dw. \tag{4.12}$$

The proof of this theorem is similar to that of Theorem 3.1.

The following corollaries are consequences of Theorem 4.1.

Corollary 4.1.1. If G is normal with variance 1, Φ , ϕ are respectively the standardized normal distribution function and the standardized normal density function and if we denote respectively with Φ_C and ϕ_C the distribution function and the density function of a bivariate normal variable with mean 0 and covariance matrix C, we have

$$d_{uv,0}(G) = (v - u)/\pi^{1/2} \tag{4.13}$$

$$\begin{aligned} K_{si}^* = \{ & \int_{-\infty}^0 \int_y^0 H_{2s-1}^*(y)[H^*(y)]^{i-1}[\Phi(-(y - x/2)/3^{1/2}) \\ & - \Phi((y/2 - x)/3^{1/2})]\phi_C(x,y)\, dx\, dy \\ & + \int_{-\infty}^0 \int_y^0 H_{2s}^*(y)[H^*(y)]^{i-1}[\Phi(-(y - x/2)/3^{1/2}) \\ & - \Phi((y/2 - x)/3^{1/2})]\phi_C(x,y)\, dx\, dy\} \end{aligned} \tag{4.14}$$

for s = 1, 2, 3,

$$K_{4i}^* = \int_{-\infty}^{0} \int_{y}^{0} H_7^*(y)[H^*(y)]^{i-1}[\Phi(-(y - x/2)/3^{1/2}) - \Phi((y/2 - x)/3^{1/2})]\phi_C(x,y)\,dx\,dy \tag{4.15}$$

where

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$H^*(w) = \int_{w/2^{1/2}}^{0} \int_{(2/3)^{1/2}(w-s^{1/2})}^{s/3^{1/2}} [\Phi(t/2^{1/2}) - \Phi((3^{1/2}/2)(w - s/2^{1/2}) - t/(2\,2^{1/2}))]\,\phi(t)\phi(s)\,dt\,ds \tag{4.16}$$

$$H_1^*(w) = 2^{-1}\pi^{-1/2}\phi_C(w/2^{1/2},0) \qquad C = \begin{bmatrix} 1 & -3^{-1/2} \\ -3^{-1/2} & 1 \end{bmatrix} \tag{4.17}$$

$$H_2^*(w) = 2^{-1}\pi^{-1/2}\int_{w/2^{1/2}}^{0} \Phi(w - u/2^{1/2})\phi(u)\,du \tag{4.18}$$

$$H_3^*(w) = 2^{-1}\pi^{-1/2}\phi_C((2/3)^{1/2}w,0)\ ,\ C = \begin{bmatrix} 1 & -1/3 \\ -1/3 & 1 \end{bmatrix} \tag{4.19}$$

$$H_4^*(w) = 2^{-1}\pi^{-1/2}\int_{(2/3)^{1/2}w}^{0} \phi(3^{1/2}w/2 - u/2^{1/2})\phi(u)\,du \tag{4.20}$$

$$H_5^*(w) = 2^{-1}\pi^{-1/2}\phi_C((2/3)^{1/2}w,0) \qquad C = \begin{bmatrix} 1 & -3^{-1/2} \\ -3^{-1/2} & 1 \end{bmatrix} \tag{4.21}$$

$$H_6^*(w) = 2^{-1}\pi^{-1/2}\int_{(2/3)^{1/2}w}^{0} \Phi(3^{1/2}w/2 - u/(2\,2^{1/2}))\phi(u)\,du \tag{4.22}$$

$$H_7^*(w) = w2^{-1}\pi^{-1/2}e^{-w^2/4}\int_0^{2^{-1/2}} \phi(wu)[2\Phi(w(1/2 - u/2^{1/2})) - 1] \times du \tag{4.23}$$

4.2. Some Results and Conclusions

Let $b_j = (j/n)^r$ for $r = 1, 2,$ and $D_j = R_j$.

From Corollaries 2.2.1, 2.2.2, 2.2.3, 2.2.4 and 2.2.5 with $f(x) = x^r$ we see that

$$e_{W_\tau,T}(G) = 1920[576]^r r^2(2r+1)\sigma^2(G)\{K_{1r}^* + K_{2r}^* + K_{3r}^* + 3K_{4r}^*\}^2/13$$

$$e_{W_\tau,\tau}(G) = \frac{16[576]^r(2r+1)r^2\{K_{1r}^* + K_{2r}^* + K_{3r}^* + 3K_{4r}^*\}^2}{[\int_{-\infty}^{\infty} g^2(x)\,dx]^2}$$

$$e_{W_\tau,\rho}(G) = \frac{200[576]^r(2r+1)r^2\{K_{1r}^* + K_{2r}^* + K_{3r}^* + 3K_{4r}^*\}^2}{13[\int_{-\infty}^{\infty} g^2(x)\,dx]^2}$$

$$e_{W_\tau,Y}(G) = \frac{(2r+1)(3+4(12\lambda(G)-3))16[576]^r r^2\{K_{1r}^* + K_{2r}^* + K_{3r}^* + 3K_{4r}^*\}^2}{13[\int_{-\infty}^{\infty} g_1^2(x)\,dx]^2}$$

$$e_{W_\tau,U}(G) = \frac{20(2r+1)(2+12(4\lambda(G)-1))[576]^r r^2\{K_{1r}^* + K_{2r}^* + K_{3r}^* + 3K_{4r}^*\}^2}{13}$$

where K_{1r}^*, K_{2r}^*, K_{3r}^* and K_{4r}^* are given by (4.3) and (4.4) respectively with $i = r$.

Some values of $e_{W_\tau,T}(G)$, $e_{W_\tau,\tau}(G)$, $e_{W_\tau,\rho}(G)$, $e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$ when G is normal are shown in Table V. For $r = 1$, $e_{W_\tau,\tau}(G) = 1.151$, $e_{W_\tau,\rho}(G) = 1.107$, $e_{W_\tau,Y}(G) = 0.872$ and $e_{W_\tau,U}(G) = 0.870$.

TABLE V

Asymptotic Efficiencies $e_{W_\tau,T}(G)$, $e_{W_\tau,\tau}(G)$, $e_{W_\tau,\rho}(G)$
$e_{W_\tau,Y}(G)$ and $e_{W_\tau,U}(G)$, when $b_j = (j/n)^r$, $m = 4$
and G is normal

r	$e_{W_\tau,T}(G)$	$e_{W_\tau,\tau}(G)$	$e_{W_\tau,\rho}(G)$	$e_{W_\tau,Y}(G)$	$e_{W_\tau,U}(G)$
1	0.845	1.151	1.107	0.872	0.870
2	0.755	1.027	0.988	0.779	0.776

5. DISCUSSION

The following observations can be made about the results discussed in Sections 3 and 4.

(i) If we wish to contrast the weighted and unweighted tests using $e_{W_\tau,\tau}(G)$ then we can conclude that for the range of distribution from normal (moderate tail) to Cauchy (very heavy tail) there is no single weighted test that is superior to an unweighted test. For $m = 3$ the best we could do is take $b_j = (1 + j/n)^{1/12}$ and achieve $e_{W_\tau,\tau}(G) \approx 1$, so there is no advantage to weighting in heavy tailed cases but that is not surprising since the sign test is more efficient than the Wilcoxon test in those situations. Weighting should be used only for moderate tailed distributions; in which case $b_j = j/n$ or $b_j = (1 + j/n)^3$ would be fairly good. So that for simplicity the former may be recommended in moderate tailed cases with $m = 3$ and similarly for $m = 4$.

We should also point out that these recommendations can be made for weighted tests in the unordered alternatives cases. For these alternatives, $m = 3$ and G normal or Cauchy we remark that

$e_{W_\tau,\tau}(G)$ reduces to the efficiency of the test proposed by Quade (1972, 1979) with respect to the Friedman test. (See Ferretti and Yohai (1986)).

(ii) For G normal, $m = 3, 4$ and $b_j = (j/n)^r$ (Tables I and V) or $b_j = (1 + j/n)^r$ (Table III) the Hollander and the Doksum test are more efficient than the W_τ-test. Results for G Cauchy, $m = 3$ and $b_j = (j/n)^r$ (Table II) or $b_j = (1 + j/n)^r$ with $r \geq 3$ (Table IV) are similar. However, the Hollander statistic and the Doksum statistic are only asymtotically distribution-free.

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