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1. Introduction.

Several problems in Physics and related fields yields to the necessity of solving elliptic problems on arbitrary domains. For instance, this kind of problems appears in potential theory, hydrodinamics and elasticity.

Finite Element Method (F.E.M.) has proved to be a very useful tool to solve them numerically, in particular, when the domain of the problem is not geometrically simple. (For a good description of the applications of F.E.M. to Mathematical Physics see [1] and references therein).

As it is well known, the starting point of F.E.M. is the subdivision of the domain into elementary subdomains; v.g. quadrilaterals or triangles for plane domains.

When quadrilaterals are used, it is useful being able to include also some triangular elements for getting a regular mesh. (For instance, if a refinement of the subdivision at a certain part of the domain is carried out).

Different algorithms ought to be used for assembling each type of element. Nevertheless, triangles are considered degenerate quadrilaterals in some codes, for the sake of computational simplicity (e.g. [2]).

The aim of this note is to provide theoretical basis for such a procedure. We shall show that, in spite of the fact that it is not a well posed method, it yields to satisfactory numerical results when it is made a suitable choice of the quadrature scheme involved.

2. Statement of the problem.

Let us consider the model problem of finding a function $u \in H_0^1(\Omega)$ such that:

$$A(u,v) = F(v) \text{ , for every } v \in H_0^1(\Omega) \qquad (1.1)$$

where Ω is an open poligonal subset of $\mathbb{R}^2$,

$$A(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij}\, \partial_i u\, \partial_j v \, dx \text{ ,} \qquad (1.2)$$

$$F(v) = \int_\Omega f v \, dx \qquad (1.3)$$

with f and a_{ij} continuous functions such that

$$\exists \alpha > 0 : \forall x \in \Omega : \forall \xi \in \Omega : \sum_{i,j=1}^{2} a_{ij}(x)\, \xi_i \xi_j \geq \alpha \sum_{i=1}^{2} \xi_i^2$$

Let $\mathcal{T}$ be a mixed subdivision of Ω composed of triangles and quadrilaterals. For each $K \in \mathcal{T}$ we shall write:

h_K = diameter of K

ρ_K = maximum of the diameters of all circles contained in K

$\sigma_K = \rho_K / h_K$.

We also denote $h = \max_{K \in \mathcal{T}} h_K$ and $\sigma = \min_{K \in \mathcal{T}} \sigma_K$

For each quadrilateral $K \in \mathcal{T}$ with vertices M_1, M_2, M_3 and M_4,

$$q_K(\hat{x},\hat{y}) = (1-\hat{x})(1-\hat{y})M_1 + \hat{x}(1-\hat{y})M_2 + \hat{x}\hat{y}M_3 + (1-\hat{x})\hat{y}M_4 \quad (1.4)$$

defines a one to one mapping from the unit square $\hat{K} = [0,1]\times[0,1]$ onto K.

To each function $\hat{\varphi}$ defined on $\hat{K}$ there corresponds a function φ defined on K by means of the canonical relation:

$$\varphi = \hat{\varphi} \circ q_K^{-1}. \quad (1.5)$$

Let $\hat{Q}_1$ be the space of functions defined on $\hat{K}$ which are linear in each coordinate and

$$Q_1(K) = \left\{ \varphi = \hat{\varphi} \circ q_K^{-1} \; ; \; \hat{\varphi} \in \hat{Q}_1 \right\}, \quad (1.6)$$

i.e. the space of functions " of type Q_1 " on K.

For each triangle $K \in \mathcal{T}$ we shall write $P_1(K)$ for the space of linear functions on K

Let V_h be the subspace of $H_0^1(\Omega)$ composed of continuous functions v_h defined on $\bar{\Omega}$ such that $v_h|_{\partial\Omega} = 0$ and $v_h|_K \in P_K$ where $P_K = P_1(K)$ or $P_K = Q_1(K)$ for $K \in \mathcal{T}$ a quadrilateral or a triangle respectively. Let $u_h \in V_h$ be the solution of the discrete problem :

$$A(u_h, v_h) = F(v_h) \quad , \text{for every} \quad v_h \in V_h \quad (1.7)$$

Since the solution u of eq.(1.1) belongs to $H^2(\Omega)$, from the results of Jamet [3] and Ciarlet-Raviart [4] and Cea's lema, one gets

$$\|u-u_h\|_{1-m,\Omega} \leq C\, h^{2-m} |u|_{2,\Omega}\ , \quad m=0,1 \tag{1.8}$$

where C is a constant depending on σ but not on h and $\|\cdot\|_{k,\Omega}$, $|\cdot|_{k,\Omega}$ are the usual Sobolev norms and seminorms.

For $\{N_i\}_{i=1}^{n}$ the set of inner nodes of $\mathcal{T}$ we take $\omega_i \in V_h$ such that $\omega_i(N_j)=\delta_{ij}$ and so the solution u_h of the problem (1.7) may be written

$$u_h = \sum_{j=1}^{n} \beta_j\, \omega_j\ ,$$

with β_j satisfying:

$$\sum_{j=1}^{n} A(\omega_j,\omega_i)\,\beta_j = F(\omega_i)\ , \quad i=1,\dots,n. \tag{1.9}$$

The matrix and the r.h.s. member of eq.(1.9) are obtained by assembling the contribution of each element as

$$A(\omega_j,\omega_i) = \sum_{\{K\in\mathcal{T}\,:\,N_i,N_j \text{ are vertices of } K\}} A_K(\omega_j,\omega_i) \tag{1.10}$$

and

$$F(\omega_i) = \sum_{\{K\in\mathcal{T}\,:\,N_i \text{ is a vertex of } K\}} F_K(\omega_i) \tag{1.11}$$

where

$$A_K(u,v) = \int_K \sum_{i,j=1}^{2} a_{ij}\, \partial_i u\, \partial_j v\, dx \tag{1.12}$$

and

$$F_K(v) = \int_K f v \, dx \qquad (1.13)$$

for $u, v \in P_K$

To do this, different algorithms should be necessary for assembling each type of element. However, in some codes (e.g. [2]), triangles are considered quadrilaterals with two identified vertices. For continuous trial functions which are linear on triangles and " of type Q_1 " on quadrilaterals we shall show that such a procedure is not well-posed and shall explain the reason why, in spite of that, it yields to a satisfactory result when an adequate quadrature method is used.

2. Interpolation of " type Q_1 " in triangles.

Let $\tilde{K}$ be the reference triangle of vertices $(0,0)$, $(1,0)$, $(0,1)$. The canonical basis functions of $P_1(\tilde{K})$ are

$$\tilde{\omega}_1 = 1 - \tilde{x} - \tilde{y} \ , \ \tilde{\omega}_2 = \tilde{x} \text{ and } \tilde{\omega}_3 = \tilde{y} \qquad (2.1)$$

Thinking of $\tilde{K}$ as a degenerate quadrilateral with vertices $M_1 = (0,0)$, $M_2 = (1,0)$, $M_3 = M_4 = (0,1)$ we may define $q_{\tilde{K}}$ according to eq.(1.4). So one gets

$$q_{\tilde{K}}(\hat{x},\hat{y}) = (\hat{x}(1-\hat{y}), \hat{y}) . \qquad (2.2)$$

This mapping has an inverse defined for every point except the vertex (0,1) which is given by

$$q_{\tilde{K}}^{-1}(\tilde{x},\tilde{y}) = (\tilde{x}/(1-\tilde{y}),\tilde{y}) \tag{2.3}$$

and so we can define $Q_1(\tilde{K})$ analogously to (1.6). (The elements of $Q_1(\tilde{K})$ will be functions well defined out of the point $(0,1)$).

From the canonical basis functions of $\hat{Q}_1$

$\hat{\omega}_1=(1-\hat{x})(1-\hat{y})$, $\hat{\omega}_2=(1-\hat{x})\hat{y}$, $\hat{\omega}_3=\hat{x}\hat{y}$)and $\omega_4=\hat{x}(1-\hat{y})$ we get, by means of the relation (1.5), a basis of $Q_1(\tilde{K})$, that is:

$\tilde{w}_1=1-\tilde{x}-\tilde{y}$, $\tilde{w}_2=\tilde{x}$, $\tilde{w}_3=\tilde{x}\tilde{y}/(1-\tilde{y})$ and $\tilde{w}_4=\tilde{y}-\tilde{x}\tilde{y}/(1$

So we have:

$$\tilde{\omega}_1=\tilde{w}_1 \quad , \quad \tilde{\omega}_2=\tilde{w}_2 \text{ and } \tilde{\omega}_3=\tilde{w}_3+\tilde{w}_4 \, . \tag{2.4}$$

Given a triangle $K\in\mathcal{T}$ with vertices N_1 , N_2 and N_3 let q_K be the affine mapping from $\tilde{K}$ onto K such that $q_K(0,0)=N_1$, $q_K(1,0)=N_2$ and $q_K(0,1)=N_3$. By direct composition with q_K^{-1} we may define the spaces $P_1(K)$ and $Q_1(K)$ and respective basis functions which obviously satisfy the same relation as (2.4):

$$\omega_1=w_1 \quad , \quad \omega_2=w_2 \text{ and } \omega_3=w_3+w_4 \tag{2.5}$$

For a continuous function v on K the natural way of defining its interpolate " of type Q_1", $\Pi_{Q_1(K)}v$, is through the reference square $\hat{K}$; thus

$$\Pi_{Q_1(K)}v = v(N_1)w_1+v(N_2)w_2+v(N_3)w_3+v(N_3)w_4 \tag{2.6}$$

Note that this may be thought as a limit of the interpolation for quadrilaterals degenerating into K .

From (2.5) we see that $\Pi_{Q_1(K)} v$ coincides with the usual linear interpolate of v on $P_1(K)$:

$$\Pi_{P_1(K)} v = v(N_1)\,\omega_1 + v(N_2)\,\omega_2 + v(N_3)\,\omega_3$$

Thereforethe assembly of triangles as degenerate quadrilaterals would yield us to compute :

$$A_K(\omega_r,\omega_s) \text{ as } \begin{cases} A_K(w_r, w_s), \text{ if } r,s = 1,2 \\ A_K(w_r, w_3) + A_K(w_r, w_4), \text{ if } r = 1,2;\ s = 3 \\ A_K(w_3, w_r) + A_K(w_4, w_s), \text{ if } r = 3;\ s = 1,2 \\ A_K(w_3, w_3) + A_K(w_3, w_4) + A_K(w_4, w_3) + A_K(w_4, w_4), \text{ if } r = s = 3 \end{cases} \quad (2.7)$$

and

$$F_K(\omega_r) \text{ as } \begin{cases} F_K(w_r), \text{ if } r = 1,2 \\ F_K(w_3) + F_K(w_4), \text{ if } r = 3. \end{cases} \quad (2.8)$$

Nevertheless we must note that $Q_1(K) \subset L^2(K)$ but w_3 and w_4 do not belong to $H^1(K)$ and then expressions as $A_K(w_r, w_s)$ are not in general well defined for r or $s = 3$ or 4.

Forasmuch we may conclude that such an assembly is quite impossible.

3. Assembly of degenerate quadrilaterals by means of numerical integration.

The effective computation of most of the problems needs to evaluate numerically integrals defining $A_K(\omega_j,\omega_i)$ and $F_K(\omega_i)$ which yields to consider new approximate forms A_{hK} and F_{hK}.

For doing that, in our case, we ought to give quadrature schemes over each reference element:

$$\int_{\hat{K}} \hat{\varphi}(\hat{x})\,d\hat{x} \sim \sum_{\ell=1}^{\hat{L}} \hat{\alpha}_\ell \hat{\varphi}(\hat{b}_\ell) \tag{3.1}$$

$$\int_{\tilde{K}} \tilde{\varphi}(\tilde{x})\,d\tilde{x} \sim \sum_{\ell=1}^{\tilde{L}} \tilde{\alpha}_\ell \tilde{\varphi}(\tilde{b}_\ell). \tag{3.2}$$

Now, changing variables we have:

$$\int_K \varphi(x)\,dx = \begin{cases} \int_{\hat{K}} \hat{\varphi}(\hat{x}) \det(q_K(\hat{x}))\,d\hat{x}, \text{ for } K \text{ a quadrilateral} \\ \int_{\tilde{K}} \tilde{\varphi}(\tilde{x}) \det(q_K(\tilde{x}))\,d\tilde{x}, \text{ for } K \text{ a triangle} \end{cases} \tag{3.3}$$

with q_K the corresponding mapping from the reference element onto K and $\hat{\varphi}$ or $\tilde{\varphi} = \varphi \circ q_K^{-1}$ in each case.

Thus, we can take

$$\int_K \varphi(x)\,dx \approx \sum_{\ell=1}^{L} \alpha_{\ell K} \varphi(b_{\ell K}) \tag{3.4}$$

where $b_{\ell K} = q_K(\hat{b}_\ell)$, $\alpha_{\ell K} = \hat{\alpha}_\ell \det(q_K(\hat{b}_\ell))$ and $L = \hat{L}$ for K a quadrilateral and the obvious analogue for K a triangle.

So the above mentioned forms result :

$$A_{hK}(u,v) = \sum_{\ell=1}^{L} \alpha_{\ell K}\left(\sum_{i,j=1}^{2} a_{ij}(b_{\ell K})\,\partial_i u(b_{\ell K})\,\partial_j v(b_{\ell K})\right) \tag{3.5}$$

$$F_{hK}(v) = \sum_{\ell=1}^{L} \alpha_{\ell K}\, f(b_{\ell K})\, v(b_{\ell K}). \tag{3.6}$$

Carlet and Raviart [5] have shown that the approximate solution of problem (1.7) obtained assembling A_{hK} and F_{hK} instead of A_K and F_K, satisfies estimates of the same order than (1.8), provided the integration schemes verify the following conditions:

- scheme (3.1) is exact to integrate constant functions.
- scheme (3.2) is exact to integrate polynomials of order two in each variable and the set $\{\hat{b}_\ell\}_{\ell=1}^{\hat{L}}$ contains a Q_1 unisolvent subset.

A gaussian quadrature scheme over $\hat{K}$ with four points satisfies the required conditions for quadrilaterals. Since for such scheme there is no evaluation of the integrand at the vertices, its application to the singular integrals arising from degenerate quadrilaterals gives finite results.

So, even when $A_K(w_r, w_s)$ is not well defined, $A_{hK}(w_r, w_s)$ actually is and the assembly of the new forms yields to:

$$A_{hK}(\omega_r,\omega_s) \text{ as } \begin{cases} A_{hK}(w_r, w_s), \text{ if } r,s = 1,2 \\ A_{hK}(w_r, w_3) + A_{hK}(w_r, w_4), \text{ if } r=1,2,\ s=3 \\ A_{hK}(w_3, w_s) + A_{hK}(w_4, w_s), \text{ if } r=3,\ s=1,2 \\ A_{hK}(w_3, w_3) + A_{hK}(w_3, w_4) + A_{hK}(w_4, w_3) + A_{hK}(w_4, w_4), \text{ if } r=s=3 \end{cases} \tag{3.7}$$

$$F_{hK}(\omega_r) \text{ as } \begin{cases} F_{hK}(w_r), \text{ if } r = 1,2 \\ F_{hK}(w_3) + F_{hK}(w_4), \text{ if } r=3 \end{cases} \tag{3.8}$$

This shows that we get the same result as if we had assembled $A_{nK}(\omega_r, \omega_s)$ and $F_{nK}(\omega_r)$ using the quadrature method induced by the gaussian scheme over the reference element $\tilde{K}$ by means of the mapping $\varphi_{\tilde{K}}$ (2.2), i.e. scheme (3.2) with $\tilde{L}=4$, $\tilde{b}_\ell = \varphi_{\tilde{K}}(\hat{b}_\ell)$, $\hat{b}_\ell$ the four gaussian points in the unit square $\hat{K}$ and $\tilde{\alpha}_\ell = \hat{\alpha}_\ell \det(\varphi_{\tilde{K}}(\tilde{b}_\ell))$, $\hat{\alpha}_\ell$ the corresponding gaussian weights.

This induced method integrates exactly constant functions. In fact, since $\det(\varphi_{\tilde{K}}(\hat{x}))$ is a polynomial of degree one, for $\tilde{\varphi}=c$ (a constant function) we have:

$$\int_{\tilde{K}} \tilde{\varphi}(\tilde{x})\,d\tilde{x} = c.\,\mathrm{area}(\tilde{K}) = c.\int_{\hat{K}} \det(\varphi_{\tilde{K}}(\hat{x}))\,d\hat{x} =$$

$$= c.\sum_{\ell=1}^{4} \hat{\alpha}_\ell \det(\varphi_{\tilde{K}}(\hat{b}_\ell)) = \sum_{\ell=1}^{4} \tilde{\alpha}_\ell \tilde{\varphi}(\tilde{b}_\ell).$$

4. Conclusions.

We have shown that in spite of the theoretical impossibility of assembling triangles as degenerate quadrilaterals, a judicious choice of the quadrature scheme allows us to get a solution of optimal order for the involved finite elements.

It is worth to remark that quadrature schemes of higher precision would give bigger values for the approximation of divergent integrals in (3.7). It would introduce significant rounding errors due to the cancellation carried out during the assembling.

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