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ON SOME LIMIT THEOREMS FOR CONTINUED FRACTIONS

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On some limit theorems for continued fractions (*)

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Abstract. As a consequence of previous results on mixing random variables, some functional limit theorems for quantities related to the continued fraction expansion of a random number in (0,1) are given, including the case of the number of solutions of a certain diophantine inequality.

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§1. Introduction.

The aim of this paper is to collect some remarks about the convergence in distribution of sums of some random variables associated to the continued fraction expansion of a random number ω in (0.1).

As discussed in Section 2, the results in [25],[27] apply directly to the sequence $\{a_j\}$ of partial quotients when ω is chosen under Gauss's measure. If it is replaced by any probability measure absolutely continuous with respect to Lebesgue measure, similar results hold (by [21, Lemma 1]; in the case of Lebesgue measure [12, Lemma 19.4.2] works). Then some theorems of Lévy [19],[20] and Doeblin [7] are obtained as corollaries and some information is added (see Examples 2.6, 2.14 and Remarks 2.7, 2.15 for references). In particular, we get necessary and sufficient conditions on a function f for the validity of a functional limit theorem (invariance principle) for sums $\Sigma_{j<n} f(a_j)$ under Lebesgue measure on (0,1); then a certain class of positive functions f of real argument is examined and we obtain (Corollaries 2.12 and 2.13) functional limit theorems for f regularly varying (and bounded on finite intervals).

In Section 4 we consider sums involving x_j, the complete quotients, and u_j, defined in (4.1), which measure the approximation of ω by its convergents. We extend some results of Section 2 (see Examples 4.1) including functional limit theorems for $\Sigma_{j<n} f(x_j)$ and $\Sigma_{j<n} f(u_j)$ for some regularly varying f; in the case of $\{x_j\}$, Corollary 4.2 generalizes [19, Theorem 4] and Corollary 4.6 contains for a certain class of regularly varying functions a result suggested in [19, pages 200-201]. Example 4.7.2 gives the functional form of a limit theorem indicated by

Doeblin. Lemma 4.5, which is used to deal with u_j, essentially contains the theorem in [15]; the proof given here is based on a relation due to Lévy (Proposition 2.1).

In order to achieve these extensions of the results of Section 2, we isolate from [5] (and [11]) some facts which lead to Corollary 3.4 (see Remark 4.4(a)).

Theorem 5.1 gives the asymptotic normality and even a functional limit theorem for the number of solutions of a certain diophantine inequality which is not covered by [18],[22].

§2. Sums of functions of the partial quotients.

Given an irrational number α, let

$$\alpha = [a_o(\alpha), a_1(\alpha), \ldots]$$

be its (infinite) simple continued fraction expansion, defined by the continued fraction algorithm

$$(2.1) \quad \alpha = a_o(\alpha) + \frac{1}{x_1(\alpha)}, \ldots, x_n(\alpha) = a_n(\alpha) + \frac{1}{x_{n+1}(\alpha)}, \ldots$$

where $a_o(\alpha) = [\alpha]$ (throughout the paper, $[.]$ denotes the integer part of a real number), $a_n(\alpha) = [x_n(\alpha)]$ (we refer to [4,§4] and [10] or [17] for the elementary facts about continued fractions). The a_n's are the partial quotients and the x_n's the complete quotients of α.

We denote by $\mathbb{N}^*$ the set of non-zero natural numbers and $\mathbb{N} = \mathbb{N}^* \cup \{0\}$. Given an integer k_o and $k_1, \ldots, k_N \in \mathbb{N}^*$, $N \in \mathbb{N}^*$, the finite continued fraction $[k_o, \ldots, k_N]$ is defined to be k_o if $N = 0$ (the two senses of $[.]$ coincide) and if $N \geq 1$ it is the rational number defined recursively by the formula $[k_o, \ldots, k_N] = k_o + ([k_1, \ldots, k_N])^{-1}$. If $[k_o, \ldots, k_N] = [k'_o, \ldots, k'_N]$ with $k_i \geq 1$ and $k'_i \geq 1$ for $i = 1, \ldots, N$ then $k_i = k'_i$ for $i = 0, \ldots, N$.

For rational α the algorithm (2.1) terminates at a certain value $N \in \mathbb{N}$ of n ($N = 0$ and $a_o(\alpha) = \alpha$ if α is an integer; $N \geq 1$ and $x_N(\alpha) = a_N(\alpha)$ otherwise) and defines $a_o(\alpha), \ldots, a_N(\alpha)$. Then

$$(2.2) \qquad \alpha = [a_o(\alpha), \ldots, a_N(\alpha)], \quad \text{with } a_N(\alpha) \geq 2 \text{ if } N \geq 1,$$

and we define $a_n(\alpha) = \infty$ if $n > N$.

We are interested in a_j and x_j as functions defined on the set of irrational numbers in $(0,1)$. Denote it by Ω and let $\mathcal{B}$ be the class of its Borel subsets. On $(\Omega,\mathcal{B})$ we will consider the Lebesgue measure λ and Gauss's measure

$$P(B) = \frac{1}{\log 2} \int_B \frac{d\omega}{1+\omega}, \quad B \in \mathcal{B}.$$

If ρ is a probability measure on $(\Omega,\mathcal{B})$ we shall write E_ρ (similarly Var_ρ , Cov_ρ) for the corresponding expectation operator and $\mathcal{L}_\rho(\xi)$ for the law of a random element ξ defined on $(\Omega,\mathcal{B},\rho)$; often we will write $E = E_P$, $\mathcal{L} = \mathcal{L}_P$. If moreover ρ is absolutely continuous with respect to λ we shall write $\rho \ll \lambda$.

Also we will deal with the functions p_n, q_n defined for $\omega \in \Omega$ by

$$p_o(\omega) = 0,\ p_1(\omega) = 1,\ p_n(\omega) = a_n(\omega)p_{n-1}(\omega) + p_{n-2}(\omega) \quad \text{if} \quad n \geq 2,$$

$$q_o(\omega) = 1,\ q_1(\omega) = a_1(\omega),\ q_n(\omega) = a_n(\omega)q_{n-1}(\omega) + q_{n-2}(\omega) \quad \text{if} \quad n \geq 2.$$

For each $\omega \in \Omega$ and $n > 0$, $p_n(\omega)/q_n(\omega) = [0,a_1(\omega),\dots,a_n(\omega)]$ is the <u>n th convergent</u> to ω.

Following Lévy [20, Chapitre IX] we write, for $n \geq 1$ and $\omega \in \Omega$

$$y_n(\omega) := \frac{q_n(\omega)}{q_{n-1}(\omega)} = [a_n(\omega),a_{n-1}(\omega),\dots,a_1(\omega)].$$

It is well known that endowing $(\Omega,\mathcal{B})$ with Gauss's measure P,

$\{a_j : j \geq 1\}$ is a (strictly) stationary and ψ-mixing sequence of r.v.'s with an exponential mixing rate and satisfies the condition $\psi^* < \infty$ ([4, page 50] or [12]; the last fact follows from the right inequality in (4.15) of [4]).

Throughout the paper, we use freely notation and concepts quoted in [25]. The dependence coefficients $\phi(k), \psi(k), \psi^*$ refer to $\{a_j\}$ defined on $(\Omega, \mathcal{B}, P)$.

The following relation, due to Lévy [20, equality (8) in §74] and called the Borel-Lévy formula by Doeblin [7], shall be useful later (the indicated dependence properties of $\{a_j\}$ can be proved starting from it [20]).

2.1. <u>Proposition</u>. If $n \geq 2$, $y = [k_{n-1}, \ldots, k_1]$ with $k_1, \ldots, k_{n-1} \in \mathbb{N}^*$ and $1 \leq a < b$ then

$$\lambda(a < x_n \leq b \,|\, y_{n-1} = y) = \lambda((a,b]) \frac{y(y+1)}{(ya+1)(yb+1)} .$$

(Apart from being stated here in Ω, this is (4.12) of [4] since $T^{n-1} = x_n^{-1}$ if $T\omega = \omega^{-1} - [\omega^{-1}]$ and $\{\omega \in \Omega : y_{n-1}(\omega) = [k_{n-1}, \ldots, k_1]\} = \{\omega \in \Omega : a_1(\omega) = k_1, \ldots, a_{n-1}(\omega) = k_{n-1}\}$).

In order to apply some of the results in [25],[27] it appears to be necessary to verify that $\phi(1) < 1$ and this can be done using Proposition 2.1. But, under the properties of $\{a_j\}$ indicated above, no further argument is needed. The following property was overlooked by us in [25], [27] and is stated by Bradley in [6, page 184]: given a probability space $(X,\mathcal{A},Q)$ and two sub-σ-algebras $\mathcal{M}$, $\mathcal{N}$ of $\mathcal{A}$ we have $\phi := \phi(\mathcal{M},\mathcal{N}) < 1$ if $\psi^* := \psi^*(\mathcal{M},\mathcal{N}) < \infty$ (see for example [27] for the definitions). For the sake of completeness, we show that $\phi < (\psi^*)^{-1}(\psi^*-1)$ if $\psi^* < \infty$ ($\psi^* \geq 1$ always). Assume $\phi > 0$; observe that for each $\varepsilon \in (0,\phi)$ there exist $A \in \mathcal{M}$, $B \in \mathcal{N}$ such that $Q(A) > 0$ and

$$\phi - \varepsilon < (Q(AB) - Q(A)Q(B))/Q(A) < 1 - Q(B)$$

(if $Q(AB) - Q(A)Q(B) < 0$, $Q(AB^c) - Q(A)Q(B^c) = -(Q(AB) - Q(A)Q(B)) > 0$) which implies $Q(B) > 0$ and

$$(1-(\phi-\varepsilon))^{-1}(\phi-\varepsilon) < Q(B)^{-1}(\phi-\varepsilon) < \psi^* - 1.$$

The inequality follows from this. We remark that in a recent preprint Philipp [24] proves the stronger fact that $\psi(1) < 0.8$ for $\{a_j\}$, thus obtaining $\phi(1) < 0.4$.

We fix some notation. In this section, H denotes a real separable Hilbert space with norm $\|\cdot\|$. $D([0,1],H)$ is the Skorohod space (see [5]) of H-valued functions on $[0,1]$ and we shall write $D = D([0,1],\mathbb{R})$. If ν is an infinitely divisible (i.d.) probability measure on H, Q_ν denotes the law on $D([0,1],H)$ of a stochastic process $\xi = \{\xi(t) : t \in [0,1]\}$ with stationary independent increments, trajectories in $D([0,1],H)$, $\xi(0) = 0$ and $\xi(1)$ having law ν.

If $\{X_{nj}\} = \{X_{nj} : j = 1,\ldots,n, n \geq 1\}$ is a double array of H-valued measurable functions on $(\Omega,\mathcal{B})$ we shall consider the property

(*) $\{r_n\} \subset \mathbb{N}^*$, $r_n < n$, $r_n/n \to 0 \Rightarrow \Sigma_{j=1}^{r_n} X_{nj} \to 0$ in measure.

In our first statements we refer directly to some assertions in [25], taking there $B = H$, $j_n = n$, $L = L_P$, $E = E_P$ and replacing the letter f by h to denote functionals.

2.2. Proposition. Let $\{f_n : n \geq 1\}$ be a sequence of functions from $\mathbb{N}$ into H and define $X_{nj} = f_n(a_j)$ if $j = 1,\dots,n, n \geq 1$. Suppose that the following conditions of [25, Corollary 6.5] are satisfied: (1), (2) modified by assuming the existence of the limits only for h in a sequentially w*-dense subset W of H', (3). Then (a) and (b) of that result hold and

(c) for any $\rho << \lambda$ and for every $\tau \in C(\mu)$,

$L_\rho(\xi_n^{(\tau)}) \to_w Q_{\gamma * c_\tau} \text{Pois}\mu$ in $D([0,1],H)$ where $\xi_n^{(\tau)}(t) = \Sigma_{1\leq j\leq [nt]}(X_{nj} - E_P X_{n1\tau})$ $(t \in [0,1])$.

Proof. Use [25, Corollary 6.5], [27, Corollary 3.3(iii)] and Lemma 2.3 below, noting that $\{X_{nj} - EX_{n1\tau}\}$ satisfies (*) (see the proof of [25, Corollary 6.5].■

2.3. Lemma. Let $\rho << \lambda$. Assume $\{f_n\}$, $\{X_{nj}\}$ are as in Proposition 2.2, $\{X_{nj}\}$ satisfying (*).

(a) Let $\xi_n(t) = \Sigma_{1\leq j\leq [nt]} X_{nj}$ $(t \in [0,1])$. If $\{L_P(\xi_n)\}$ or $\{L_\rho(\xi_n)\}$ converges weakly (in $D([0,1],H)$) then both sequences have the same limit.

(b) Part (a) holds with $\Sigma_{j=1}^{n} X_{nj}$ in place of ξ_n.

Proof. (a) Take $\{r_n\}$ as in the definition of (*) with $r_n \to \infty$; write $\tilde{\xi}_n(t) = \Sigma_{r_n < j \le [nt]} X_{nj}$ $(t \in [0,1])$. First we observe that

$$\sup_{t\in[0,1]} \|\xi_n(t)-\tilde{\xi}_n(t)\| = \max_{k\le r_n} \|\Sigma_{j=1}^{k} X_{nj}\| \to 0 \text{ in measure}$$

(this follows from (*) and a well-known maximal inequality quoted, for example, in [25, Proposition 2.2]).

On the other hand, if g is any bounded continuous real function on $D([0,1],H)$, Lemma 1 of [21] shows that $\lim_n (E_P g(\tilde{\xi}_n) - E_\rho g(\tilde{\xi}_n)) = 0$ since $\tilde{\xi}_n$ is $\sigma(a_j;\ j > r_n)$-measurable. ∎

2.4. Proposition. Let $\{f_n\}$ and $\{X_{nj}\}$ be as in Proposition 2.2. Suppose that for some $\rho \ll \lambda$, $\{L_\rho(\Sigma_{j=1}^{n} X_{nj})\}$ converges weakly to a probability measure ν on H.

(I) If $\{X_{nj}\}$ satisfies (*) then ν is i.d. and if $\nu = \delta_{z_\tau} * \gamma * c_\tau \mathrm{Pois}\,\mu$, $\tau \in C(\mu)$, is its Lévy-Khintchine representation, assertions (a)-(c) of [25, Theorem 6.2] hold and also we have (b') of [25, Corollary 6.3] if the second part of (ii) of that result is satisfied.

(II) Let ξ_n be the random function $\xi_n(t) = \Sigma_{1\le j\le [nt]} X_{nj}$ $(t\in[0,1])$ and suppose that $\{L_\lambda(\xi_n)\}$ is relatively compact in $D([0,1],H)$. Then

$\{X_{nj}\}$ satisfies (*), ν is i.d. and $L_\rho(\xi_n) \to_w Q_\nu$.

Proof. (I) Lemma 2.3 and [25].

(II) The argument in [27, Theorem 3.2, proof of (III) $\Rightarrow$ (II)] shows that $\{X_{nj}\}$ satisfies (*). Then use (I), Lemma 2.3 and [27, Theorem 3.2].■

2.5. Remark. In the real valued case, the convergence in law of $\xi_n^{(\tau)}(1)$ in Proposition 2.2 also follows from the main theorem in [16], which improves [3]; it gives necessary and sufficient conditions (under certain preliminary assumptions) even in the non-stationary case. See [27, Remark 3.4.2] for another reference (convergence to stable laws).

Next we give examples which are related to some results in [7].

2.6. Examples.

2.6.1. Let ℓ^2 be the Hilbert space of square summable real sequences and let $\{e_p : p \geq 1\}$ be its canonical orthonormal basis. Define $\Gamma^{(n)} : \Omega \to \ell^2$ by

$$\Gamma_p^{(n)}(\omega) = \text{card}\{j \leq n : a_j(\omega) = p\}, \quad p \geq 1, \omega \in \Omega,$$

and $\gamma = (\gamma_p)_{p\geq 1}$ by

$$(2.3) \qquad \gamma_p = P(a_1 = p) = \frac{1}{\log 2}\log\left(1 + \frac{1}{p(p+2)}\right), \; p \geq 1.$$

Then, if ξ_n is the random function

$$\xi_n(t) = n^{-1/2}(\Gamma^{([nt])} - [nt]\gamma) \qquad (t \in [0,1]),$$

for any $\rho \ll \lambda$ we have $\mathcal{L}_\rho(\xi_n) \to_w Q_\nu$ where ν is the centered Gaussian measure on ℓ^2 whose covariance operator S satisfies

$$(2.4) \qquad (Se_p, e_q) = \delta_{pq}\gamma_p - \gamma_p\gamma_q + 2\sum_{j=1}^{\infty} \{P(a_1=p, a_{j+1}=q) - \gamma_p\gamma_q\},$$

$p \geq 1$, $q \geq 1$; here $\delta_{pq} = 1$ if $p = q$, $= 0$ if $p \neq q$, and $(.,.)$ denotes the inner product of ℓ^2.

Proof. Let $f(p) = e_p$ and take $f_n(p) = n^{-1/2}(f(p)-\gamma)$. Since $E_p\|f(a_1)\|^2 < \infty$, by the same arguments which led from [25, Corollary 4.5] to [25, Corollary 4.7] we can verify that $\{f_n\}$ satisfies the hypotheses of Proposition 2.2 with $\mu = 0$ and $\Phi(h) = \mathrm{Var}_P h(a_1) + 2\sum_{j=1}^{\infty} \mathrm{Cov}_P(h(a_1), h(a_{j+1}))$ (see also [25, Remark on page 405]). Concerning (2.4), we remark that $P(a_1 = p, a_{j+1} = q) = P(a_1 = q, a_{j+1} = p)$ (see [19, page 182]).■

2.6.2. Let $\theta > 0$ and $\alpha \in \mathbb{R}$. For each $n \geq 1$ define ξ_n by

$$\xi_n(t) = n^{-\alpha} \sum_{1 \leq j \leq [nt]} a_j^\alpha I_{\{a_j > \theta n\}} \quad (t \in [0,1]).$$

Then for any $\rho \ll \lambda$, $\mathcal{L}_\rho(\xi_n) \to_w Q_\nu$ where

a) if $\alpha > 0$, $\nu = c_{\theta^\alpha}$ Pois μ with $\mu(dx) = I_{(\theta^\alpha,\infty)}(\alpha \log 2)^{-1} x^{-\frac{1}{\alpha} - 1}\, dx$;

b) if $\alpha < 0$, $\nu = \delta_z * c_{\theta^\alpha}\text{Pois}\,\mu$ with $z = ((1-\alpha)\theta^{1-\alpha}\log 2)^{-1}$,

$$\mu(dx) = I_{(0,\theta^\alpha)}(x)(-\alpha \log 2)^{-1}x^{-\frac{1}{\alpha}-1}\,dx;$$

c) if $\alpha = 0$, then $\xi_n(t) = \text{card}\{j < [nt] : a_j > \theta n\}$ $(t \in [0,1])$ and $\nu = \text{Pois}\,((\theta \log 2)^{-1}\delta_1)$.

Proof. Take $f_n(p) = (p/n)^\alpha I_{(\theta n,\infty)}(p)$ in Proposition 2.2. Condition (1) there, is satisfied with the corresponding μ because, for positive x,

$$(2.5) \qquad P(a_1 > x) = \frac{1}{\log 2}\log(1+\frac{1}{[x]+1}) \sim \frac{1}{\log 2}\cdot\frac{1}{x} \quad \text{as} \quad x \to \infty .$$

On the other hand, observe that if $\alpha > 0$, $X_{n1\delta} = 0$ for $\delta \in (0, \theta^\alpha]$. For b), note that $\sup_n nEX^2_{n1\delta} < \delta^2 \sup_n nP(a_1 > \theta n) = O(\delta^2)$ and that $\lim_n nEX_{n1\theta^\alpha} = z$. If $\alpha = 0$ then $X_{n1\delta} = 0$ for $\delta \in (0,1)$. ∎

2.6.3. Fix a sequence $\{\theta_r\}$ such that $0 < \theta_1 < \theta_2 < \ldots$ and $\lim_r \theta_r = \infty$. Define $L^{(n)} : \Omega \to \ell^2$ by

$$L_r^{(n)}(\omega) = \text{card}\{j < n : \theta_r n < a_j(\omega) < \theta_{r+1}n\}, \quad r \geq 1, \quad \omega \in \Omega$$

and ξ_n by $\xi_n(t) = L^{([nt])}$, $t \in [0,1]$. Then for any $\rho \ll \lambda$, $\mathcal{L}_\rho(\xi_n) \to_w Q_\nu$ where

$$\nu = \text{Pois}\,\mu \quad \text{with} \quad \mu = \Sigma_{r=1}^{\infty}\frac{1}{\log 2}\left(\frac{1}{\theta_r} - \frac{1}{\theta_{r+1}}\right)\delta_{e_r} .$$

Moreover, $(\mathrm{Pois}\,\mu)(F) = 1$ where

$$F = \{ (x_1, x_2, \ldots) \in \ell^2 : x_r \in \mathbb{N} \text{ and only a finite number of } x_r\text{'s is non-zero} \}$$

and

$$(\mathrm{Pois}\,\mu)(\{x\}) = \exp\left(-\frac{1}{\theta_1 \log 2}\right) \prod_{r\geq 1} \frac{1}{x_r!} \left(\frac{1}{\log 2}\left(\frac{1}{\theta_r} - \frac{1}{\theta_{r+1}}\right)\right)^{x_r}$$

if $x \in F$.

<u>Proof</u>. Take $f_n(p) = \sum_{r=1}^{\infty} I_{(\theta_r n, \theta_{r+1} n]}(p) e_r$ in Proposition 2.2. Note that for every $\delta \in (0,1)$, $X_{n1\delta} = 0$ and that for any subset A of H we have

$$\left| (nL(X_{n1}) | B_\delta^c)(A) - \mu(A) \right| < \sum_{r=1}^{\infty} \left| nP(\theta_r n < a_1 < \theta_{r+1} n) - (\log 2)^{-1} (\theta_r^{-1} - \theta_{r+1}^{-1}) \right|$$

which goes to zero as $n \to \infty$ because each term tends to zero and

$$\Sigma_{r=1}^{\infty} nP(\theta_r n < a_1 < \theta_{r+1} n) = nP(a_1 > \theta_1 n) \to (\theta_1 \log 2)^{-1}$$
$$= \Sigma_{r=1}^{\infty} (\log 2)^{-1} (\theta_r^{-1} - \theta_{r+1}^{-1}).$$

The expression for $\mathrm{Pois}\,\mu$ follows by direct calculation of μ^{*n}, $n > 1$. ∎

2.7. Remarks. Example 2.6.1 gives a natural extension of the result in [7, §2, n°5]. The limit laws of $\xi_n(1)$ given in a) and c) of 2.6.2 appear in [7, §4, §3] where a), case $\alpha = 1$, is used for deriving the limit law of $\xi_n(1)$ in Example 2.14.2 below. The proofs presented in [7] of both results have been objected and the last one established in [13] by using [8].

Now we are interested in sums of the form $\Sigma_{j<n} f(a_j)$.

2.8. Proposition. Let f be a function from $\mathbb{N}^*$ into $\mathbb{R}$ and let $\{x(n)\} \subset \mathbb{R}$ and $\{b(n)\} \subset (0,\infty)$ with $b(n) \to \infty$. Assume that for some $\rho << \lambda$, $L_\rho(b(n)^{-1}(\Sigma_1^n f(a_j) - nx(n))) \to_w \nu$, a non degenerate probability measure. Then ν is stable.

Proof. Since $b(n) \to \infty$ we can find $\{r_n\} \subset \mathbb{N}^*$, $r_n < n$, $r_n \to \infty$ such that $b(n)^{-1} \Sigma_1^{r_n} f(a_j) \to 0$ in measure. Arguing as in the proof of Lemma 2.3 we can replace ρ by P in our hypothesis and then [27, Remark 3.4.3.1] or [23, Theorem 2] concludes the proof. ■

A function $R : [r,\infty) \to (0,\infty)$ $(r > 0)$ is *regularly varying (at ∞) with exponent* $\alpha \in \mathbb{R}$ ([28],[2]) if it is Borel measurable and $\lim_{x\to\infty} R(tx)(R(x))^{-1} = t^\alpha$ for every $t > 0$. If $\alpha = 0$, R is *slowly varying*.

2.9. Proposition. Let $f : \mathbb{N}^* \to \mathbb{R}$.

a) Let $\{x(n)\} \subset \mathbb{R}$ and $\{b(n)\} \subset (0,\infty)$ with $b(n) \to \infty$. The following assertions are equivalent:

(I) The random functions ξ_n defined by

$$(2.6) \qquad \xi_n(t) = b(n)^{-1} \Sigma_{1 \le j \le [nt]} (f(a_j) - x(n)) \quad (t \in [0,1])$$

satisfy

$$(2.7) \qquad L_\lambda(\xi_n) \to_w W, \text{ the Wiener measure on } D.$$

(II) $L_\lambda(b(n)^{-1} \Sigma_1^n (f(a_j) - x(n))) \to_w N(0,1)$, the standard normal distribution, and $\{X_{nj}\} := \{b(n)^{-1}(f(a_j) - x(n)) : 1 \le j \le n, n \ge 1\}$ satisfies (*).

b) The assertion

(A) there exist a bounded sequence $\{x(n)\} \subset \mathbb{R}$ and $\{b(n)\} \subset (0,\infty)$ with $b(n) \to \infty$ such that (I) is satisfied, holds if and only if

$$(2.8) \qquad \lim_{x \to \infty} \frac{x^2 \Sigma_{k:|f(k)| > x} k^{-2}}{\Sigma_{k:|f(k)| \le x} f^2(k) k^{-2}} = 0$$

or, equivalently, if

$$(2.9) \qquad U(x) := (\log 2)^{-1} \Sigma_{k:|f(k)| \le x} f^2(k) k^{-2}$$

is slowly varying. If this is the case and $U(x) \to \infty$ as $x \to \infty$, we can take $x(n) = E_P f(a_1)$ and any $\{b(n)\}$ such that $\lim_n nb(n)^{-2} U(b(n)) = 1$.

c) If (I) holds for some $\{x(n)\}$, $\{b(n)\}$, then (2.7) holds with λ replaced by any $\rho \ll \lambda$.

<u>Proof</u>. a) and c): [27, Corollary 3.3(iii)] and Lemma 2.3.

b) First we observe that $U'(x) := E_P(f^2(a_1); |f(a_1)| < x)$ is slowly varying if and only if U is (U and U' both have a finite limit as $x \to \infty$ or both tend to ∞; in the later case, $U \sim U'$ by (2.3)); moreover this holds if and only if (2.8) is satisfied.

Then, that (2.8) implies (I) (with $\{x(n)\}$ and $\{b(n)\}$ as indicated in the case $U(x) \to \infty$) follows from [27, Corollary 3.7] and its proof (see [26]) noting that, with the notation there, $\Phi_1^{(0)} = 1$ and $\Phi_j^{(0)} = 0$ if $j \geq 2$ (use that $\psi^* < \infty$ and [25, Proposition 2.7]).

For the converse, suppose that (II) holds and that $U'(x) \to \infty$ as $x \to \infty$. Fix $\delta \in (0,1)$ and write $Y_{nj\delta} = X_{nj\delta} - E_P X_{nj\delta}$. By (a) and (b) of Proposition 2.4 (I) (or [25, Theorem 4.2]), $\lim_n E(\Sigma_1^n Y_{nj\delta}) = 1$; moreover, $E(\Sigma_1^n Y_{nj\delta})^2 \leq (1+4\Sigma_1^\infty \phi^{1/2}(j))nE(X_{n1}^2; |X_{n1}| < \delta)$ by an inequality of Ibragimov. Then, using that $\{x(n)\}$ is bounded and $b(n) \to \infty$ we obtain

$$\frac{1}{2} \leq M\, n\, b(n)^{-2}\, E(f^2(a_1); |f(a_1)| < b(n))$$

if $n > 1$ for some $M > 0$ and $n_1 \in \mathbb{N}^*$. Since $\lim_n nP(|f(a_1)| > b(n)) = 0$ (use (a) of Proposition 2.4 (I) and that $x(n)/b(n) \to 0$) and $b(n+1)/b(n) \to 0$ we can conclude that $x^2 P(|f(a_1)| > x)(E(f^2(a_1); |f(a_1)| < x))^{-1} \to 0$ as $x \to \infty$, which says that U' is slowly varying. ∎

2.10. Proposition. Let $f : \mathbb{N}^* \to \mathbb{R}$ and $\kappa_1, \kappa_2, \beta$ be such that $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa_1+\kappa_2 > 0$, $\beta \in (0,2)$. Denote by $\nu(\kappa_1,\kappa_2,\beta)$ the stable law c_1 Pois $(\mu(\kappa_1,\kappa_2,\beta))$ with Lévy measure

$$\mu(\kappa_1,\kappa_2,\beta)(dx) = \{ I_{(-\infty,0)}(x)\kappa_2|x|^{-1-\beta} + I_{(0,\infty)}(x)\kappa_1 x^{-1-\beta}\} dx.$$

a) Let $\{x(n)\} \subset \mathbb{R}$ and $\{b(n)\} \subset (0,\infty)$ with $b(n) \to \infty$. The following assertions are equivalent:

(I) ξ_n defined as in (2.6) satisfy

$$L_\lambda(\xi_n) \to_w Q_{\nu(\kappa_1,\kappa_2,\beta)} \quad . \tag{2.10}$$

(II) $L_\lambda(b(n)^{-1}\Sigma_1^n(f(a_j) - x(n))) \to_w \nu(\kappa_1,\kappa_2,\beta)$ and $\{X_{nj}\} :=$ $\{b(n)^{-1}(f(a_j)-x(n)) \; : \; 1 \leq j \leq n, \; n \geq 1\}$ satisfies (*).

b) The assertion

(A) there exist $\{x(n)\} \subset \mathbb{R}$ and $\{b(n)\} \subset (0,\infty)$ with $b(n) \to \infty$ such that (I) of a) is satisfied,

hols if and only if

$$R(x) := \Sigma_{k:|f(k)|>x} k^{-2} \quad \text{is regularly varying with exponent } -\beta, \tag{2.11}$$

$$\lim_{x\to\infty} \frac{\Sigma_{k:f(k)>x} k^{-2}}{\Sigma_{k:|f(k)|>x} k^{-2}} = \frac{\kappa_1}{\kappa_1+\kappa_2}$$

and

$$\lim_{x\to\infty} \frac{\Sigma_{k:f(k)< -x} k^{-2}}{\Sigma_{k:|f(k)|>x} k^{-2}} = \frac{\kappa_2}{\kappa_1+\kappa_2} .$$

If this is the case we can take $x(n) = E_p(f(a_1); |f(a_1)| < b(n))$ and any $\{b(n)\}$ such that $\lim_n nb(n)^{-2}U(b(n)) = (\kappa_1+\kappa_2)(1-\beta)^{-1}$ (with U defined in (2.9)).

c) If (I) holds for some $\{x(n)\},\{b(n)\}$, then (2.10) holds with λ replaced by any $\rho \ll \lambda$.

<u>Proof</u>. b) Assume that (II) holds. Proposition 2.4 implies that $n\mathcal{L}_p(b(n)^{-1}f(a_1))| B_\tau^c \to_w \mu(\kappa_1,\kappa_2,\beta)|B_\tau^c$ for every $\tau > 0$. To conclude the proof of the "only if" part see [2, pages 81 and 84-85] and use (2.3). For the converse, apply Proposition 2.2 and argue as in [2, pages 87-88]. ■

We point out that if $x(n) = nx$ for some $x \in R$ then the condition that $\{X_{nj}\}$ satisfies (*) can be omitted in II of Propositions 2.9 and 2.10 ([23 , Theorem 2] and [27, Remark 3.4.3.1]).

Next we make some remarks about the validity of (2.8) or (2.11) for certain positive functions f of real argument.

Suppose $f : [1,\infty) \to (0,\infty)$ is bounded on finite intervals and $\lim_{x\to\infty} f(x) = \infty$; then the following functions are well defined for $y \in [f(1),\infty)$

$$\bar{f}_0(y) = \inf\{x>1 : f(x)> y\},$$

$$\bar{f}_1(y) = \inf\{x>1 : f(x)> y\},$$

$$\bar{f}_2(y) = \sup\{x>1 : f(x)< y\}.$$

We have $1 < \bar{f}_0 < \bar{f}_1 < \bar{f}_2$; each $\bar{f}_i$ is non-decreasing and $\lim_{x\to\infty} \bar{f}_i(x) = +\infty$ for such an f. We will say that $\underline{f \in F}$ if f is Borel measurable, satisfies the preceding conditions and $\bar{f}_1(y) \sim \bar{f}_2(y)$ as $y \to \infty$.

2.11. Lemma. i) if $f : [1,\infty) \to (0,\infty)$ is non-decreasing and $\lim_{x\to\infty} f(x) = \infty$ then $f \in F$.

ii) If $f : [1,\infty) \to (0,\infty)$ is bounded on finite intervals and regularly varying with exponent $\alpha > 0$ then $f \in F$. Moreover $\bar{f}_0(y) \sim \bar{f}_2(y)$ as $y \to \infty$ and $\bar{f}_i$ is regularly varying with exponent $1/\alpha (i = 0,1,2)$.

iii) If $f \in F$ and $\bar{f}_1$ is regularly varying with exponent $1/\alpha$ for some $\alpha > 0$ then f is regularly varying with exponent α .

Proof. i) $\bar{f}_1 = \bar{f}_2$ if f is non-decreasing.

ii) First we prove that $\bar{f}_0 \sim \bar{f}_2$. We will show that for every $t > 1$ we have $\bar{f}_2(y) < t\,\bar{f}_0(y)$ for all sufficiently large y by using the Karamata representation: $f(x) = x^{\alpha} c(x) \exp\left(\int_1^x s^{-1} \varepsilon(s)\,ds\right)$, c and ε being measurable functions with $\lim_{x\to\infty} c(x) = c > 0$, $\lim_{s\to\infty} \varepsilon(s) = 0$ (see [28]). Fix $t > 1$ and take $r \in (0,1)$ such that $rt^{\alpha/2} > 1$. There exists y_0 such that for every $y > y_0$ we have $\bar{f}_0(y)+1 < t\bar{f}_0(y)$,

$r(t\bar{f}_o(y)(\bar{f}_o(y)+1)^{-1})^{\alpha/2} > 1$, $|\varepsilon(s)| < \alpha/2$ if $s > \bar{f}_o(y)$ and $c(x)/c(x')$ $> r$ if $x, x' > \bar{f}_o(y)$. Then if $y > y_o$ and $x > t\bar{f}_o(y)$, taking x' such that $\bar{f}_o(y) < x' < \bar{f}_o(y)+1$ and $f(x') > y$, we have $f(x)/y > r(x/x')^{\alpha/2} > 1$; this implies that $\bar{f}_2(y) < t\bar{f}_o(y)$ if $y > y_o$.

Now fix $t > 0$. Given $r > 1$, by hypothesis we have $\lim_{y\to\infty} f(r^{-1}t^{-1/\alpha}\bar{f}_1(ty))/f(\bar{f}_1(ty)-1) = r^{-\alpha}t^{-1}$ which implies that, for all sufficiently large y, $f(r^{-1}t^{-1/\alpha}\bar{f}_1(ty)) < t^{-1}f(\bar{f}_1(ty)-1) < y$ by the definition of $\bar{f}_1$ and $r^{-1}t^{-1/\alpha}\bar{f}_1(ty) < \bar{f}_2(y)$ by the definition of $\bar{f}_2$. Then $\limsup_{y\to\infty}\bar{f}_1(ty)/\bar{f}_2(y) < t^{1/\alpha}$. By a similar argument we can deduce from the fact that $\lim_{y\to\infty} f(rt^{1/\alpha}\bar{f}_1(y))/f(\bar{f}_1(y)-1) = r^{\alpha}t$ for each $r \in (0,1)$ that $\liminf_{y\to\infty}\bar{f}_2(ty)/\bar{f}_1(y) > t^{1/\alpha}$. This implies that $\bar{f}_i$ varies regularly with exponent $1/\alpha$ because $\bar{f}_o, \bar{f}_1, \bar{f}_2$ are asymptotically equivalent.

iii) Take $t > 0$. For any $r > 1$, the hypotheses give that $\lim_{x\to\infty}\bar{f}_1(rt^{\alpha}f(x))/\bar{f}_2(f(x)) = r^{1/\alpha}t$ which implies that, for all sufficiently large x, $\bar{f}_1(rt^{\alpha}f(x)) > t\bar{f}_2(f(x)) > tx$ by the definition of $\bar{f}_2$ and $f(tx) < rt^{\alpha}f(x)$ by the definition of $\bar{f}_1$. Then $\limsup_{x\to\infty} f(tx)/f(x) < t^{\alpha}$. On the other hand, $\lim_{x\to\infty}\bar{f}_1(r^{-1}t^{-\alpha}f(tx))/\bar{f}_2(f(tx)) = r^{-1/\alpha}t^{-1}$ for each $r \in (0,1)$ and an analogous argument shows that $\liminf_{x\to\infty} f(tx)/f(x) > t^{\alpha}$. ∎

2.12. <u>Corollary</u>. a) Let $f \in F$. Then assertion (A) of Proposition 2.9 holds if and only if

$$(2.12)\qquad \lim_{x\to\infty} \frac{x^{-1}f^2(x)}{\Sigma_{k:k<x} f^2(k)k^{-2}} = 0.$$

Moreover, in this case U (defined in (2.9)) is asymptotically equivalent to

$$\tilde{U}(x) = (\log 2)^{-1} \Sigma_{k:k<\bar{f}_2(x)} f^2(k)k^{-2} ;$$

here $\bar{f}_2$ can be replaced by $\bar{f}_1$.

b) If $f : [1,\infty) \to (0,\infty)$ is regularly varying with exponent $\alpha = 1/2$ and bounded on finite intervals then (A) of Proposition 2.9 holds.

Proof. a) Assume that f satisfies (2.12). We claim

$$(2.13)\qquad \Sigma_{k<\bar{f}_1(y)} f^2(k)k^{-2} \sim \Sigma_{k<\bar{f}_2(y)+1} f^2(k)k^{-2} \quad \text{as} \quad y\to\infty .$$

Write $g(y) = (\Sigma_{k<\bar{f}_2(y)+1} f^2(k)k^{-2})/(\Sigma_{k<\bar{f}_1(y)} f^2(k)k^{-2})$. Let $\varepsilon \in (0,1/2)$. There exists y_o such that if $y > y_o$ then $f^2(z)z^{-2} < \varepsilon z^{-1}\Sigma_{h<z} f^2(h)h^{-2}$ for $z > \bar{f}_1(y)$ and $\log((\bar{f}_2(y)+1)/(\bar{f}_1(y)-1)) < 2$. Therefore if $y > y_o$

$$\Sigma_{\bar{f}_1(y)<k<\bar{f}_2(y)+1} f^2(k)k^{-2} < 2\varepsilon\, \Sigma_{h<\bar{f}_2(y)+1} f^2(h)h^{-2}$$

which implies $1 < g(y) < 1+2\varepsilon g(y)$, that is $1 < g(y) < (1-2\varepsilon)^{-1}$. This proves (2.13).

By the definitions of $\bar{f}_1$ and $\bar{f}_2$ we have

$$y^2 \Sigma_{k:f(k)>y} k^{-2} < y^2 \Sigma_{k>\bar{f}_1(y)} k^{-2} < \frac{f^2(\bar{f}_2(y)+1)}{\bar{f}_1(y)-1}$$

and $\Sigma_{k:f(k)<y} f^2(k)k^{-2} > \Sigma_{k<\bar{f}_1(y)} f^2(k)k^{-2}$. Then using that $\bar{f}_1 \sim \bar{f}_2$, (2.13) and (2.12) we obtain (2.8). That $U \sim \tilde{U}$ follows from (2.13) and the inequalities

$$\Sigma_{k<\bar{f}_1(y)} f^2(k)k^{-2} < \Sigma_{k:f(k)<y} f^2(k)k^{-2} < \Sigma_{k<\bar{f}_2(y)} f^2(k)k^{-2}.$$

Now suppose that f satisfies (2.8). Write $u(x)$ for the quotient in (2.12). First observe that

$$\Sigma_{k:f(k)<f(x)-1} f^2(k)k^{-2} < \Sigma_{k<x} f^2(k)k^{-2} + 2\,\frac{f^2(x)}{x}$$

$(x > 1)$ and that for some constant C

$$\frac{1}{x} < C\,\Sigma_{k:f(k)>f(x)-1} k^{-2}$$

for all sufficiently large x (by the definitions of $\bar{f}_1$ and $\bar{f}_2$ we have $x > \bar{f}_1(f(x)-1)$ and $(\bar{f}_2(f(x)-1)+1)^{-1} < \Sigma_{k>\bar{f}_2(f(x)-1)} k^{-2} <$

$\Sigma_{k:f(k)>f(x)-1} k^{-2}$; moreover, $\bar{f}_1 \sim \bar{f}_2$ and $f(x) \to \infty$ as $x \to \infty$). Therefore we have for such x's

$$u(x) \leq C \frac{f^2(x) \Sigma_{k:f(k)>f(x)-1} k^{-2}}{\Sigma_{k<x} f^2(k) k^{-2}}$$

$$\leq C(1+2u(x)) \frac{f^2(x) \Sigma_{k:f(k)>f(x)-1} k^{-2}}{\Sigma_{k:f(k)<f(x)-1} f^2(k) k^{-2}} .$$

Since $f(x) \to \infty$ as $x \to \infty$, (2.8) implies that for any $\varepsilon \in (0,(2C)^{-1})$ we have $u(x) \leq C(1+2u(x))\varepsilon$ and hence $u(x) \leq \varepsilon C(1-2\varepsilon C)^{-1}$ for all sufficiently large x. This implies (2.12).

b) Use a), Lemma 2.11(ii) and [2, Chapter 2, Lemma 6.15].■

2.13. Corollary. a) Let $\kappa_1, \kappa_2, \beta$ be as in Proposition 2.10 with $\beta = 1/\alpha$, $\alpha > 1/2$. Let $f \in F$. Then (A) of Proposition 2.10 holds if and only if f is regularly varying with exponent α.

b) Assume $f : [1,\infty) \to (0,\infty)$ is regularly varying with exponent $\alpha > 1/2$, bounded on finite intervals. Let

$$\text{(2.14)} \qquad \nu_\alpha = \begin{cases} \delta_{((\alpha-1)\log 2)^{-1}} * \nu\left(\frac{1}{\alpha \log 2}, 0, \frac{1}{\alpha}\right) & \text{if } \alpha \neq 1 \\ \nu\left(\frac{1}{\log 2}, 0, 1\right) & \text{if } \alpha = 1 \end{cases}$$

($\nu(.,.,.)$ defined as in Proposition 2.10) and define ξ_n by

$$(2.15)\qquad \xi_n(t) = f(n)^{-1}\,\Sigma_{1\le j\le [nt]} f(a_j) \qquad\qquad \text{if} \quad \alpha > 1,$$

$$(2.16)\qquad \xi_n(t) = f(n)^{-1}\,\Sigma_{1\le j\le [nt]}\{f(a_j) - E_P(f(a_1); f(a_1)\le f(n))\} \qquad \text{if} \quad \alpha = 1,$$

$$(2.17)\qquad \xi_n(t) = f(n)^{-1}\,\Sigma_{1\le j\le [nt]}\{f(a_j) - E_P f(a_1)\} \quad \text{if} \quad \tfrac{1}{2} < \alpha < 1.$$

Then for any $\rho \ll \lambda$, $L_\rho(\xi_n) \to_w Q_{\nu_\alpha}$.

Proof. a) Since $f \in F$, by Lemma 2.11 it is sufficient to show

$$(2.18)\qquad \Sigma_{k:f(k)>x}\, k^{-2} \sim \frac{1}{\bar{f}_1(x)} \quad \text{as} \quad x \to \infty.$$

By the definitions of $\bar{f}_1$ and $\bar{f}_2$

$$1 \le \frac{\Sigma_{k:f(k)>x}\, k^{-2}}{\Sigma_{k>\bar{f}_2(x)}\, k^{-2}} \le 1 + \frac{\Sigma_{k:\bar{f}_1(x)\le k\le \bar{f}_2(x)}\, k^{-2}}{\Sigma_{k>\bar{f}_2(x)}\, k^{-2}} = 1 + v(x) \ \text{(say)};$$

moreover $\Sigma_{k:\bar{f}_1(x)\le k\le\bar{f}_2(x)}\, k^{-2} \le (\bar{f}_1(x)-1)^{-1} - (\bar{f}_2(x))^{-1}$ and $\Sigma_{k>\bar{f}_2(x)}\, k^{-2} \ge (\bar{f}_2(x)+1)^{-1}$. Then $\lim_{x\to\infty} v(x) = 0$ and (2.18) holds since $\Sigma_{k>\bar{f}_2(x)}\, k^{-2} \sim (\bar{f}_2(x))^{-1} \sim (\bar{f}_1(x))^{-1}$ as $x \to \infty$.

b) From [2, Chapter 2, Lemma 6.15] we obtain

$$\lim_{x\to\infty} \frac{\Sigma_{k<x}\, f^2(k)k^{-2}}{x^{-1}f^2(x)} = \frac{1}{2\alpha-1}. \tag{2.19}$$

On the other hand

$$1 < \frac{\Sigma_{k:f(k)<x}\ f^2(k)k^{-2}}{\Sigma_{k<\bar f_1(x)}\ f^2(k)k^{-2}} < \frac{\Sigma_{k<\bar f_2(x)}\ f^2(k)k^{-2}}{\Sigma_{k<\bar f_1(x)}\ f^2(k)k^{-2}}$$

which by (2.19) goes to one as $x\to\infty$ because $\bar f_1 \sim \bar f_2$ and f is regularly varying. Then by (2.9) and (2.19)

$$\frac{n}{f^2(n)}\, U(f(n)) \sim \frac{1}{\log 2}\,\frac{n}{f^2(n)}\ \Sigma_{k<\bar f_1(f(n))} f^2(k)k^{-2} \sim \frac{1}{(2\alpha-1)\log 2}$$

because f is regularly varying and $\bar f_1(f(n)) \sim n$ as $n\to\infty$ (observe that $\bar f_1(f(n))(\bar f_2(f(n)))^{-1} < \bar f_1(f(n))n^{-1} < \bar f_1(f(n))\bar f_1(f(n)-1)$ and $\bar f_1$ is regularly varying). Therefore we can take $b(n) = f(n)$, $\kappa_1 = (\alpha \log 2)^{-1}$, $\kappa_2 = 0$ in (2.10). ∎

This result implies that if f satisfies the assumptions in b) with $\alpha\in(1/2,1)$ then

$$L_p\Big(\frac{1}{n}\operatorname{card}\{k<n : \tfrac{1}{k}\Sigma_1^k f(a_j) > E_p f(a_1)\}\Big)$$

converges to the law given in [1, Theorem 5.2] (observe that for such an α, ν_α is strictly stable and satisfies $0<\nu_\alpha((0,\infty))<1$ - use [9, Chapter IV, §1, Theorem 7]).

2.14. Examples.

2.14.1. Let $f(x) = x^{1/2}$ and take $b(n) = (n \log n/\log 2)^{1/2}$, $x(n) = E_p(a_1^{1/2})$ in (2.6). Then (2.7) holds with λ replaced by any $\rho << \lambda$ (observe that $\tilde{U}(x) \sim (\log 2)^{-1}/2 \log x)$.

2.14.2. If ξ_n is defined by

$$(2.20) \qquad \xi_n(t) = \frac{1}{n} \Sigma_{1<j<[nt]}\{a_j - \frac{\log n}{\log 2}\} \qquad (t \in [0,1]),$$

then for any $\rho << \lambda$, $L_\rho(\xi_n) \to_w Q_\nu$, where

$$(2.21) \qquad \nu' = \delta_x * \nu(\frac{1}{\log 2}, 0, 1)$$

with $x = \lim_n \frac{1}{\log 2} (\Sigma_{k=1}^n k \log(1 + \frac{1}{k(k+2)}) - \log n)$.

As a consequence, if $L(\xi) = Q_{\nu'}$,

$$L_\rho(\frac{1}{n} \operatorname{card}\{k < n : \frac{1}{k}\Sigma_1^k a_j > \frac{\log n}{\log 2}\}) \to_w L(\lambda\{t \in [0,1] : \xi(t) > 0\}) = \sigma$$

(say). We do not know an explicit expression for σ (observe that ν' is not strictly stable; on the other hand, [9, Chapter IV, §1, Theorem 7] shows that $\sigma \neq \delta_0$, $\sigma \neq \delta_1$).

2.14.3. Let $\alpha > 1/2$ and $c > 0$ with $c\alpha > (\alpha^2+1)^{1/2}$; then $f(x) = x^\alpha(c + \operatorname{sen}(\log x))$ belongs to F and is not regularly varying. Hence if $\alpha > 1/2$, f does not verify (A) of Proposition 2.10 (this is related to [19, footnote on page 199]). If $\alpha = 1/2$, f satisfies (2.12) and (A) of Proposition (2.9) holds with $x(n) = E_p f(a_1)$, $b(n) = ((c^2-\frac{1}{2})(\log 2)^{-1} n \log n)^{1/2}$ (we have $\tilde{U}(x) \sim (c^2-\frac{1}{2})(\log 2)^{-1}\log(f^{-1}(x))$; writing $h(x) = \log(f^{-1}(x))$ we obtain $h(x) + 2\log(c+\text{sen}\, h(x)) = 2\log x$ which implies $h(x) \sim 2\log x$. Then $\tilde{U}(x) \sim (2c^2-1)(\log 2)^{-1}\log x)$.

2.15. __Remarks__. Lévy [19] proves the convergence of $\mathcal{L}_\rho(\xi_n(1))$ of Corollary 2.13 for non-decreasing regularly varying functions (see also [20, Chapitre IX]); the case $f(x) = x$ (which improves a result of Khintchine [14, page 377]) was also given by Doeblin [7] (for $\rho = \lambda$) and by Philipp [24] (using [25]). The assertion that $\mathcal{L}_\lambda(\xi_n(1))$ of Example 2.14.1 converges to the normal law is stated in [20, Chapitre IX] without indicating the norming constants.

§3. Comparison with other sums.

Throughout this section, $\{\eta_{nj} : 1 \leq j \leq n,\ n \geq 1\}$ denotes a double array of measurable real functions on $(\Omega, \mathcal{B})$.

Define

$$(3.1)\qquad \mathcal{M}_{j\ell} = \begin{cases} \sigma(a_{j-\ell}, \dots, a_{j+\ell}) & \text{if } j-\ell \geq 1 \\ \sigma(a_1, \dots, a_{j+\ell}) & \text{if } j-\ell < 1. \end{cases}$$

For the proof of the following inequality see [5, pages 188-190].

3.1. Lemma. Assume $E_P\, \eta_{nj}^2 < \infty$ for all n, j. If

$$(3.2)\qquad \mu_n(p) := \Sigma_{\ell=p}^{\infty} \max_{1 \leq j \leq n} E_P^{1/2} (\eta_{nj} - \eta_{nj\ell})^2$$

where $\eta_{nj\ell} := E_P(\eta_{nj} \mid \mathcal{M}_{j\ell})$, and

$$\beta_n(p, \varepsilon) := \max_{0 \leq k \leq n-2p} P(\Sigma_{j=k+1}^{k+2p} |\eta_{nj}| > \varepsilon)$$

then for any $\varepsilon > 0$, $n \geq 1$, $1 \leq p \leq n/2$ we have

$$P(\max_{1 \leq i \leq n} |\Sigma_{j=1}^{i} \eta_{nj}| > 6\varepsilon)$$

$$\leq \phi(2p) + 4(2/\varepsilon)^2\, n\mu_n^2(p) + 4n\beta_n(p, \varepsilon/2)$$

$$+ 2 \max_{1 \leq i \leq n} P(|\Sigma_{j=i}^{n} \eta_{nj}| > \varepsilon).$$

3.2. Lemma. Assume

(1) $E_P\, \eta_{nj}^2 < \infty$ for all n,j.

(2) $\lim_p \sup_n n\mu_n^2(p) = 0$ (μ_n defined in (3.2)).

(3) $\lim_n n \max_{1<j<n} P(|\eta_{nj}|>\varepsilon) = 0$ for each $\varepsilon > 0$.

(4) $\lim_n \max_{1<i<n} P(|\Sigma_{j=i}^n \eta_{nj}|> \varepsilon) = 0$ for each $\varepsilon > 0$.

Then $\max_{1<i<n} |\Sigma_{j=1}^i \eta_{nj}| \to 0$ in measure.

Proof. Let $\varepsilon > 0$. By Lemma 3.1 it suffices to find $p_n \to \infty$, $p_n < n/2$ such that $\lim_n n\beta_n(p_n,\varepsilon) = 0$. This can be obtained from (3), noting that $n\beta_n(p,\varepsilon) < 2pn \max_{j<n} P(|\eta_{nj}| > \varepsilon/(2p))$ for each p (this is an argument in [5, page 175]). ■

3.3. Proposition. Assume

(1) $E_P\, \eta_{nj}^2 < \infty$, $E_P\, \eta_{nj} = 0$ for all n,j.

(2) $\lim_n n \max_{1<j<n} E_P\, \eta_{nj}^2 = 0$.

(3) $\lim_p \sup_n n\mu_n^2(p) = 0$ (μ_n defined in (3.2)).

Then $\max_{1<i<n} |\Sigma_{j=1}^i \eta_{nj}| \to 0$ in measure.

Proof. In order to verify that (4) of Lemma 3.2 holds it is sufficient to show that

$$(3.3) \qquad \lim_n \max_{1 \le i \le n} E_P(\Sigma_{j=i}^n \eta_{nj})^2 = 0.$$

Write $M_n = \max_{j \le n} E_P \eta_{nj}^2$ and $\nu_{nj}(\ell) = E_P(\eta_{nj} - \eta_{nj\ell})^2$. Let $1 \le j < k \le n$ with $k-j > 3$. If $\ell = [(k-j)/3]$ arguing as in [11, page 369] (or [5, page 185]) by conditioning with respect to $M_{j\ell}$ and $M_{k\ell}$ we obtain

$$\begin{aligned} |E_P \eta_{nj} \eta_{nk}| &\le 2\phi^{1/2}(\left[\tfrac{k-j}{3}\right]) M_n \\ &+ 2(M_n \max_{i \le n} \nu_{ni}(\left[\tfrac{k-j}{3}\right]))^{1/2} \\ &+ \max_{i \le n} \nu_{ni}(\left[\tfrac{k-j}{3}\right]). \end{aligned}$$

Therefore, writing $K_{no} = K_{n1} = K_{n2} = M_n$ and

$$\begin{aligned} K_{nh} &= 2\phi^{1/2}(\left[\tfrac{h}{3}\right]) M_n + 2(M_n \max_{i \le n} \nu_{ni}(\left[\tfrac{h}{3}\right]))^{1/2} \\ &+ \max_{i \le n} \nu_{ni}(\left[\tfrac{h}{3}\right]) \end{aligned}$$

for $h > 3$, we get $|E_P \eta_{nj} \eta_{nk}| \le K_{n,k-j}$ if $1 \le j < k \le n$. Then if $1 \le i \le n$

$$\begin{aligned} E_P(\Sigma_{j=i}^n \eta_{nj})^2 &\le n(K_{no} + 2\Sigma_{h=1}^\infty K_{nh}) \\ &= 5nM_n + 4nM_n \Sigma_{h=3}^\infty \phi^{1/2}(\left[\tfrac{h}{3}\right]) \\ &+ 4(nM_n)^{1/2} n^{1/2} \Sigma_{h=3}^\infty \max_{i \le n} \nu_{ni}^{1/2}(\left[\tfrac{h}{3}\right]) \end{aligned}$$

$$+ 2n \Sigma_{h=3}^{\infty} \max_{i \leq n} \nu_{ni} \left(\left[\frac{h}{3}\right]\right).$$

From this one can obtain (3.3). ■

We will use only the following.

3.4. Corollary. Let $\{\eta_j : j \geq 1\}$ be a sequence of measurable real functions on (Ω, E) and $\{b(n) : n \geq 1\} \subset (0,\infty)$. Assume

(1) $\sup_j E_P \eta_j^2 < \infty$ y $E_P \eta_j = 0$ for every $j \geq 1$.

(2) $\lim_n nb(n)^{-2} = 0$.

(3) $\Sigma_{\ell=1}^{\infty} \sup_j E_P^{1/2} (\eta_j - E_P(\eta_j \mid M_{j\ell}))^2 < \infty$.

Then $\max_{1 \leq i \leq n} |b(n)^{-1} \Sigma_{j=1}^{i} \eta_j| \to 0$ in measure.

Proof. Write $\eta_{nj} = b(n)^{-1} \eta_j$ and observe that

$$\sup_n n\mu_n^2(p)$$

$$\leq (\sup_n nb(n)^{-2}) (\Sigma_{\ell=p}^{\infty} \sup_j E_P^{1/2} (\eta_j - E_P(\eta_j \mid M_{j\ell}))^2)^2. \blacksquare$$

§4. Complete quotients and the sequence $\{u_j\}$.

Following Doeblin [7, page 365] we write for $\omega \in \Omega$, $j \geq 1$,

$$\frac{1}{u_j(\omega)} = \left| \omega - \frac{p_{j-1}(\omega)}{q_{j-1}(\omega)} \right| q_{j-1}^2(\omega). \tag{4.1}$$

Then $u_1(\omega) = x_1(\omega)$ and $u_j(\omega) = x_j(\omega) + (y_{j-1}(\omega))^{-1}$ if $j \geq 2$.

We will try to extend some results of §2 to $\{x_j\}$ and $\{u_j\}$. In our first statements, if ξ is a random element defined in terms of the a_j's, $\tilde{\xi}$ denotes that one obtained by replacing the a_j's by the x_j's; $\tilde{\tilde{\xi}}$ is similarly defined when considering the u_j's. For instance, if ξ_n is as in Example 2.6.2(c), $\tilde{\tilde{\xi}}_n(t) = \text{card}\{j \leq [nt] : u_j > \theta n\}$.

4.1. Examples.

4.1.1. Let θ, α and ξ_n be as in Example 2.6.2. Then the conclusion there, remains valid for $\tilde{\xi}_n$ and $\tilde{\tilde{\xi}}_n$.

Proof. We have $\sup_{t \in [0,1]} |\tilde{\tilde{\xi}}_n(t) - \xi_n(t)| \leq \Sigma_{j=1}^n |\eta_{nj}|$ where

$$\eta_{nj} = n^{-\alpha}(u_j^\alpha I_{\{u_j > \theta n\}} - a_j^\alpha I_{\{a_j > \theta n\}}).$$

Write

$$\Sigma_{j=1}^n |\eta_{nj}| \leq n^{-\alpha} \Sigma_{j=1}^n u_j^\alpha I_{\{u_j > \theta n, a_j \leq \theta n\}}$$

$$+ n^{-\alpha} \Sigma_{j=1}^n |u_j^\alpha - a_j^\alpha| I_{\{a_j > \theta n\}} = X_n + Y_n \quad \text{(say)}.$$

Note that $P(X_n > 0) < nP(\theta n-2 < a_1 < \theta n) \to 0$ (observe that $a_j < u_j < a_j+2$) and, since $Y_n < c_\alpha n^{-1} n^{-(\alpha-1)} \Sigma_{j<n} a_j^{\alpha-1} I_{\{a_j>\theta n\}}$ with $c_\alpha = 2\alpha 3^{\alpha-1}$ if $\alpha > 1$, $= 2|\alpha|$ if $\alpha < 1$, 2.6.2 shows that $Y_n \to 0$ in measure. The proof for $\tilde{\xi}_n$ is similar. ∎

4.1.2. The statement of Example 2.6.3 is true if we put everywhere $\sim$ (or $\approx$) over the random elements there.

<u>Proof</u>. (Case $\approx{\xi}_n$) Let $\eta_{nj} = f_n(u_j) - f_n(a_j)$, f_n being defined as in Example 2.6.3; it is sufficient to show that $\Sigma_{j=1}^n \|\eta_{nj}\| \to_P 0$. We have

$$\|\eta_{nj}\|^2 < 2\Sigma_{r=1}^{\infty}(I_{A_{njr}} + I_{B_{njr}}) \quad \text{where}$$

$$A_{njr} = \{\theta_r n < u_j < \theta_{r+1} n,\ a_j < \theta_r n\} \quad ,$$

$$B_{njr} = \{\theta_r n < a_j < \theta_{r+1} n,\ a_j > \theta_{r+1}\ n-2\} \ .$$

Then $\Sigma_{j=1}^n P(\|\eta_{nj}\| > 0) < \Sigma_{j=1}^n \Sigma_{r=1}^{\infty} P(A_{njr}) + \Sigma_{j=1}^n \Sigma_{r=1}^{\infty} P(B_{njr})$

$$= \alpha_n + \beta_n \quad \text{(say)}.$$

Writing $\alpha_{nr} = \Sigma_{j=1}^n P(A_{njr})$ we have $\alpha_n = \Sigma_{r=1}^{\infty} \alpha_{nr}$. Note that $\alpha_{nr} < nP(\theta_r n-2 < a_1 < \theta_r n) \to 0$ as $n \to \infty$ for each r. Moreover, given $r_o > 1$,

$$\Sigma_{r>r_o} \alpha_{nr} < \Sigma_{j=1}^n P(u_j > \theta_{r_o} n) < nP(a_1 > \theta_{r_o} n-2)$$

which tends to $(\theta_{r_o} \log 2)^{-1}$ as $n \to \infty$. Then for every $r_o > 1$, $\limsup_n \alpha_n < (\theta_{r_o} \log 2)^{-1}$; hence $\lim_n \alpha_n = 0$. Analogously $\lim_n \beta_n = 0$. ■

Now we turn to sums of the form $\Sigma_1^n f(x_j), \Sigma_1^n f(u_j)$. From Corollary 2.13 we obtain

4.2. Corollary. Assume f is as in Corollary 2.13(b) with $\alpha > 1$. If ξ_n is defined by (2.15) then $\tilde{\xi}_n$ and $\tilde{\tilde{\xi}}_n$ satisfy the conclusion there.

Proof. (Case $\tilde{\tilde{\xi}}_n$) We will show that $X_n := f(n)^{-1} \Sigma_{j=1}^n |f(u_j) - f(a_j)| \to_P 0$. Write $f(x) = x^\alpha L(x)$, L being slowly varying. We have

$$X_n < f(n)^{-1} \Sigma_{j=1}^n u_j^\alpha |L(u_j) - L(a_j)| + f(n)^{-1} \Sigma_{j=1}^n (u_j^\alpha - a_j^\alpha) L(a_j)$$

$$= X_{n1} + X_{n2} \quad \text{(say)}.$$

Since $u_j^\alpha - a_j^\alpha < \alpha 3^{\alpha-1} a_j^{\alpha-1}$, then $X_{n2} \to_P 0$ will follow if we show that $M_{n2} := f(n)^{-1} \Sigma_{j=1}^n a_j^{-1} f(a_j) \to_P 0$. Observe that if $K > 1$

$$M_{n2} < \left(\max_{i \in \{1,\ldots,K\}} \frac{f(i)}{i}\right) \frac{n}{f(n)} + \frac{1}{K} \frac{1}{f(n)} \Sigma_{j=1}^n f(a_j)$$

(write $1 = I_{\{a_j \le K\}} + I_{\{a_j > K\}}$ in each term). Then given $\varepsilon > 0$ we conclude by Corollary 2.13 that for every $K > 1$

$$\overline{\lim}_n P(M_{n2} > \varepsilon) < \overline{\lim}_n P(\xi_n(1) > K(\varepsilon/2))$$

$$< \nu_\alpha(\{x : x > K(\varepsilon/2)\})$$

which goes to zero as $K \to \infty$. Hence $M_{n2} \to_P 0$.

On the other hand, for each $K > 1$

$$X_{n1} < (1+3^{\alpha})C_K \frac{n}{f(n)} + \frac{3^{\alpha}}{f(n)} \Sigma_{j=1}^{n} f(a_j) \left| \frac{L(u_j)}{L(a_j)} - 1 \right| I_{\{a_j > K\}}$$

where $C_K = \sup_{1<x<K+2} f(x)$ (finite by hypothesis). Let $\varepsilon > 0$. Given $\eta > 0$ take $K > 1$ such that $\sup_{0<s<2} |L(x)^{-1}L(x+s)-1| < \eta$ if $x > K$ (possible by the Karamata representation of L); then

$$\overline{\lim}_n P(X_{n1} > \varepsilon) < \overline{\lim}_n P(\xi_n(1) > (\eta 3^{\alpha})^{-1}(\varepsilon/2))$$

$$< \nu_{\alpha}(\{x : x > (\eta 3^{\alpha})^{-1}(\varepsilon/2)\})$$

which tends to zero as $\eta \to 0$. Then $X_{n1} \to_P 0$.■

4.3. Lemma. Assume $f : [1,\infty) \to (0,\infty)$ is Borel measurable and satisfies

(4.2) there exist $r > 0$ and $M : \mathbb{N}^* \to [0,\infty)$ with $E_P M^2(a_1) < \infty$ such that for every $k \in \mathbb{N}^*$, $|f(x)-f(y)| < M(k)|x-y|^r$ if $x,y \in [k,k+2]$.

Let $\{b(n)\} \subset (0,\infty)$ such that $\lim_n nb(n)^{-2} = 0$. Then if $\eta_j = f(u_j)-f(a_j)-E_P(f(u_j)-f(a_j))$ $(j > 1)$ we have $\max_{1<i<n} |b(n)^{-1} \Sigma_{j=1}^{i} \eta_j| \to 0$ in measure. The same result holds if we replace everywhere u_j by x_j in the definition of η_j.

<u>Proof</u>. (Case $\{u_j\}$) Since $|f(u_j)-f(a_j)| < 2^r M(a_j)$, (1) and (2) of Corollary 3.4 are satisfied; it remains to verify (3). First, fix $j \geq 1$, $\ell \geq 1$, and $k_{j-\ell},\ldots,k_{j+\ell} \in \mathbb{N}^*$ and take $\omega,\omega' \in \Delta := \Delta_{j\ell}(k_{j-\ell},\ldots,k_{j+\ell})$ where

$$(4.3) \qquad \Delta_{j\ell}(k_{j-\ell},\ldots,k_{j+\ell}) = \begin{cases} \{a_{j-\ell} = k_{j-\ell},\ldots,a_{j+\ell} = k_{j+\ell}\} & \text{if } j-\ell \geq 1 \\ \{a_1 = k_1,\ldots,a_{j+\ell} = k_{j+\ell}\} & \text{if } j-\ell < 1; \end{cases}$$

we claim that

$$(4.4) \qquad |u_j(\omega) - u_j(\omega')| < 6\ 2^{-\ell}.$$

We have

$$|x_j(\omega)-x_j(\omega')| = |a_j(\omega) + x_{j+1}(\omega)^{-1} - (a_j(\omega') + x_{j+1}(\omega')^{-1})|$$

$$= |x_{j+1}(\omega)^{-1} - x_{j+1}(\omega')^{-1}| < 2.2^{-\ell}$$

because $x_{j+1}(\omega)^{-1} = [0,k_{j+1},\ldots,k_{j+\ell},a_{j+\ell+1}(\omega),\ldots]$ and $x_{j+1}(\omega')^{-1} = [0,k_{j+1},\ldots,k_{j+\ell},a_{j+\ell+1}(\omega'),\ldots]$ both are in the fundamental interval of rank ℓ, $\{\alpha \in [0,1) : a_1(\alpha) = k_{j+1},\ldots,a_\ell(\alpha) = k_{j+\ell}\}$ whose length is less than $2^{-(\ell-1)}$. This proves (4.4) when $j = 1$. Now suppose $j \geq 2$ and recall that $u_j = x_j + (y_{j-1})^{-1}$. If $j-\ell < 1$ we have $y_{j-1}(\omega)^{-1} = [0,k_{j-1},\ldots,k_1] = y_{j-1}(\omega')^{-1}$ and (4.4) holds. Suppose $j-\ell \geq 2$. If $j=3$ merely observe that $\ell = 1$ and $|y_{j-1}(\omega)^{-1} - y_{j-1}(\omega')^{-1}| < 1 < 4\ 2^{-\ell}$.

If $j \geq 4$ then, writing

$$\tilde{\omega} = y_{j-1}(\omega)^{-1} = [0, k_{j-1}, \ldots, k_{j-\ell}, a_{j-\ell-1}(\omega), \ldots, a_1(\omega)]$$

and

$$\tilde{\omega}' = y_{j-1}(\omega')^{-1} = [0, k_{j-1}, \ldots, k_{j-\ell}, a_{j-\ell-1}(\omega'), \ldots, a_1(\omega')],$$

we conclude that

$$a_o(\tilde{\omega}) = 0 = a_o(\tilde{\omega}'), \; a_i(\tilde{\omega}) = k_{j-i} = a_i(\tilde{\omega}') \quad \text{if} \quad 1 \leq i \leq \ell-1$$

(we have used the following fact, whose proof involves (2.2): if $\alpha = [k_o, \ldots, k_N]$ with $k_o \in \mathbb{Z}$, $k_1, \ldots, k_N \in \mathbb{N}^*$ and $N \geq 2$ then $a_i(\alpha) = k_i$ if $0 \leq i \leq N-2$). Thus $\tilde{\omega}$ and $\tilde{\omega}'$ both belong to $\{\alpha \in [0,1) : a_1(\alpha) = k_{j-1}, \ldots, a_{\ell-1}(\alpha) = k_{j-(\ell-1)}\}$ whose length is $\leq 2^{-(\ell-2)}$. Therefore $|y_{j-1}(\omega)^{-1} - y_{j-1}(\omega')^{-1}| \leq 4\ 2^{-\ell}$ and (4.4) holds. By (4.2) we obtain

$$|\eta_j(\omega) - \eta_j(\omega')| \leq 6^r (2^r)^{-\ell} M(a_j(\omega));$$

this implies

$$\begin{aligned} \left|\eta_j(\omega) - \frac{1}{P(\Delta)} \int_\Delta \eta_j dP\right| &= \left|\frac{1}{P(\Delta)} \int_\Delta (\eta_j(\omega) - \eta_j) dP\right| \\ &\leq 6^r (2^r)^{-\ell} M(a_j(\omega)). \end{aligned}$$

Hence

$$E_P^{1/2}(\eta_j - E_P(\eta_j |\ \mathcal{M}_{j\ell}))^2 \leq 6^r (2^r)^{-\ell} E_P^{1/2}\ M^2(a_1)$$

for every $\ell \geq 1$, $j \geq 1$ and (3) of Corollary 3.4 is verified. ∎

4.4. Remarks. (a) In the proof of the preceding lemma, $\{x_j\}$ case, Corollary 3.4 can be replaced by the (functional version of the) theorem in [5, page 192] (consider the function $\tilde{f}(\omega) = f(\omega^{-1})-f([\omega^{-1}])$).

(b) Let $\alpha > 1/2$ and $c > 0$. The function $f(x) = x^{\alpha}(c-(\log x)^{-1/2}\cos((\pi/3)^{x-[x]}))$ is regularly varying with exponent α but does not satisfy (4.2) (for some $b > 0$, $f'(x) > bx^{\alpha}(\log x)^{-1/2}$ if $x \in (k,k+1)$ and $k > 1$; then $E_p(f(x_1)-f(a_1))^2 = \infty$). Hence (K_o) below is not satisfied; we do not know whether the law of $\Sigma_1^n f(x_j)$, suitable normalized, converges.

For x_j we have $L_p(x_j)(dt) = I_{(1,\infty)}(t)(t(t+1)\log 2)^{-1}dt$ for every j. For u_j the next result is useful. (4.6) is proved in [7, page 365] and (4.7) is (apart from the specification of r) a reformulation of the theorem in [15]; by (a) both are consequences of a result of Lévy. Our proof follows an indication in [7]. We use (a) in the proof of Theorem 5.1.

4.5. Lemma. Denote $G_n(t) = \lambda(y_n > t)$ for real t and $n > 1$.

(a) If $n > 2$ then

$$H_n(t) := \lambda(u_n < t) = \begin{cases} \frac{1}{t}\int_0^{t-1} G_{n-1}(\frac{1}{s})ds & \text{if } t \geq 1 \\ 0 & \text{if } t < 1, \end{cases}$$

and

$$h_n(t) = I_{(1,\infty)}(t)\{t^{-1}G_{n-1}((t-1)^{-1})-t^{-2}\int_0^{t-1} G_{n-1}(s^{-1})ds\}$$

is a density function for $L_\lambda(u_n)$; moreover $H_n(t) = 1-t^{-1}(1+E_\lambda(1/y_{n-1}))$ if $t > 2$ and $h_n(t) = t^{-2}(1+E_\lambda(1/y_{n-1}))$ if $t > 2$.

(b) Let H be the distribution function with density

$$(4.5) \qquad h(t) = (\log 2)^{-1}\{I_{(1,2]}(t)t^{-1}(1-t^{-1})+I_{(2,\infty)}(t)t^{-2}\}.$$

Then there exists $r \in (0,1)$ such that

$$(4.6) \qquad \sup_t |H_n(t)-H(t)| = O(r^n),$$

$$(4.7) \qquad \sup_t |h_n(t)-h(t)| = O(r^n).$$

Proof. (a) Let $n > 2$. By Proposition 2.1 we have if $1 < t < 2$

$$H_n(t) = \Sigma_y \lambda(x_n + \frac{1}{y_{n-1}} < t | y_{n-1} = y)\lambda(y_{n-1} = y)$$

$$= \Sigma_{y:y>(t-1)^{-1}} \lambda(1 < x_n < t-y^{-1} | y_{n-1} = y)\lambda(y_{n-1} = y)$$

$$= \Sigma_{y:y>(t-1)^{-1}} (1-t^{-1}-t^{-1}y^{-1})\lambda(y_{n-1} = y)$$

$$= (1-t^{-1})\lambda(y_{n-1} > (t-1)^{-1}) - t^{-1}E_\lambda(\frac{1}{y_{n-1}}; y_{n-1} > (t-1)^{-1})$$

and if $t > 2$

$$\lambda(2 < u_n < t) = \Sigma_y \lambda(2-y^{-1} < x_n < t-y^{-1} | y_{n-1} = y)\lambda(y_{n-1} = y)$$

$$= (2^{-1}-t^{-1})\Sigma_y (1+y^{-1})\lambda(y_{n-1} = y)$$

$$= (2^{-1}-t^{-1})(1+E_\lambda(1/y_{n-1})).$$

But an integration by parts shows that

$$E_\lambda(\frac{1}{y_{n-1}} \; ; \; \frac{1}{y_{n-1}} < t-1) = (t-1)\lambda(\frac{1}{y_{n-1}} < t-1) - \int_0^{t-1} \lambda(\frac{1}{y_{n-1}} < s)ds$$

if $t > 1$, which implies (since $y_{n-1} > 1$)

$$E_\lambda(\frac{1}{y_{n-1}}) = 1 - \int_0^1 \lambda(\frac{1}{y_{n-1}} < s)ds.$$

From the preceding relations we can easily obtain the indicated expressions for H_n. The property of h_n follows from the equality

$$\int_1^u (\int_0^{t-1} t^{-2}\lambda(\frac{1}{y_{n-1}} < s)ds)dt =$$

$$= \int_1^u \frac{1}{t}\lambda(\frac{1}{y_{n-1}} < t-1)dt - \frac{1}{u}\int_0^{u-1} \lambda(\frac{1}{y_{n-1}} < s)ds$$

where $u > 1$. On the other hand, note that $h_n(t) = t^{-2}(2 - \int_0^1 \lambda(y_{n-1} > s^{-1})ds)$ if $t > 2$.

(b) It is proved in [20, Chapitre IX] that the function $F(x) := (\log 2)^{-1}\log(2x/(x+1))$ if $x > 1$, $= 0$ if $x \le 1$ satisfies

(4.8) $$\sup_x |\lambda(y_n < x) - F(x)| \le C r^n$$

for some $C>0$ and $r\in(0,1)$. Now (4.6) and (4.7) follow from (a) since H and h are related to $1-F$ just as H_n and h_n are to G_{n-1}. ∎

4.6. <u>Corollary</u>. Assume $f:[1,\infty)\to(0,\infty)$ is regularly varying with exponent $\alpha\in[1/2,1]$, $E_p f^2(a_1)=+\infty$ and satisfies

(K_o) $f(x)=x^{\alpha}L(x)$ where $L(x)=c\exp\{\int_1^x \varepsilon(t)t^{-1}dt\}$ with $c>0$, $\varepsilon:[1,\infty)\to\mathbb{R}$ measurable, bounded and $\lim_{t\to\infty}\varepsilon(t)=0$.

Let ν_α be defined by (2.14) if $\alpha\in(1/2,1]$ and write $\nu_{1/2}=N(0,1)$. Let

$$m(f)=(\log 2)^{-1}\int_1^\infty\int_1^\infty (f(x+y^{-1})-f([x]))(xy+1)^{-2}dx\,dy$$

if $\alpha=1$, $m(f)=\int_1^\infty f(t)h(t)dt$ (h being the density in (4.5)) if $\alpha\in[1/2,1)$ and define ξ_n by

$$(4.9)\qquad \xi_n(t)=f(n)^{-1}\Sigma_{1\le j\le[nt]}\{f(u_j)-m(f)-E_p(f(a_1);f(a_1)\le f(n))\}$$

if $\alpha=1$,

$$(4.10)\qquad \xi_n(t)=f(n)^{-1}\Sigma_{1\le j\le[nt]}\{f(u_j)-m(f)\}\qquad\text{if}\quad \alpha\in(1/2,1),$$

$$(4.11)\qquad \xi_n(t)=b(n)^{-1}\Sigma_{1\le j\le[nt]}\{f(u_j)-m(f)\}\qquad\text{if}\quad \alpha=1/2,$$

where $\{b(n)\}$ is any sequence satisfying $\lim_n nb(n)^{-2}\tilde{U}(b(n)) = 1$ (with $\tilde{U}$ defined as in Corollary 2.12). Then for any $\rho << \lambda$, $\mathcal{L}_\rho(\xi_n) \to_w Q_{\nu_\alpha}$. The same result holds if ξ_n is defined by replacing in (4.9)-(4.11) u_j by x_j and $m(f)$ by $m'(f)$ where $m'(f) = E_p(f(x_1)-f(a_1))$ if $\alpha = 1$, $= E_p f(x_1)$ if $\alpha \in [1/2,1)$.

Proof. By (K_o) f is bounded on finite intervals and, as we will show, it satisfies (4.2). Writing $M = \max\{1, \sup_{t>1}|\varepsilon(t)|\}$, by (K_o) we have if $k \in \mathbb{N}^*$, $k < x < y < k+2$,

$$|L(x)-L(y)| < L(x)((y/x)^M - 1) < M'L(x)x^{-1}|x-y|$$

where $M' = M3^{M-1}$ $(x^{-1}y < 3)$; then, since $\alpha < 1$,

$$|f(x)-f(y)| < x^\alpha |L(x)-L(y)| + |x^\alpha - y^\alpha| L(y)$$

$$< M'(L(x) + L(y))|x-y|.$$

On the other hand, there exists C such that $L(x) < Cx^{1/4}$ for every $x > 1$. Thus if $k \in \mathbb{N}^*$ and $x,y \in [k,k+2]$

$$(4.12) \qquad |f(x)-f(y)| < M'2C(3k)^{1/4}|x-y| = M(k)|x-y| \qquad \text{(say)}$$

which proves (4.2). Corollaries 2.12 and 2.13 and Lemma 4.3 now imply the assertion about $\{x_j\}$. For $\{u_j\}$ we conclude that $\mathcal{L}_\rho(\xi_n') \to_w Q_{\nu_\alpha}$.

ξ_n' being defined by (4.9) with $m(f)$ replaced by $E_P(f(u_j)-f(a_j))$ (which depends on j) in the case $\alpha = 1$ and by (4.10)-(4.11) with $m(f)$ replaced by $E_P f(u_j)$ if $\alpha \in [1/2,1)$.

Suppose $\alpha \in [1/2,1)$. By Lemma 4.5 we have

$$|E_\lambda f(u_n)-m(f)| < (\sup_{t\in[1,2]} f(t)) \sup_t |h_n(t)-h(t)|$$

$$+ (\int_2^\infty f(t)t^{-2}dt) |1 + E_\lambda(\frac{1}{y_{n-1}}) - \frac{1}{\log 2}|$$

and hence, for some constant C_1,

$$(4.13) \qquad |E_\lambda f(u_n)-m(f)| < C_1 r^n \quad \text{for every } n > 1.$$

Write $g_{n\ell} = E_P(f(u_n)|M_{n\ell})$. As in the proof of Lemma 4.3, using (4.4) and (4.12), we obtain that for some C_2

$$(4.14) \qquad E_P^{1/2}(f(u_n)-g_{n\ell})^2 < C_2 2^{-\ell} \quad \text{for every } n > 1 \text{ and } \ell > 1.$$

On the other hand, since there exist constants K and $r' \in (0,1)$ such that $|P(A)-\lambda(A)| < K(r')^k P(A)$ for any $A \in \sigma(a_k, a_{k+1}, \ldots)$, $k > 1$ (argue as in the proof of [12, Lemma 19.4.2] using (7) of [21]), we have for some C_3

$$(4.15) \qquad |E_P g_{n\ell} - E_\lambda g_{n\ell}| < C_3 (r')^{n-\ell} \quad \text{if } n > \ell > 1$$

$(|\int_0^\infty (P(g_{n\ell} > x) - \lambda(g_{n\ell} > x))\,dx| < KE_P f(u_n)(r')^{n-\ell})$. Taking $\ell_n = [n/2]$ we get from (4.13)-(4.15)

$$|E_P f(u_n) - m(f)| < C_2 2^{-\ell_n} + C_3 (r')^{n-\ell_n}$$
$$+ (2 \log 2)^{1/2} C_2 2^{-\ell_n} + C_1 r^n.$$

Thus $|E_P f(u_n) - m(f)| = O(s^n)$ for some $s \in (0,1)$ which implies that $\sup_t |\xi_n(t) - \xi'_n(t)| \to 0$ pointwise and so the proof in the case $\alpha < 1$ is complete.

Now assume $\alpha = 1$. First observe that Proposition 2.1 implies that for any Borel measurable function h

$$\int_{\{y_{n-1}=y\}} h(x_n)\,d\lambda = (\int_1^\infty h(x) \frac{y(y+1)}{(xy+1)^2}\,dx)\lambda(y_{n-1} = y)$$

provided one of the two members exists, y being a possible value of y_{n-1}. Thus, writing

$$\text{(4.16)} \qquad K(y) = \int_1^\infty (f(x+y^{-1}) - f([x]))y(y+1)(xy+1)^{-2}dx, \; y > 1,$$

we have (by (4.12) K is bounded and the following integrals exist)

$$E_\lambda(f(u_n) - f(a_n)) = \Sigma_y \int_{\{y_{n-1}=y\}} (f(x_n + \frac{1}{y}) - f([x_n]))\,d\lambda$$
$$= \int_{[1,\infty)} K \, dL_\lambda(y_{n-1}).$$

On the other hand, $m(f) = \int_{[1,\infty)} K\, dF$ where F is the distribution function appearing in (4.8).

Denote $g(x,y)$ the integrand in (4.16) and $v(x,y) = y(y+1)(xy+1)^{-2}$. If $x > 1$ and $y,y' > 1$ we get by (4.12)

$$\begin{aligned}|g(x,y)-g(x,y')| &< M([x])|y^{-1}-(y')^{-1}||v(x,y)| \\ &\quad + M([x])|x+(y')^{-1}-[x]||v(x,y)-v(x,y')| \\ &< 10\ M([x])(xy)^{-2}|y-y'|.\end{aligned}$$

Hence if $y,y' > 1$ we have

$$\begin{aligned}|K(y)-K(y')| &< 10\Big(\int_1^{\infty} M([x])x^{-2}dx\Big)\ y^{-2}|y-y'| \\ &= A\ y^{-2}|y-y'| \qquad \text{(say)}\end{aligned}$$

and K is absolutely continuous. Then

$$|E_\lambda(f(u_n)-f(a_n))-m(f)| < A\int_1^{\infty}|\lambda(y_{n-1} > t)-(1-F(t))|\,t^{-2}dt$$

and (4.8) gives that $|E_\lambda(f(u_n)-f(a_n))-m(f)| = O(r^n)$. In order to complete the proof, observe that analogous relations to (4.14) and (4.15) are valid and argue as above. ■

4.7. <u>Examples.</u>

4.7.1. If $f(x) = x^\alpha$ where $\alpha \in [1/2,1)$ then $m(f) =$

$(\alpha(1-\alpha)\log 2)^{-1}(2^{\alpha}-1)$ and we can take $b(n) = (n \log n/\log 2)^{1/2}$ in (4.11).

4.7.2. Let $f(x) = x$. Then $m'(f) = (\log 2)^{-1}-1$ and $m(f) = m'(f) + (\log 2)^{-1}\int_1^{\infty} y^{-2}(y+1)^{-1}dy = 2((\log 2)^{-1}-1)$. If ξ_n is defined by (2.20) then for any $\rho << \lambda$, $L_\rho(\tilde{\xi}_n) \to_w Q_{\tilde{\nu}'}$ and $L_\rho(\tilde{\tilde{\xi}}_n) \to_w Q_{\tilde{\tilde{\nu}}'}$, where $\tilde{\nu}' = \delta_{((\log 2)^{-1}-1)} * \nu'$, $\tilde{\tilde{\nu}}' = \delta_{2((\log 2)^{-1}-1)} * \nu'$, ν' being defined by (2.21) (we use the notation at the beginning of this section). Similar remarks to those made in 2.14.2 apply. We point out that the convergence of $L_\lambda(\tilde{\tilde{\xi}}_n(1))$ was indicated by Doeblin [7, page 365].

§5. An application to Diophantine approximation.

5.1. Theorem. Let $c \in (0,1/2]$. Given $n > 1$ and $\omega \in \Omega$ define

$$\Lambda_{c,n}(\omega) = \text{card}\{(p,q) \in \mathbb{N}\times\mathbb{N}^* : |q\omega - p| < \frac{c}{q},\ 1 < q < n\}$$

$$= \text{card}\{q \in \mathbb{N}^* : \langle q\omega\rangle < \frac{c}{q},\ 1 < q < n\},$$

where $\langle\alpha\rangle$ is the distance between α and the nearest integer, and

$$\Lambda^{+}_{c,n}(\omega) = \text{card}\{(p,q) \in \mathbb{N}\times\mathbb{N}^* : 0 < q\omega - p < \frac{c}{q},\ 1 < q < n\}$$

$$= \text{card}\{q \in \mathbb{N}^* : q\omega - [q\omega] < \frac{c}{q},\ 1 < q < n\}.$$

Consider the random functions

$$\xi_n(t) = \left(\frac{\pi^2/6}{2\ c \log n \log(\log n)}\right)^{1/2} \{\Lambda_{c,[n^t]} - 2\ c \log [n^t]\},$$

$$\xi^{+}_n(t) = \left(\frac{\pi^2/6}{c \log n \log(\log n)}\right)^{1/2} \{\Lambda^{+}_{c,[n^t]} - c \log [n^t]\},$$

$t \in [0,1]$, $n > 3$. Then for any $\rho \ll \lambda$, $L_\rho(\xi_n) \to_w W$ and $L_\rho(\xi^{+}_n) \to_w W$ in D, W being Wiener measure.

Define $\Lambda_{f,n}$ as above by replacing the constant c by a function f such that $\Sigma f(q)q^{-1}$ diverges. For fairly general f, the theorems of Le Veque, Erdős and Schmidt show that $\Lambda_{f,n} \sim 2\,\Sigma_{q<n} f(q)q^{-1} =: b(n)$ a.e.

(see [17]). Philipp [22] proved, in accordance with a conjecture of Le Veque [18 b], the asymptotic normality of $b(n)^{-1/2}(\Lambda_{f,n}-b(n))$ for f satisfying, besides other conditions, $\lim_q f(q) = 0$; here different norming constants are needed. We remark that [18] and [22], which contain several related results, use central limit theorems for weakly dependent variables having finite variances. Relation (5.4) was suggested by an estimation made in [17, page 35].

<u>Proof</u>. Fix $c \in (0,1/2]$. We will write $\Lambda_n (\Lambda_n^+)$ for $\Lambda_{c,n}$ $(\Lambda_{c,n}^+)$.

(I) <u>Convergence of ξ_n</u>. <u>Step 1</u>. Let $n > 1$ and $\omega \in \Omega$. It is well known that

(5.1) $$|q'\omega - p'| < (2q')^{-1}, \text{ g.c.d. } (p',q') = 1 \text{ imply } p' = p_k(\omega),$$
$$q' = q_k(\omega) \text{ for some } k > 0.$$

Then

$$\Lambda_{q_n(\omega)-1}(\omega) = \#\{(k,r) \in \mathbb{N}\times\mathbb{N}^* : |rq_k(\omega)\omega - rp_k(\omega)| < \frac{c}{rq_k(\omega)}, 1 < rq_k(\omega) < q_n(\omega)\}$$
$$= \Sigma_{k=0}^{n-1} \#\{r \in \mathbb{N}^* : r < (c\, u_{k+1}(\omega))^{1/2}\} ;$$

for the proof of the second equality, use that $|q_k(\omega)\omega - p_k(\omega)| = (x_{k+1}(\omega)q_k(\omega) + q_{k-1}(\omega))^{-1}$ if $k > 1$ and

(5.2) $$r < (c\, u_{k+1}(\omega))^{1/2},\ k > 0, \text{ implies } rq_k(\omega) < q_{k+1}(\omega)$$

(observe that if $k \geq 1$ and $r^2 < c\, u_{k+1}$ then $(rq_k)^2 <$ $c(x_{k+1}\, q_k^2 + q_k\, q_{k-1}) < a_{k+1}\, q_k^2 + (1/2) q_k\, q_{k-1} < (a_{k+1}\, q_k + q_{k-1})^2 =$ q_{k+1}^2).

Now consider the denumerable set Ω_o of those ω for which $(c\, u_{k+1}(\omega))^{1/2}$ is an integer for some $k \geq 0$. Then if $\omega \in \Omega \setminus \Omega_o$

$$\Lambda_{q_n(\omega)-1}(\omega) = \Sigma_{j=1}^{n} \left[(c\, u_j(\omega))^{1/2} \right] .$$

Define the r.v. $\tau_n : \Omega \to \mathbb{N}^*$ by

$$\tau_n(\omega) = k \quad \text{if} \quad q_{k-1}(\omega) \leq n < q_k(\omega) \qquad (k = 1,2,\ldots). \tag{5.3}$$

Therefore we have for $\omega \in \Omega \setminus \Omega_o$

$$\Lambda_n(\omega) = \Sigma_{j=1}^{\tau_n(\omega)} \left[(c\, u_j(\omega))^{1/2} \right] - \beta_n(\omega) \tag{5.4}$$

where

$$\beta_n(\omega) := \#\{(p,q) \in \mathbb{N} \times \mathbb{N}^* : |q\omega - p| < \frac{c}{q},\ n < q < q_{\tau_n}(\omega)\} \leq (\tfrac{3}{2}\, a_{\tau_n}(\omega))^{1/2} \tag{5.5}$$

(we write $q_{\tau_n}(\omega) = q_{\tau_n(\omega)}(\omega)$, etc.). In order to prove the inequality in (5.5), note that if (p,q) belongs to the set defining β_n then, by (5.1) and (5.2), $p = r\, p_{\tau_n - 1}(\omega)$, $q = r\, q_{\tau_n - 1}(\omega)$ with a positive integer $r < (c\, u_{\tau_n}(\omega))^{1/2}$.

<u>Step 2</u>. From now on fix $\rho << \lambda$. By Example 2.14.1 we know that, writing

$$\xi_n^{(1)}(t) = \left(\frac{\log 2}{c\ n\ \log n}\right)^{1/2} \Sigma_{j\leq[nt]}\{(ca_j)^{1/2} - E_P(ca_1)^{1/2}\} \quad (t\in[0,1]),$$

$L_\rho(\xi_n^{(1)}) \to_w W$. We will deduce that if

$$\xi_n^{(2)}(t) = \left(\frac{\log 2}{c\ n\ \log n}\right)^{1/2} \Sigma_{j\leq[nt]}\{[(cu_j)^{1/2}] - E_P([(cu_j)^{1/2}])\}$$

we have $L_\rho(\xi_n^{(2)}) \to_w W$; for this purpose, it is sufficient to show that if

$$f_j = [(cu_j)^{1/2}] - (ca_j)^{1/2}, \quad \eta_j = f_j - E_P f_j \ , \ j \geq 1,$$

(observe that $|f_j| \leq 1$) then $\{\eta_j\}$ verifies (3) of Corollary 3.4.

In order to prove this, fix $j \geq 1$, $\ell \geq 1$ and write $f_{j\ell} = E_P(f_j | M_{j\ell})$, $r_j = [(cu_j)^{1/2}]$ and

$$A_{j\ell} = \{c^{-1}r_j^2 + 62^{-\ell} < u_j < c^{-1}(r_j+1)^2 - 62^{-\ell}\} .$$

Given positive integers $k_{j-\ell},\dots,k_{j+\ell}$ and $\Delta := \Delta_{j\ell}(k_{j-\ell},\dots,k_{j+\ell})$ defined by (4.3) we have that if $\omega \in A_{j\ell} \cap \Delta$ and $\omega' \in \Delta$ then $f_j(\omega) = f_j(\omega')$ (since $\omega \in A_{j\ell}$, using (4.4) we get $c^{-1}r_j^2(\omega) < u_j(\omega') < c^{-1}(r_j(\omega)+1)^2$ which says that $r_j(\omega') = r_j(\omega)$); hence $f_j(\omega) = P(\Delta)^{-1}\int_\Delta f_j\, dP$ if $\omega \in A_{j\ell} \cap \Delta$. This implies that $f_j = f_{j\ell}$ almost surely on $A_{j\ell}$.

On the other hand

$$P(A^c_{j\ell}) < (\log 2)^{-1}\Sigma^{\infty}_{r=0}\{\lambda(c^{-1}r^2 < u_j < c^{-1}r^2 + 62^{-\ell})$$

$$+ \lambda(c^{-1}(r+1)^2 - 62^{-\ell} < u_j < c^{-1}(r+1)^2)\}.$$

If $j > 2$ and ℓ satisfies $62^{-\ell} < 1$ we obtain, using that $H_j(b) - H_j(a) < 2b^{-1}(b-a)$ if $1 < a < b$ (this follows from Lemma 4.5(a)),

$$P(A^c_{j\ell}) < 12c(\log 2)^{-1}(1+2\Sigma^{\infty}_{r=1} r^{-2})2^{-\ell} = C_1 2^{-\ell} \quad \text{(say)}$$

and a similar bound is valid for $j = 1$. Hence there exists a constant C such that $P(A^c_{j\ell}) < C2^{-\ell}$ for every j and ℓ.

Now we can write

$$(5.6) \qquad E_P(\eta_j - E_P(\eta_j | M_{j\ell}))^2 = E_P(f_j - f_{j\ell})^2$$

$$= \int_{A^c_{j\ell}} (f_j - f_{j\ell})^2 \, dP < 4C2^{-\ell}$$

for every $j > 1$ and $\ell > 1$. Thus Corollary 3.4 implies that $L_p(\xi^{(2)}_n) \to_w W$.

<u>Step 3</u>. If $j > 2$, by Lemma 4.5(a)

$$E_\lambda([(cu_j)^{1/2}]) = \Sigma^{\infty}_{r=1} r(H_j(c^{-1}(r+1)^2) - H_j(c^{-1}r^2))$$

$$= c(1+E_\lambda(\frac{1}{y_{j-1}})) \; \Sigma^{\infty}_{r=1} r(\frac{1}{r^2} - \frac{1}{(r+1)^2})$$

and this series is $\sum_{r=1}^{\infty} r^{-2} = \pi^2/6$; then Lemma 4.5(b) gives, for some constant C_1,

$$(5.7)\qquad |E_\lambda([(cu_j)^{1/2}]) - \frac{c\pi^2}{6 \log 2}| < C_1 r^j \quad \text{for every} \quad j \geq 1.$$

Writing $r_{j\ell} = E_P(r_j | M_{j\ell})$, r_j being defined as above, and arguing as in the proof of the inequality in (5.6) we get for some C_2

$$(5.8)\qquad E_P^{1/2}(r_j - r_{j\ell})^2 < C_2 (2^{1/2})^{-\ell} \quad \text{for every} \quad j \geq 1 \quad \text{and} \quad \ell \geq 1$$

(note that $|r_j - r_{j\ell}| < 2$ almost surely. It is enough to verify that $|r_j(\omega) - r_j(\omega')| < 2$ if $\omega, \omega' \in \Delta(k_{j-\ell}, \ldots, k_{j+\ell})$ - defined by (4.3) - with positive integers $k_{j-\ell}, \ldots, k_{j+\ell}$; but in this case we have $|r_j(\omega) - r_j(\omega')| < (c(k_j+2))^{1/2} - (ck_j)^{1/2} + 1 < 2$). Moreover, the argument which led to (4.15) shows that for some C_3

$$(5.9)\qquad |E_P\, r_{j\ell} - E_\lambda\, r_{j\ell}| < C_3 (r')^{j-\ell} \quad \text{if} \quad j > \ell \geq 1$$

with $r' \in (0,1)$.

By (5.7)-(5.9) we have now $|E_P([(cu_j)^{1/2}]) - (6 \log 2)^{-1} c \pi^2| = O(s^j)$ with $s \in (0,1)$, which implies that if

$$\xi_n^{(3)}(t) = \left(\frac{\log 2}{c\, n \log n}\right)^{1/2} \sum_{j \leq [nt]} \{[(cu_j)^{1/2}] - \frac{c\pi^2}{6 \log 2}\}$$

then $L_\rho(\xi_n^{(3)}) \to_w W$ (see proof of Corollary 4.6).

<u>Step 4</u>. By a well known theorem of Khintchine $\lim_k k^{-1} q_k = B := \pi^2/(12 \log 2)$ a.e. This and (5.3) give

$$\lim_n \frac{\tau_n}{\log n} = B^{-1} \quad \text{a.e.} \tag{5.10}$$

Now we claim

$$\eta_n := (\log n \log(\log n))^{-1/2} \max_{1<k<n} a_{\tau_k}^{1/2} \to 0 \text{ in measure.} \tag{5.11}$$

For every $\varepsilon > 0$, since $\tau_k < \tau_n$ if $k < n$,

$$P(\eta_n > \varepsilon) < P(\tau_n > (1+\varepsilon) B^{-1} \log n)$$

$$+ P((\log n \log(\log n))^{-1/2} \max_{j<(1+\varepsilon)B^{-1}\log n} a_j^{1/2} > \varepsilon)$$

and the second term on the right is bounded by $(1+\varepsilon)B^{-1}(\log n) P(a_1 > \varepsilon^2 \log n \log(\log n))$; then (5.10) and (2.5) imply (5.11).

We also need

$$\zeta_n := (\log n \log(\log n))^{-1/2} \max_{1<k<n} |B\tau_k - \log k| \to 0 \text{ in measure.} \tag{5.12}$$

First observe that, since $q_{\tau_k - 1} < k < q_{\tau_k}$, $0 < \log q_{\tau_k} - \log k$

$< \log(y_{\tau_k}) < \log(a_{\tau_k}+1) < a_{\tau_k}^{1/2}$, and we get $|B\tau_k - \log k| < |B\tau_k - \log q_{\tau_k}| + a_{\tau_k}^{1/2}$. Moreover, if $Y_n(t) = \tau_n^{-1/2}(\log q_{[t\tau_n]} - [t\tau_n]B)$ $(t \in [0,1])$, we have

$$\max_{k \leq n} |\log q_{\tau_k} - B\tau_k| = \tau_n^{1/2} \sup_{t \in [0,1]} |Y_n(t)|$$

and (5.12) follows from (5.10), (5.11) and a theorem of Billingsley [5, page 194 and Theorem 17.1].

We claim that if

$$\xi_n^{(4)}(t) = \left(\frac{\pi^2/12}{c \log n \log(\log n)}\right)^{1/2} \sum_{j=1}^{\tau_{[n^t]}} \left\{[(cu_j)^{1/2}] - \frac{c\,\pi^2}{6 \log 2}\right\}$$

then $L_\rho(\xi_n^{(4)}) \to_w W$. Here we will use the notation and results of [5, §17]. Define

$$\Phi_n(t,\omega) = \begin{cases} [\log n]^{-1} \tau_{[n^t]}(\omega) & \text{if } [\log n]^{-1}\tau_n(\omega) < 1 \\ tB^{-1} & \text{otherwise} \end{cases}$$

and $g(t) = tB^{-1}$ $(t \in [0,1],\ \omega \in \Omega)$. We have $\Phi_n(.,\omega) \in D_o$ for each ω, $g \in D_o$ and Φ_n converges in ρ-measure to g in D_o; this follows from (5.12) since

$$\sup_{t \in [0,1]} |\Phi_n(t,.) - g(t)|$$

$$< [\log n]^{-1}\{ \max_{1 < k \leq n} |\tau_k - B^{-1} \log k| + B^{-1}\}.$$

Starting from the weak convergence of $L_\rho(\xi^{(3)}_{[\log n]}, \Phi_n)$ (use [5, Theorem 4.4]), noting that if ω satisfies $[\log n]^{-1}\tau_n(\omega) < 1$ and $t \in [0,1]$

$$(\xi^{(3)}_{[\log n]} \circ \Phi_n)(t,\omega) =$$

$$= \left(\frac{\log 2}{c[\log n]\log([\log n])}\right)^{1/2} \Sigma_{j=1}^{\tau_{[n^t]}(\omega)} \{[(cu_j(\omega))^{1/2}] - \frac{c\,\pi^2}{6\log 2}\}$$

and using (5.10), we can argue as in [5, proof of Theorem 17.1] to conclude that $L_\rho(\xi_n^{(4)}) \to_w W$.

In order to complete the proof, observe that by (5.4) and (5.5)

$$\sup_{t\in[0,1]}|\xi_n(t)-\xi_n^{(4)}(t)| < (\pi^2/(12c))^{1/2}((3/2)^{1/2}\eta_n+2c\zeta_n)\,. \quad \text{a.e.}$$

and use (5.11) and (5.12).

(II) <u>Convergence of ξ_n^+</u>. Slight modifications in (I) are needed. We only make two remarks. Since a convergent $p_k(\omega)/q_k(\omega)$ is less than ω if and only if k is even, (5.4) is true for Λ_n^+ if in the sum of the right member j runs through odd values and β_n is suitable defined; then the inequality in (5.5) also holds. On the other hand, we have for $t \in [0,1]$

$$|\Sigma_{1<j<[nt]}\, a_j^{1/2} - \Sigma_{1<h<[[n/2]t]}\, a_{2h-1}^{1/2}| <$$

$$< a^{1/2}_{[[n/2]t]} + a^{1/2}_{[[n/2]t]+2} \quad .$$

From (the $C[0,1]$ version of) the preceding result one can deduce (see [5]) the following. Given $c \in (0,1/2]$ let $T_{c,n}(\omega)$ be the first i, $1 < i < n$, such that

$$\Lambda_{c,j}(\omega) < 2\ c \log j \qquad \text{for every} \quad j \in \{i,\ldots,n\}$$

or

$$\Lambda_{c,j}(\omega) > 2\ c \log j \qquad \text{for every} \quad j \in \{i,\ldots,n\}$$

if $\Lambda_{c,n}(\omega) \neq 2\ c \log n$; otherwise define $T_{c,n}(\omega) = n$. Then for any $\rho << \lambda$, $\mathcal{L}_\rho((\log T_{c,n})/\log n)$ converges weakly to the arc sine law; this says that if $0 < a < b < 1$ then

$$\lim_n \rho(n^a < T_{c,n} < n^b) = (2/\pi)(\text{arc}\ \sin\sqrt{b} - \text{arc}\ \sin\sqrt{a}).$$

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