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ONE-PARAMETER FAMILIES OF MEASURE-THEORETIC AND TOPOLOGICAL ENTROPIES

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One-parameter families of measure-theoretic and topological entropies

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ABSTRACT. We consider one-parameter families of measure-theoretic and topological entropies associated with dynamical systems. These families are such that, for a particular parameter value, well stablished quantities in Ergodic Theory are recovered. Concerning the family of measure-theoretic entropies, which was introduced by the authors in a previous work, we review the definition and some of its properties, particularly the isomorphism theorems, and also present new results and perform calculations for some particular dynamical systems. With respect to the topological entropies two families, which are shown to be equivalent for a set of values of their parameter, are defined and their properties, such as the topological invariance, studied. We also outline the relationship between the members of both, the topological and the measure-theoretic, families. Finally, within the same spirit, we introduce families of topological pressures.

1. INTRODUCTION

A family of entropies of a measure-preserving transformation which depends on a parameter q was introduced by the authors in[15]. In that work we have generalized a physically motived family of entropies[28] to abstract dynamical systems in a similar form as Kolmogorov[11] did with Shannon's entropy[23] and have demonstrated the isomorphism invariance of the entropies so defined. Our family contains as a member (when $q \to 1$) Kolmogorov's definition and the above mentioned results generalize meaningful theorems by Sinai, Melshalkin and Ornstein[24],[14],[17].

Here we continue with that study in diverse ways. In the first place, after recalling our definition of the measure-theoretic q-entropies and their invariance under isomorphisms, we calculate them for several dynamical systems. The entropies of i) measure-preserving transformations of finite spaces; ii) rotations of the unit circle; iii) rotations of compact metric abelian groups and iv) endomorphisms of compact

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groups are evaluated for arbitrary values of the parameter q so that standard results for $q \to 1$ are generalized.

In the second place we give two definitions for the topological q-entropies of a continuous mapping from a compact topological space X to itself: one using open covers in the sense of[1] and the other one by generalizing to arbitrary q's, Bowen's definition of topological entropy[3],[5]. We show the equivalence of both families for some values of q as well as their topological invariance and estimate bounds for the topological q-entropy of differentiable maps between compact Riemannian manifolds and specially of the geodesic flow in such a manifold. We also introduce within the same idea two families of topological pressures that reproduce, when the function of which they depend is identically null, one or the other of the above mentioned families of topological entropies, respectively.

Finally, we study the relationship between the topological and the measure-theoretic q-entropies following a variational approach.

2. MEASURE-THEORETIC q-ENTROPY

2.1. **Reviewing** A dynamical system or, more precisely, a measure-preserving dynamical system, also simply called a measure-preserving transformation, is a quadruple $(X, \mathcal{A}, \mu, T)$ where X is a set, $\mathcal{A}$ a σ-algebra, μ a measure on $\mathcal{A}$ and T is a mapping of the underlying set X to itself which is measurable and preserves the measure μ, *i.e. for any* $A \in \mathcal{A}$, $T^{-1}A \in \mathcal{A}$ *and* $\mu(T^{-1}A) = \mu(A)$ ($\mu(.)$ denotes the measure of an element . of $\mathcal{A}$).

Bernoulli schemes are particular cases of measure-preserving dynamical systems of great importance in ergodic theory. Thus, a Bernoulli scheme is also a quadruple $(X, \mathcal{A}, \mu, T,)$where X is now a *sample space* associated with the possible results arising in a probabilistic experiment, say $1, 2, ...n$, having respective probabilities $p_1, p_2, ...p_n$ (we use the notation of [9]):

$$X = \{1, 2, ...n\}^{\mathbf{Z}}$$

$$= \{x = (....x_{-1}, x_0, x_1, ...) : x_i \in \{1, 2, ...n\} \ for\ all\ i \in \mathbf{Z}\} . \tag{1}$$

The σ-algebra $\mathcal{A}$ on X and the measure μ are the standard ones for this kind of probabilistic experiment. Thus, $\mu = (p_1, p_2, ...p_n)^{\mathbf{Z}}$, assuming that the individual experiments are independent. Finally, the measure-preserving transformation T is the *shift* on X defined $Tx = x'$ where $x'_i = x_{i+1}$. As usual we denote the above Bernoulli scheme by $BS(p)$.

Let also recall the concept of isomorphism between two dynamical system $S = (X, \mathcal{A}, \mu, T)$ and $S' = \left(X', \mathcal{A}', \mu', T'\right)$. S and S' are isomorphic if there exists a measurable mapping $f : X \to X'$ which is a bijection such that $i)$ *for any* $A' \in \mathcal{A}', \mu(f^{-1}A') = \mu'(A')$ and $ii)$ *for all* x, $f(Tx) = T'(fx)$.

The Kolmogorov-Shannon entropy is constructed considering that, within Shannon's information theory, the amount of information we have if we know that a point $x \in X$ belongs to some fixed set of a partition $C = \{C_1, C_2, ..., C_m\}$of X is

$$H_1(C) = -\sum_{i=1}^{m} \mu(C_i) \log \mu(C_i). \tag{2}$$

A generalization of this magnitude has been given in[28]. In our context it reads

$$H_q(C) = (q-1)^{-1} \left(1 - \sum_{i=1}^{m} [\mu(C_i)]^q\right), \tag{3}$$

where q is any real number. We observe that for $q \to 1$, Shannon's expression (2) is recovered.

To define our one-parameter family of entropies we follow [9]. We firstly consider a finite partition $B = \{B_1, B_2, ..., B_k\}$ of the sample space, *i.e.* $\bigcup_{i=1}^{k} B_i = X$; $B_i \bigcap B_j = \phi$ $\forall i \neq j$. Then we take n points on the orbit of X : x, Tx, T^2x,.....,$T^{n-1}x$. Being B a partition, for each of these points there exists only one set B_i to which it belongs. We associate to each x a string $\ell = (\ell_0, \ell_1, ..., \ell_{n-1})$, called the name of x, where $T^i x \in B_{\ell_i}$. From B we construct a new partition $B^n = \{B^n(\ell) : \ell \text{ is any name of length } n\}$ where $B^n(\ell)$ is the set of x with name ℓ.

Thus, we define a mean entropy of the measure-preserving transformation T by

$$h_q(T) = (q-1)^{-1}\left[1 - \exp\left(\tilde{h}_q(T)\right)\right], \tag{4}$$

where

$$\tilde{h}_q(T) = \sup_{B} \tilde{h}_q(B, T) \tag{5}$$

and

$$\tilde{h}_q(B,T) = \overline{\lim_{n\to\infty}} \frac{1}{n} \left\{\log\left[1 + (1-q)\, H_q(B^n)\right]\right\}. \tag{6}$$

Eqs.(4)-(6) define a family of entropies depending on the parameter q.

In particular for a Bernulli scheme $BS(p)$ we have $\mu(B^n(\ell)) = p_{\ell_0} p_{\ell_1} ......p_{\ell_{n-1}}$ for each name ℓ of length n. Therefore, using for the partition B^n the generator $B_i = \{x : x_0 = i\}$ with $1 \leq i \leq k$, we obtain

$$\left[\tilde{h}_q(T)\right]_{BS(p)} = \log \sum_{i=1}^{n} p_i^q \tag{7}$$

which yields Tsallis's entropy[28] for arbitrary q

$$[h_q(T)]_{BS(p)} = (q-1)^{-1}\left[1 - \sum_{i=1}^{n} p_i^q\right] \tag{8}$$

and Shannon's expression[23]

$$[h_1(T)]_{BS(p)} = -\sum_{i=1}^{n} p_i \log(p_i) \tag{9}$$

for $q \to 1$.

The following results were proved in Ref.[15]

Theorem 2.1. *If two measure-preserving dynamical systems are isomorphic then the associated family of entropies depending of the parameter* q, *as defined by* Eqs.(4-6), *are equal.*

The complete isomorphism is proved just for Bernoulli schemes and reads as follows:

Theorem 2.2. *Two Bernoulli schemes with the same associated family of entropies depending of the parameter* q, as given by Eq.(8) *are finitarily isomorphic.*

Remark 2.3. We recall that an isomorphism f between two Bernoulli schemes is *finitary* if, given an element x of the sample space X (see Eq.(1)), there exists two integers $n_1 \leq n_2$ such that for any other $x' \in X$ that verifies $x'[n_1, n_2] = x[n_1, n_2]$, the zero coordinates $[f(x)]_0$, $\left[f(x')\right]_0$ and $[f^{-1}(x)]_0$, $\left[f^{-1}(x')\right]_0$ are, respectively, equal. Here we denote with $x[n_1, n_2]$ the *word* $x_{n_1} x_{n_1+1} \ldots . x_{n_2-1} x_{n_2}$.

2.2. Results and calculations

Here we present some results for the measure theoretic q-entropy.

i) $h_q(id : X \to X) = 0$. It directly follows from the definition Eqs.(4-6) since $H_q(B^n) = 0$ for any partition B.

ii)For any integer positive $k \geq 2$ we have that $h_q(T^k) \leq (q-1)^{-1}\left[1 - \exp\left(k\tilde{h}_q(T)\right)\right]$ for $q > 1$ and the corresponding result obtained by changing $\leq$ and $>$ by $\geq$ and $<$, respectively.

Remark 2.4. For $q = 1$ we have $h_1(T^k) = k\ h_1(T)$, a well known result.

Proof. Let B^k be the partition with names of length k obtained from a given partition B and the transformation T. Let call $\tilde{H}_q\left((B^k)^n\right) = 1 + (1-q)\, H_q\left((B^k)^n\right)$ where now $(B^k)^n$ is the partition with names of length n obtained from the partition B^k and the transformation T^k. We have

$$\tilde{h}_q(B^k, T^k) = \overline{\lim_{n\to\infty}} \frac{1}{n}\left\{\log\left[\tilde{H}_q\left((B^k)^n\right)\right]\right\} = \overline{\lim_{n\to\infty}} \frac{k}{kn}\left\{\log\left[\tilde{H}_q\left(B^{kn}\right)\right]\right\} = k\tilde{h}_q(B, T). \tag{10}$$

Here B^{kn} the partition with names of length kn constructed from the partition$B$ and the transformation T. It follows that $k\tilde{h}_q(T) = k \sup_B \tilde{h}_q(B, T) = \sup_B \tilde{h}_q(B^k . T^k) \leq \tilde{h}_q(T^k)$. From this inequality the statement is proved. □

iii) We apply ii) to coment on the measure theoretic q-entropy of rotations of the unit circle.

Let $T_g : S^1 \to S^1$ with $T_g(z) = gz$ and $g = e^{2\pi i/r}$ for some positive integer r. Together with the σ-algebra of Borel subsets and the Haar measure it defines our dynamical system. It is not ergodic because g is a root of the unity. Since $(T_g)^r \equiv id_{S^1}$ then by i) we have $h_q((T_g)^r) = 0$ for any q. We also have that, for $q < 1$ and any measure-preserving transformation T is $\tilde{h}_q(T) \geq 0$. Thus, in the present case. we see from ii) that $\tilde{h}_q(T_g) = 0$ for $q \leq 1$.

iv) A bit more general situation can be considered if the rotation is taken in an abstract topological compact group instead of the unity circle, say $T_g : \Gamma \to \Gamma$ where $T_g(\gamma) = g\gamma$ with g such that g^r is the identity in Γ for some positive integer r. Let $\chi(\Gamma)=\{\rho : \Gamma \to S^1 / \ \rho$ is a continuous homomorphism$\}$ ($\chi(\Gamma)$ is ussually called the set of characters of Γ). since Γ is compact, $\chi(\Gamma)$ is discret. We set $\chi(\Gamma)=\{\rho_1, \rho_2, \rho_3 \ldots\}$ and we call $F_n = \bigcap_{i=1}^{n} \ker(\rho_i)$. Γ/F_n factorizes in the product of a finite group G and a torus T^{i_n}[8]. We now consider the induced application to the quotients

$$\Pi_n : \Gamma/F_n \to \Gamma/F_n$$

$$\gamma F_n \mapsto g\gamma F_n.$$

Because Π_n is a rotation in Γ/F_n, it can be written as $\Pi_n \equiv \Pi_{n_1} \times \Pi_{n_2}$ whith Π_{n_1} rotation in G and Π_{n_2} rotation in T^{i_n}. In v) we shall see that $\tilde{h}_q(\Pi_n) \geq \tilde{h}_q(\Pi_{n_1}) + \tilde{h}_q(\Pi_{n_2})$. In particular, iii) implies that $\tilde{h}_q(\Pi_n) \geq \tilde{h}_q(\Pi_{n_2})$ for $q \leq 1$.

v) Here we will prove, in general, the above mentioned inequality, say $\tilde{h}_q(T_1 \times T_2) \geq \tilde{h}_q(T_1) + \tilde{h}_q(T_2)$. To this end we consider two dynamical systems $S_1 = (X_1, \mathcal{A}_1, \mu_1, T_1)$, $S_2 = (X_2, \mathcal{A}_2, \mu_2, T_2)$ and define the product $S_1 \times S_2 = (X_1 \times X_2, \mathcal{A}, \mu, T_1 \times T_2)$. Here $\mathcal{A}$ is the $\sigma-$algebra generated by rectangles $A_{1i} \times A_{2j}$ where $A_{\alpha k} \in \mathcal{A}_\alpha$ ($\alpha = 1, 2$) and μ is the product measure $\mu(A_{1i} \times A_{2j}) = \mu_1(A_{1i}) \cdot \mu_2(A_{2j})$. Let B' and B'' be partitions of X_1 and X_2, respectively, and B the product partition $B' \times B''$. Thus $\tilde{H}_q\left(B^{(n,n)}\right) = \sum_{\ell'\ell''} \left[\mu_1\left(B'^n(\ell')\right) \cdot \mu_2\left(B''^n(\ell'')\right)\right]^q$ where the sum is extended over the names ℓ' and ℓ'', for T_1 and T_2, respectively. Let consider the partitions B' and B''

such that (see Eq.(5)) $\widetilde{h'_q}(T_1) = \widetilde{h'_q}(B', T_1)$ and $\widetilde{h''_q}(T_2) = \widetilde{h''_q}(B'', T_2)$. We have

$$\tilde{h}_q(B, T_1 \times T_2) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sum_{\ell'} \left[\mu_1 \left(B'^n(\ell')\right)\right]^q + \overline{\lim_{n\to\infty}} \frac{1}{n} \log \sum_{\ell''} \left[\mu_2 \left(B''^n(\ell'')\right)\right]^q$$

$$= \widetilde{h'_q}(B', T_1) + \widetilde{h''_q}(B'', T_2) = \widetilde{h'_q}(T_1) + \widetilde{h''_q}(T_2),$$

so that

$$\tilde{h}_q(T_1 \times T_2) \geq \widetilde{h'_q}(T_1) + \widetilde{h''_q}(T_2). \tag{11}$$

For those values of q for which $\tilde{h}_q(T_1 \times T_2) \leq 0$ and $\widetilde{h'_q}(T_1) = \widetilde{h''_q}(T_2) = 0$ Eq. (11) obviously implies $h_q(T_1 \times T_2) = 0$. For example, let $T = T_1 \times T_2 \times .... \times T_n$ where $T_i : S^1 \to S^1$with $T_i(z) = g_i z$ and g_i a root of the unity $(i = 1, 2, ...n)$ so that T is a rotation on a $n - Torus$. By the previous result and iii) we conclude that for some values of q rotations on the $n - Torus$ have zero entropy.

3. Topological q-entropy

3.1. **Definition using open coverings**

In order to define topological q-entropies we start specifying what we will understand, in the following, by a dynamical system. It will be a pair (X, T) with X a compact topological space and T a continuous map from X to X.

Let $\mathcal{U} = (U_\alpha)_{\alpha\in\Lambda}$ be an open covering of X and $N(\mathcal{U})$ be the number of sets in a finite subcovering of $\mathcal{U}$ with the smallest cardinality, i.e.$N(\mathcal{U}) = \min\left\{k : (U_{\alpha_i})_{i=1,2...k}\right\}$ is a finite subcovering of $\mathcal{U}$. From $\mathcal{U}$ we obtain a new covering

$$\mathcal{U}^n = \left\{U_{\alpha_{i_0}} \bigcap T^{-1}\left(U_{\alpha_{i_1}}\right) \bigcap .... \bigcap T^{-(n-1)}\left(U_{\alpha_{i_{n-1}}}\right) : U_{\alpha_{i_j}} \in \mathcal{U}\right\} \tag{12}$$

Analogously to Eq.(3) we set, for any covering $\mathcal{U}$ and arbitrary $q \neq 1$,

$$H_q(\mathcal{U}) = (q-1)^{-1}\left[1 - (N(\mathcal{U}))^{1-q}\right]. \tag{13}$$

Thus we define

$$h_q(T) = (q-1)^{-1}\left[1 - \exp\left(\tilde{h}_q(T)\right)\right], \tag{14}$$

where

$$\tilde{h}_q(T) = \sup\left\{\tilde{h}_q(\mathcal{U}, T) : \mathcal{U} \text{ is a covering of } X\right\} \tag{15}$$

and

$$\tilde{h}_q(\mathcal{U}, T) = \overline{\lim_{n\to\infty}} \frac{1}{n}\left\{\log\left[1 + (1-q)\, H_q(\mathcal{U}^n)\right]\right\}. \tag{16}$$

We call the quantity $h_q(T)$ defined by Eqs.(14-16) the *topological q-entropy* associated to the dynamical system (X, T).

Remark 3.1. For $q = 1$ we have $H_1(\mathcal{U}) = \lim_{q \to 1} H_q(\mathcal{U}) = \log N(\mathcal{U})$; $h_1(\mathcal{U},T) = \overline{\lim}_{n\to\infty} \frac{1}{n} \{\log [H_1(\mathcal{U}^n)]\}$ and $h_1(T) = \sup \{h_1(\mathcal{U},T) : \mathcal{U}\ is\ a\ covering\ of\ X\}$

Bernoulli schemes and Markov systems We use previous definition to calculate topological q-entropies for a couple of basic dynamical systems, namely Bernoulli schemes (BS) and Markov systems (MS).

A Bernoulli scheme associated to k points $(0, 1, \ldots.k-1)$,denoted by $BS(0, 1, \ldots k-1)$, is defined in the same way as in Section 1 but now the measure is misleading. Like before, sample space X is $X = \left\{x = (x_n)_{n\in\mathbf{Z}} : x_i \in \{0, 1, \ldots..k-1\}, \ \forall i \in \mathbf{Z}\right\}$. The given topology is the product of the discret topology in $\{0, 1, \ldots..k-1\}$. A basis for this topology are the "cylinders"

$$\left\{x = (x_n)_{n\in\mathbf{Z}} : x_i = \bar{x}\ (fixed)\ for\ each\ i \in I\ where\ I\ is\ some\ set\ of\ indices\right\}.$$

The transformation is again the shift $\sigma : X \to X$ with $(\sigma x)_n = x_{n+1}$ which is an homeomorphism.

Notice that this space is metrizable by

$$d(x, y) = \sum_{n=-\infty}^{\infty} \delta_{x_n, y_n} 2^{-|n|}. \tag{17}$$

Here $\delta_{x,y}$ denotes the Kroenecker delta.

For calculating the topological q-entropy we take as the covering the partition $\mathcal{U} = \{U_0, U_1, \ldots..U_{k-1}\}$ where $U_i = \{x : x_0 = i\}$ $(i = 0, 1, \ldots.k-1)$. Thus, $N(\mathcal{U}^n) = k^n$ and

$$\tilde{h}_q(\mathcal{U}, \sigma) = \overline{\lim_{n\to\infty}} \frac{1}{n} \left\{\log \left[k^{n(1-q)}\right]\right\} = \log \left[k^{1-q}\right], \qquad for\ q \neq 1. \tag{18}$$

In Section 4 we shall prove that, for $q < 1$, $\tilde{h}_q(\mathcal{V}, \sigma) \leq \tilde{h}_q(\mathcal{U}, \sigma)$ for any covering $\mathcal{V}$. Therefore

$$[h_q(\sigma)]_{BS(0,1,\ldots k-1)} = (q-1)^{-1} \left[1 - k^{1-q}\right], \qquad for\ q < 1, \tag{19}$$

and

$$[h_q(\sigma)]_{BS(0,1,\ldots k-1)} \leq (q-1)^{-1} \left[1 - k^{1-q}\right], \qquad for\ q > 1. \tag{20}$$

For $q = 1$ $[h_1(\sigma)]_{BS(0,1,\ldots k-1)} = \log(k)$.

A Markov system is defined in a similar way. We take bi-infinite sequences and we add a $k \times k$ matrix A with entries 0 or 1, so the phase space is

$$\Sigma_A = \left\{ x = (x_n)_{n\in \mathbf{Z}} : x_i \in \{0,1,.....k-1\}\,,\ \forall i \in \mathbf{Z}; \right.$$

$$\left. A = (A_{ij}) \,\middle|\, A_{ij} \in \{0,1\} \text{ and } A_{x_i,x_{i+1}} = 1 \right\} \tag{21}$$

The transformation and the topology are the same as for BS.

Remark 3.2. When the entries of A are all 1, we have BS (all sequences are allowed).

Remark 3.3. By Perron-Frobenius theorem, the matrix A has a positive eigenvalue E such that $E > |E_i|$ where E_i is any other eigenvalue of A.

We take again the partition $\mathcal{U} = \{U_0, U_1, ...U_{k-1}\}$ where $U_i = \{x : x_0 = i\}$ $(i = 0,1,....k-1)$. Let $F \subset \Sigma_A$ a closed subset and $\sigma(F) = F$ (We shall only consider transitive subshifts of finite type). Let $\lambda_n(F)$ be the cardinality of the set of n-uples $(\ell_0, \ell_1, ...\ell_{n-1})$ such that there exists a point $x = (x_n)_{n\in Z} \in F$ with $x_i = \ell_i (i = 0,1,...n-1)$. Therefore, $N(\mathcal{U}^n, \sigma|_F) = \lambda_n(F)$ and $\tilde{h}_q(\mathcal{U}, \sigma|_F) = \overline{\lim_{n\to\infty}} \frac{1}{n}\left\{\log\left[(\lambda_n(F))^{1-q}\right]\right\} = \tilde{h}_q(\sigma|_F)$ where the second equality holds for $q < 1$.

For the n-uples $(\ell_0, \ell_1, ...\ell_{n-1})$, the existence of a bi-infinite sequence $(x_n)_{n\in\mathbf{Z}}$ with $x_i = \ell_i$ is equivalent to $\prod_{i=0}^{n-2} A_{\ell_i,\ell_{i+1}} = 1$. Then, if $F = \Sigma_A$ we have

$$\lambda_n(\Sigma_A) = \sum_{\ell_0=0}^{k-1}\sum_{\ell_1=0}^{k-1}\cdots\sum_{\ell_{n-1}=0}^{k-1}\prod_{i=0}^{n-2} A_{\ell_i,\ell_{i+1}} = \sum_{\ell_0=0}^{k-1}\sum_{\ell_1=0}^{k-1}\cdots\sum_{\ell_{n-1}=0}^{k-1}\left(A^{n-1}\right)_{\ell_0,\ell_{n-1}}. \tag{22}$$

We set $\|A\| = \sum_{i,j=0}^{k-1} |A_{i,j}|$ so $\lambda_n(\Sigma_A) = \|A^{n-1}\|$. Hence, for $q < 1$,

$$\begin{aligned}\tilde{h}_q(\sigma) &= \overline{\lim_{n\to\infty}}\frac{1}{n}\left\{\log\left[(\lambda_n(\Sigma_A))^{1-q}\right]\right\} \\ &= (1-q)\,\overline{\lim_{n\to\infty}}\left\{\log\left[\left\|A^{n-1}\right\|^{\frac{1}{n}}\right]\right\} = (1-q)\log(E).\end{aligned} \tag{23}$$

Thus,

$$h_q(\sigma) = (q-1)^{-1}\left[1 - E^{1-q}\right], \qquad for\ q < 1 \tag{24}$$

and

$$h_q(\sigma) \leq (q-1)^{-1}\left[1 - E^{1-q}\right], \qquad for\ q > 1 \tag{25}$$

whereas $h_1(\sigma) = \log(E)$. First equality in Eq.(23) will be justified later.

The final result in Eq. (23), say $\tilde{h}_q(\sigma) = (1-q)\log(E)$ for $q < 1$, can be alternatively obtained if, instead of growth rate of $\|A^{n-1}\|^{\frac{1}{n}}$ as given by the second equality.

we use the growth rate of some expression related with distribution of fixed points of σ^n.

Let $P_n(\sigma) = Card\{x \in \Sigma_A : \sigma^n x = x\}$. We have $\sigma^n x = x$ iff $x_n = x_{n+j}$ for each $n \in \mathbf{Z}$. So

$$P_n(\sigma) = \sum_{\ell_0=0}^{k-1} \sum_{\ell_1=0}^{k-1} \cdots \sum_{\ell_{n-1}=0}^{k-1} A_{\ell_0,\ell_1} A_{\ell_1,\ell_2}, \ldots A_{\ell_{n-1},\ell_0}$$

$$= Tr(A^n) = \sum_{i=0}^{k-1} E_i^n. \tag{26}$$

Then, we see that $P_n(\sigma)$ goes assymptotically as E^n. Therefore

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \left\{\log\left[(P_n(\sigma))^{1-q}\right]\right\} = (1-q)\log(E). \tag{27}$$

We know that the expression in the right side member equals $\tilde{h}_q(\sigma)$ for $q < 1$. For $q = 1$, we have $\overline{\lim_{n\to\infty}} \frac{1}{n} \{\log[P_n(\sigma)]\} = \log(E) = h_1(\sigma)$.

3.2. **Definition using span sets** Another family of topological q-entropies can be defined following Bowen's ideas[3]. Let (X, d) be a compact metric space and $T : X \to X$ a continous map. We define a metric

$$d_n(x, y) = \max_{i=0,1,\ldots n-1} \left\{d\left(T^i x, T^i y\right)\right\}. \tag{28}$$

The open ball of radius R with respect to d_n is $\bigcap_{i=0}^{n-1} T^{-i}(\mathcal{B}_R(T^i x))$ where $\mathcal{B}_R(x)$ means the open ball with respect to d.

Definition. A subset $Y$of X is said (n, ϵ)-*span respect to* T if, for each $x \in X$. exists an element $y \in Y$ such that $d_n(x, y) < \epsilon$. Here $n \in \mathbf{Z}^+$, $\epsilon > 0$.

We call $\alpha_{n,\epsilon} = \min\{Card(Y) : Y \text{ is } (n, \epsilon) - \textit{span respect to } T\}$ and put. for $q \neq 1$,

$$H_{q;n,\epsilon} = (q-1)^{-1}\left[1 - \alpha_{n,\epsilon}^{1-q}\right]. \tag{29}$$

Thus we define

$$h'_q(T) = (q-1)^{-1}\left[1 - \exp\left(\widetilde{h'_q}(T)\right)\right] \tag{30}$$

where

$$\widetilde{h'_q}(T) = \lim_{\epsilon\to 0} \tilde{h}_{q;\epsilon} \tag{31}$$

with

$$\tilde{h}_{q;\epsilon} = \overline{\lim_{n\to\infty}} \frac{1}{n} \{\log [1 + (1-q)\, H_{q;n,\epsilon}]\} \,. \tag{32}$$

Remark 3.4. For $q = 1$ we have $H_{1;n,\epsilon} = \lim_{q\to 1} H_{q;n,\epsilon} = \log(\alpha_{n,\epsilon})$so we recover $h'_{1;\epsilon} = \overline{\lim_{n\to\infty}} \frac{1}{n} \{\log [H_{1;n,\epsilon}]\}$ and $h'_1(T) = \lim_{\epsilon\to 0} h_{1;\epsilon}$, a known result.

Remark 3.5. If X is not compact, we take, for a compact subset K of X,

$$\tilde{h}_{q;\epsilon;K} = \overline{\lim_{n\to\infty}} \frac{1}{n} \{\log [1 + (1-q)\, H_{q;n,\epsilon;K}]\} \,. \tag{33}$$

where $H_{q;n,\epsilon;K}$ is defined in the same way as before, but using (n,ϵ)-*span sets* for K and then

$$\widetilde{h'_q}(T) = \sup_{K \text{ compact}} \left\{ \lim_{\epsilon\to 0} \tilde{h}_{q;\epsilon} \right\} \tag{34}$$

We call the quantity $h'_q(T)$ given by Eqs.(30-32), for $q \neq 1$, and by *Remark 1* for $q = 1$, the *topological q-entropy defined by using span-sets.*

Another definition, dual to the above one, can be given by using (n,ϵ)-*separated sets.*

Definition. A subset Y of X is said (n,ϵ)-*separated respect to* T if, for each $x, y \in Y$, with $x \neq y$, $d_n(x,y) > \epsilon$.

We now call $\beta_{n,\epsilon} = \max\{Card(Y) : Y \text{ is } (n,\epsilon) - separated\ respect\ to\ T\ \}$ and put, for $q \neq 1$,

$$H_{q;n,\epsilon} = (q-1)^{-1} \left[1 - \beta_{n,\epsilon}^{1-q}\right] . \tag{35}$$

Then we obtain the dual to equations (30-32) and *Remarks 1* and *2* by changing $\alpha_{n,\epsilon}$ by $\beta_{n,\epsilon}$.

The following relations hold[3]:

i) $\alpha_{n,\epsilon} \leq \beta_{n,\epsilon} \leq \alpha_{n,\epsilon/2}$.

ii) If $\epsilon_1 < \epsilon_2$, then $\beta_{n,\epsilon_1} \geq \beta_{n,\epsilon_2}$.

Remark 3.6 With these definitions, it is directly proved that, if T is an isometry in (X,d), its entropy is zero.

At this point a natural question arises: is the entropy invariant with respect to the metric considered ? The answer is affirmative in case of considering uniformely equivalent metrics.

Proposition 3.7. If $id : (X,d_1) \to (X,d_2)$ and $id : (X,d_2) \to (X,d_1)$ are continous, then $h_q(T,d_1) = h_q(T,d_2)$, where we have explicitly indicated the dependence of the entropy on the metrics.

Proof: We choose ϵ_1, ϵ_2, $\epsilon_3 > 0$ and $\epsilon_2 \geq \epsilon_3$ such that $d_1(x,y) \leq \epsilon_1$ whenever $d_2(x,y) \leq \epsilon_2$ and $d_2(x,y) \leq \epsilon_2$ whenever $d_1(x,y) \leq \epsilon_3$.

A (n, ϵ_2)-span set for X with respect to d_2 is a (n, ϵ_1)-span set for X with respect to d_1. So $\alpha_{n,\epsilon_1,d_1} \leq \alpha_{n,\epsilon_2,d_2}$ and $\alpha_{n,\epsilon_2,d_2} \leq \alpha_{n,\epsilon_3,d_1}$. For $q > 1$ we have $(\alpha_{n,\epsilon_1,d_1})^{1-q} \geq (\alpha_{n,\epsilon_2,d_2})^{1-q}$ and $(\alpha_{n,\epsilon_2,d_2})^{1-q} \geq (\alpha_{n,\epsilon_3,d_1})^{1-q}$. Therefore,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \left\{ \log \left[(\alpha_{n,\epsilon_1,d_1})^{1-q} \right] \right\} \geq \overline{\lim_{n\to\infty}} \frac{1}{n} \left\{ \log \left[(\alpha_{n,\epsilon_2,d_2})^{1-q} \right] \right\}$$

$$\geq \overline{\lim_{n\to\infty}} \frac{1}{n} \left\{ \log \left[(\alpha_{n,\epsilon_3,d_1})^{1-q} \right] \right\} \qquad (36)$$

and $\tilde{h}_{q;\epsilon_1} \geq \tilde{h}_{q;\epsilon_2} \geq \tilde{h}_{q;\epsilon_3}$ (For $q < 1$, inequalities invert). If $\epsilon_i \to 0$, then $\tilde{h}_q(T, d_1) = \tilde{h}_q(T, d_2)$ and $h_q(T, d_1) = h_q(T, d_2)$. □

3.3. **Relationship between both two definitions** Here we shall gather for which values of the parameter q previous definitions are equivalent.

The diameter of a covering $\mathcal{U}$ is defined by $diam(\mathcal{U}) = \sup_{A\in\mathcal{U}} \{diam(A)\}$. Notice that, for $\mathcal{U}$ and $\mathcal{V}$ open coverings of X (compact) and δ a Lebesgue number for $\mathcal{V}$, if $diam(\mathcal{U}) < \delta$ then $\mathcal{U}$ is a refinament of $\mathcal{V}$.

Proposition 3.8. Let $q \leq 1$ and let $\{\mathcal{U}_m\}_{m=1}^{\infty}$ be a family of coverings of X with $diam(\mathcal{U}_m) \underset{m\to\infty}{\longrightarrow} 0$, then $\lim_{m\to\infty} h_q(\mathcal{U}_m, T) = h_q(T)$.

Proof: Let $q < 1$ and suppose firstly that $h_q(T) < \infty$. For each $\epsilon > 0$ there exists a covering $\mathcal{V}$ with $\tilde{h}_q(\mathcal{V}, T) > \tilde{h}_q(T) - \epsilon$. Let δ be a Lebesgue number for $\mathcal{V}$, since $diam(\mathcal{U}_m) \underset{m\to\infty}{\longrightarrow} 0$, there is a natural N such that for $m \geq N$, $diam(\mathcal{U}_m) < \delta$. So $\mathcal{U}_m \prec \mathcal{V}$ whenever $m \geq N$. It follows that $N(\mathcal{U}_m^n) \geq N(\mathcal{V}^n)$ and, because $q < 1$, $\tilde{h}_q(\mathcal{U}_m, T) \geq \tilde{h}_q(\mathcal{V}, T) > \tilde{h}_q(T) - \epsilon$. Then $\tilde{h}_q(T) \geq \tilde{h}_q(\mathcal{U}_m, T) > \tilde{h}_q(T) - \epsilon$ and $\lim_{m\to\infty} \tilde{h}_q(\mathcal{U}_m, T) = \tilde{h}_q(T)$. So $\lim_{m\to\infty} h_q(\mathcal{U}_m, T) = h_q(T)$.

If $\tilde{h}_q(T) = \infty$ for each M arbitrarily large we can choose a covering $\mathcal{V}$ such that $\tilde{h}_q(\mathcal{V}, T) > M$ and we follow like before.

For $q = 1$ we proceed similarly by using the definition for this value.□

Remark 3.9. For any value of q we have $h_q(T) \leq (q-1)^{-1} \left\{ 1 - \exp\left[\lim_{m\to\infty} h_q(\mathcal{U}_m, T) \right] \right\}$.

Lemma 3.10. (c.f. [29]) Let $T : X \to X$ continous with (X, d) a compact metric space.

i) If $\mathcal{U}$ is an open covering of X and δ is a Lebesgue number for $\mathcal{U}$, then

$$N(\mathcal{U}^n) \leq \alpha_{n,\delta/2} \leq \beta_{n,\delta/2} \qquad (37)$$

ii) For each $\epsilon > 0$ and $\mathcal{V}$ an open covering with $diam(\mathcal{V}) \leq \epsilon$, we have $\alpha_{n,\epsilon} \leq \beta_{n,\epsilon} \leq N(\mathcal{V}^n)$.

Corollary 3.11. (c.f. [29]) Let $T : X \to X$ continous with (X, d) a compact metric space. Let $\mathcal{U}_\epsilon$ be a covering by open balls of radius 2ϵ and let $\mathcal{V}_\epsilon$ be another covering by open balls of radius $\epsilon/2$, then

$$N\left(\mathcal{U}_\epsilon^n\right) \le \alpha_{n,\epsilon} \le \beta_{n,\epsilon} \le N\left(\mathcal{V}_\epsilon^n\right) \tag{38}$$

Based on the previous results we can state the

Theorem 3.12. If (X,d) is a compact metric space and $T : X \to X$ a continous map, then

i) for $q \le 1$ is $h_q(T) = h'_q(T)$

ii) for $q > 1$ is $h_q(T) \le h'_q(T)$,

where $h_q(T)$ and $h'_q(T)$ are the topological q-entropies defined by coverings and span sets, respectively.

Proof: i) For $q < 1$, by the Lemma 3.11, $\tilde{h}_q(\mathcal{U}_\epsilon, T) \le \tilde{h}_{q;\epsilon} \le \tilde{h}_q(\mathcal{V}_\epsilon, T)$. If we put $\epsilon = r_n$ with $r_n \to 0$, we have, by the Proposition 3.8, $h_q(T) = h'_q(T)$ because $\tilde{h}_q(\mathcal{U}_\epsilon, T)$ and $\tilde{h}_q(\mathcal{V}_\epsilon, T)$ tend to $\tilde{h}_q(T)$ and $\tilde{h}_{q;\epsilon}$ goes to $\widetilde{h'_q}(T)$. For $q = 1$ the result follows by a similar argument but using the special definition for this case.

ii) From Eq.(38) we have

$$\left[N\left(\mathcal{U}_\epsilon^n\right)\right]^{1-q} \ge \left[\alpha_{n,\epsilon}\right]^{1-q} \ge \left[\beta_{n,\epsilon}\right]^{1-q} \ge \left[N\left(\mathcal{V}_\epsilon^n\right)\right]^{1-q}. \tag{39}$$

so

$$h_q(T) \le (q-1)^{-1}\left\{1 - \exp\left[\lim_{n\to\infty} \tilde{h}_{q;r_n}\right]\right\} = h'_q(T). \tag{40}$$

□

3.4. **Properties and calculations** We define now what we understand by an isomorphism between two topological dynamical systems.

Definition. Let $S_i = (X_i, T_i)$, $i = 1,2$ be two dynamical systems. We say that $S_1$is isomorphic to S_2 if there exists an homeomorphism $\varphi : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ \varphi\downarrow & & \downarrow\varphi \\ X_2 & \xrightarrow{T_2} & X_2 \end{array}$$

Theorem 3.13. Topological q-entropy defined by coverings is an isomorphism invariant.

Proof: Let $S_i = (X_i, T_i)$, $i = 1,2$ be two isomorphic dynamical systems and $\mathcal{U}$ be an open covering of X_1. We consider the subcovering $(U_{\alpha_i})_{i=1,2\ldots N(\mathcal{U})}$. As before

$$\mathcal{U}^n = \left\{U_{\alpha_{i_0}} \bigcap T_1^{-1}\left(U_{\alpha_{i_1}}\right) \bigcap \cdots \bigcap T_1^{-(n-1)}\left(U_{\alpha_{i_{n-1}}}\right) : U_{\alpha_{i_j}} \in \mathcal{U}\right\}.$$

Let the covering $\mathcal{V} = \varphi(\mathcal{U})$ (φ is homeo) and

$$\mathcal{V}^n = \left\{ V_{\alpha_{j0}} \bigcap T_2^{-1}\left(V_{\alpha_{j1}}\right) \bigcap \cdots \bigcap T_2^{-(n-1)}\left(V_{\alpha_{j_{n-1}}}\right) : V_{\alpha_{j_k}} \in \mathcal{V} \right\}.$$

If A is a member of $\mathcal{U}^n$ then $A \equiv U_{\alpha_{i_0}} \bigcap T_1^{-1}\left(U_{\alpha_{i_1}}\right) \bigcap \cdots \bigcap T_1^{-(n-1)}\left(U_{\alpha_{i_{n-1}}}\right)$ for some sequence $(i_0, i_1, ..., i_{n-1})$. Therefore $A \subset U_{\alpha_{i_0}}$ and $\varphi(A) \subset \varphi\left(U_{\alpha_{i_0}}\right) = V_{\alpha_{i_0}}$: $T_1(A) \subset U_{\alpha_{i_1}}$ and $T_2[\varphi(A)] \subset V_{\alpha_{j_1}}$ (for some j_1) but $T_2[\varphi(A)] = \varphi[T_1(A)]$ so $i_1 = j_1$. By continuing in this way we finally get $(i_0, i_1, ..., i_{n-1}) \equiv (j_0, j_1, ..., j_{n-1})$.

$N(\mathcal{U}^n) = Card\left\{(i_0, i_1, ..., i_{n-1}) : U_{\alpha_{i_0}} \bigcap T_1^{-1}\left(U_{\alpha_{i_1}}\right) \bigcap \cdots \bigcap T_1^{-(n-1)}\left(U_{\alpha_{i_{n-1}}}\right) \neq \emptyset\right\}$.

Let $x \in U_{\alpha_{i_0}} \bigcap T_1^{-1}\left(U_{\alpha_{i_1}}\right) \bigcap \cdots \bigcap T_1^{-(n-1)}\left(U_{\alpha_{i_{n-1}}}\right)$, $x \in U_{\alpha_{i_0}}$ iff $\varphi(x) \in \varphi\left(U_{\alpha_{i_0}}\right) = V_{\alpha_{i_0}}$.

Besides $T_1(x) \in U_{\alpha_{i_1}}$ iff $\varphi[T_1(x)] = T_2[\varphi(x)] \in \varphi(U_{\alpha_{i_1}})$, then $\varphi(x) \in T_2^{-1}\left(V_{\alpha_{i_1}}\right)$and so on. Thus, if $(i_0, i_1, ..., i_{n-1})$ is such that $U_{\alpha_{i_0}} \bigcap T_1^{-1}\left(U_{\alpha_{i_1}}\right) \bigcap \cdots \bigcap T_1^{-(n-1)}\left(U_{\alpha_{i_{n-1}}}\right) \neq \emptyset$ then $V_{\alpha_{i_0}} \bigcap T_2^{-1}\left(V_{\alpha_{i_1}}\right) \bigcap \cdots \bigcap T_2^{-(n-1)}\left(V_{\alpha_{i_{n-1}}}\right) \neq \emptyset$ so $N(\mathcal{U}^n) = N(\mathcal{V}^n)$ and $\tilde{h}_q(\mathcal{U}, T_1) = \tilde{h}_q(\mathcal{V}, T_2)$. From this we have $\tilde{h}_q(T_1) \geq \tilde{h}_q(\mathcal{V}, T_2)$ i.e. $\tilde{h}_q(T_1)$ is an upper bound of $\tilde{h}_q(\mathcal{V}, T_2)$ and $\tilde{h}_q(T_1) \geq \tilde{h}_q(T_2)$. Anagously $\tilde{h}_q(T_1) \leq \tilde{h}_q(T_2)$ by symmetry.□

Remark 3.14. If φ is continous and surjective then just holds $h_q(T_1) \geq h_q(T_2)$ for $q \geq 1$ and $h_q(T_1) \leq h_q(T_2)$ for $q < 1$.

Remark 3.15. Since for $q \leq 1$ the topological q-entropies defined by the two ways previously introduced are equivalent the statement of the theorem is true also for the definition using span sets. For $q > 1$, the theorem can be proved for this last definition by using a similar idea.

Now we find a bound for the topological q-entropy using span sets for a differentiable map $f : M^d \to M^d$ where M is a compact, Riemannian, d-dimensional manifold.

Remark 3.16. The calculation will be done using the metric induced by the Riemaniann structure. However, it is not relevant due to the compactness of the manifold. In case that the manifold were not compact, the bound works but, of course, it will depend on the metric.

Let $\mathcal{D}_x f : T_x(M) \to T_{f(x)}(M)$ be the differential map at the point $x \in M$ and

$K = \sup_{x \in M} \|\mathcal{D}_x f\|$. Suppose K finite (otherwise the result that we will show would be trivial). If $K \leq 1$, by the mean value theorem, $d_n(x, y) \leq d(x, y)$ for each n, and so $h'_q(f) = 0$.

For $K > 1$ we use a Bowen's construction. Let $\mathcal{U} = \{f_\lambda(B)\}_{\lambda \in \Lambda}$ a covering obtained by selecting differentiable maps: $f_\lambda : B \to M$ where $B = \mathcal{B}_R(0) \subset \mathbf{R}^d$ with the radius R convenently choosen.

Let now $\left\{f_{\lambda_j}(B)\right\}_{j=1}^{N}$ be a finite subcovering of $\mathcal{U}$. We consider a constant C such that $d\left[f_{\lambda_j}(x),\ f_{\lambda_j}(y)\right] \leq C\left\|x-y\right\|$ for $x, y \in \mathcal{B}_{R'}(0)$, $R' < R$ and $j = 1, 2, \ldots, N$.

For $0 < r \leq 1$, we put $H(r) = \left\{(n_1 r, n_2 r, \ldots., n_d r) : n_i \in \mathbf{Z},\ |n_i| < R'/r\right\}$. So. $Card(H(r)) \leq \left(2R'/r\right)^d$. Let $G(r) = \bigcup_{j=1}^{N} f_{\lambda_j}(H(r))$; $G(r)$ is a $(n, K^n C r)$*-span set respect to* f.

For any $\epsilon > 0$, we set $r = \epsilon/K^n C$ and then $\alpha_{n,\epsilon} \leq N\left(2R'K^nC/\epsilon\right)^d = (K^n)^d \times N\left(2R'C/\epsilon\right)^d$.

For $q > 1$

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\left\{\log\left[(\alpha_{n,\epsilon})^{1-q}\right]\right\}$$

$$\geq \overline{\lim_{n\to\infty}}\frac{1}{n}\left\{\log\left[\left(K^{nd}\right)^{1-q}\right] + \log\left[\left(2NR'C/\epsilon\right)^{d(1-q)}\right]\right\} = d(1-q)\log(K). \quad (41)$$

So

$$\widetilde{h'_q}(f) \geq d(1-q)\log(K) \quad (42)$$

and

$$h'_q(f) \leq \frac{1-K^{d(1-q)}}{q-1}. \quad (43)$$

For $q < 1$ Eq.(43) also holds because the unequality in Eq.(42) inverts.

For $q = 1$ we have $h'_1(f) \leq \max\{0, d\log(K)\}$

Remark 3.17. For $q \leq 1$, the topological q-entropy by coverings $h_q(f)$ is equivalent to $h'_q(f)$; for $q > 1$ we have $h_q(f) \leq h'_q(f)$ (c.f. **Theorem 3.12.** ii). the bound works but we do not know if there exists a better one.

Flows Here we present some results about the q-entropy of flows. First we relate topological q-entropy of a geodesic flow in a manifold M with a volume in the universal covering $\tilde{M}$ by using Manning's constructions[12].

Let M be a closed Riemannian manifold, and let $\mathcal{B}_R(x) = \left\{y : \tilde{d}(x,y) < R\right\}$ where $\tilde{d}$ is the metric in $\tilde{M}$ inherited from the Riemannian structure of M. We call $V_{x,R}$ the volume of the ball $\mathcal{B}_R(x)$.

Fact. $L = \lim_{R\to\infty} \frac{1}{R}\log(V_{x,R})$ exits and is independent of x. For proving this claim let F a fundamental domain in $\tilde{M}$, i.e. $\tilde{M} = \bigcup_{\gamma\in\Gamma} \gamma\left(\bar{F}\right)$, $\Gamma \subset Isom(\tilde{M})$ and $\gamma\left(\bar{F}\right) \bigcap \lambda\left(\bar{F}\right) = \emptyset$ for $\lambda \neq \gamma$. Let D de diameter of F (the metric can be normalized in order to do $D = 1$). For each $y \in F$

$$\mathcal{B}_{R-D}(x) \subset \mathcal{B}_R(y) \subset \mathcal{B}_{R+D}(x) \qquad for\ R > D$$

and for all $x, y \in \tilde{M}$

$$V_{x,R-D} \leq V_{y,R} \leq V_{x,R+D}$$

because the translation of the point from or towards F is by isometries. So the limit in case of existing is independent of x. For the existence we have[12]

$$\overline{\lim_{R\to\infty}} \frac{1}{R}\log(V_{x,R}) \leq \underline{\lim_{S\to\infty}} \frac{1}{S}\log(V_{x,S})$$

and the *Fact* follows.□

If M has constant sectional curvature $\mathcal{K} \leq 0$, is a standard result $V_{x,R} \sim const \times \exp\left[(n-1)\sqrt{|\mathcal{K}|}R\right]$ ($n = \dim M$), so in this case

$$\lim_{R\to\infty} \frac{1}{R}\log(V_{x,R}) = (n-1)\sqrt{|\mathcal{K}|} \tag{44}$$

Let $\Phi = \{\varphi_t : M \to M \,/\, t \in \mathbf{R}\}$ a flow in a compact manifold. The topological q-entropy (using span or separated sets) of Φ is defined by

$$h_q^{'}(\Phi) = h_q^{'}(\varphi_{t=1}) \equiv h_q^{'}(\varphi)$$

An alternative definition can be given for flows:

Definition. Let t and ϵ two real strictly positive numbers. A set $Y \subset M$ is (t,ϵ)-*span respect to* Φ if for any $y \in Y$ there exists a $x \in M$ such that $d(\varphi_s(x), \varphi_s(y)) < \epsilon$ for $0 \leq s \leq t$.

Definition. Let t and ϵ two real strictly positive numbers. A set $Y \subset M$ is (t,ϵ)-*separated respect to* Φ if for $x, y \in Y$, $x \neq y$ is $d(\varphi_s(x), \varphi_s(y)) > \epsilon$ for some $s \in [0,t]$.

We call $\eta_{t,\epsilon} = \min\{Card(Y) : Y \text{ is } (t,\epsilon) - span\ respect\ to\ \Phi\ \}$ and put, for $q \neq 1$

$$\widetilde{h_q^{'}}(T) = \lim_{\epsilon\to 0} \tilde{h}_{q;\epsilon} \tag{45}$$

with

$$\tilde{h}_{q;\epsilon} = \overline{\lim_{t\to\infty}} \frac{1}{t} \left\{ \log\left[1 + (1-q) H_{q;t,\epsilon}\right] \right\} . \tag{46}$$

where $H_{q;t,\epsilon}$ is defined like in Eq.(29) by changing $\alpha_{n,\epsilon}$ by $\eta_{t,\epsilon}$.

Anagolously let $\chi_{t,\epsilon} = \max\{Card(Y) : Y \text{ is } (t,\epsilon) - \textit{separated respect to } \Phi \}$ We define $\widetilde{h'_q}(T)$ and $\tilde{h}_{q;\epsilon}$ as in Eqs(45) and (46) bu now $H_{q;t,\epsilon}$ is defined like in Eq.(35) by changing $\beta_{n,\epsilon}$ by $\chi_{t,\epsilon}$.

Let M a closed Riemaniann manifold and let $S(M)$ its sphere tangent bundle. i.e

$$S(M) = \{(x,v) \in T(M) / g_x(v,v) = 1 \}$$

where $g_x$is the bilinear form in $T_x(M)$ which gives the Riemaniann structure.

We particulary considerer the geodesic flow on $S(M)$: $\varphi_t(x,v) = \dot{\gamma}_v(t)$ where γ_v is a geodesic in M with $\gamma_v(0) = x$ and $\dot{\gamma}$ is its derivative such that $\dot{\gamma}_v(0) = v$. Now we use Manning's ideas. For any small $\delta > 0$, let $\Omega = \mathcal{B}_{R+\delta/2}(x) - \mathcal{B}_R(x)$ and for any arbitrary small ϵ we have

$$\exp\left[(L-\epsilon) R\right] \le V_{x,R} \le \exp\left[(L+\epsilon) R\right], \qquad \textit{for } R \ge R_0(\epsilon). \tag{47}$$

We can choose a sequence of values of $R = (R_n)_{R_n\to\infty}$ such that if we slightly increase one value of the sequence (say R_N) we have$\subset$

$$V_{x,R_N+\delta/2} - V_{x,R_N} \ge \exp\left[(L-\epsilon) R_N\right]. \tag{48}$$

We take a set $\mathcal{T}_{R_N}$ (maximal) in the annulus Ω whose points are pointwise 2δ apart, $Card(\mathcal{T}_{R_N}) \ge const. \times \exp\left[(L-\epsilon) R_N\right]$.

Considering the unitary initial vectors of the geodesics (with length between $\delta/2$ and δ) which join x with any point of $\mathcal{T}_{R_N}$, we get a set $\tilde{\mathcal{T}} \subset S(\tilde{M})$ that is $(R_N,\delta) -$ *separated* for the geodesic flow on $S(\tilde{M})$ (Manning). These geodesics exist because $\tilde{M}$ is complete. We have so

$$d'\left[\tilde{\varphi}_{R_N}(\tilde{v}_1), \tilde{\varphi}_{R_N}(\tilde{v}_2)\right] > \delta \qquad \textit{for any } \tilde{v}_1,\, \tilde{v}_2 \in \tilde{\mathcal{T}}_{R_N}; \tilde{v}_1 \ne \tilde{v}_2 \qquad 49)$$

where $\tilde{\varphi}$ is the flow and d' is a metric in $S(\tilde{M})$. The image of these vectors by the natural map $S(\tilde{M}) \to S(M)$ gives a $(R_N,\delta) - \textit{separated set}$ for φ in $S(M)$ (Manning).

Let $\beta_{R_N} = \max\{Card(E) : E \text{ is } (R_N,\delta) - \textit{separated}\}$. Thus for $q > 1$,

$$\beta_{R_N}^{1-q} \le \left\{const. \times \exp\left[(L-\epsilon) R_N\right]\right\}^{1-q}. \tag{50}$$

Then, $\widetilde{h'_q}(\Phi) \le (1-q)(L-\epsilon)$. Therefore, since ϵ is arbitraly small:

$$h'_q(\Phi) \ge (q-1)^{-1} \left\{1 - \exp\left[(1-q) L\right]\right\}. \tag{51}$$

For $q < 1$ unequality (50) inverts and, consequently, Eq.(51) also holds for $q < 1$.

When $q = 1$, $h_1'(\Phi) \geq L$.

Remark 3.18. If M has all sectional curvatures equal zero, the bound is irrelevant. However, we believe that for a flow of a Hamiltonian system in the $2n$-dimensional euclidean space a genuine bound could be obtained. We expect to report results on this matter elsewhere.

Now we consider a geodesic flow in a manifold with all sectional curvatures negatives (in particular the flow is Anosov).

In $S(M)$ we take the metric:

$$d_1(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) = \sup_{t\in[0,1]} \{d(\gamma_1(t), \gamma_2(t))\} \tag{52}$$

where γ_1, γ_2 are geodesics in M parametrized by arc length. The main argument used here is that two geodesics apart one other exponentially in negatively curved spaces.

One span set for φ in $S(M)$ is obtained by taking the images of tangent vectors to geodesics in $\tilde{M}$ by the map $S(\tilde{M}) \to S(M)$. Again we use a Manning's construction for these types of spaces.

Let $\Omega_R = \left\{x \in \tilde{M} \ : R - D \leq d(x, F) \leq R\right\} \subset \mathcal{B}_{R+D}(x)$. Here F is as before a fundamental domain in $\tilde{M}$ and $D = diam(F)$. We work in a similar way as above: we can find a maximal set $\mathcal{H}_{R_N}$ in the annulus Ω_R, with $Card(\mathcal{H}_{R_N}) \leq const. \times \exp\left[(L-\epsilon)(R_N + D + \delta/2)\right]$ for small positve δ, ϵ. We take again geodesics parametrized by arc length which join any point in a maximal $\delta - separated\ set$ G in F with any point in $\mathcal{H}_{R_N}$ (R_N a sequence selected like before). The image of the initial vectors to such geodesics by the map $S(\tilde{M}) \to S(M)$ forms a $(R_N - 1, 4\delta) - span\ set$. The result used for prooving this is the following:

Lemma 3.19. ([12]): If γ_1, γ_2 are geodesics in a negatively curved manifold M, parametrized by arc length, then $d(\gamma_1(t), \gamma_2(t)) \leq d(\gamma_1(0), \gamma_2(0)) + d(\gamma_1(\lambda), \gamma_2(\lambda))$ where $\gamma_1, \gamma_2 : [0, \lambda] \to M$.

By taking the metric defined by Eq.(52), and by using the previous Lemma, it can be proved that the tangent vectors to the initial points of the geodesics mentioned in the paragraph just before the Lemma 3.4.1, are within distance 4δ apart, so the corresponding vectors in $S(M)$ are 4δ apart too.

Let $\alpha_{R_N} = \max\{Card(E) : E \text{ is } (R_N - 1, 4\delta) - span\ set\}$. Therefore for $q > 1$,

$$\alpha_{R_N}^{1-q} \geq \{Card(\mathcal{H}_{R_N}) \cdot Card(G)\}^{1-q} \tag{53}$$

where G is the maximal set in F mentioned above. Thus $h_q'(\Phi) \geq (1-q)(L-\epsilon)$ for any arbitrary small ϵ. So

$$h'_q(\Phi) \leq (q-1)^{-1}\left\{1-\exp\left[(1-q)\,L\right]\right\}. \tag{54}$$

For $q < 1$, the unequality is true because Eq.(53) inverts, and with the previous result we have

$$h'_q(\Phi) = (q-1)^{-1}\left\{1-\exp\left[(1-q)\,L\right]\right\}. \tag{55}$$

in manifolds with all sectional curvatures negatives.

Remark 3.20. Let Σ a surface with constant negative curvature $\mathcal{K}$ and let call

$$\mathcal{C}(R) = Card\left\{\gamma : \text{is a closed geodesic in } \Sigma \text{ with length}(\gamma) \leq R\right\}.$$

Inmediatly from Margulis[13] and Eq.(44) we have

$$\lim_{R\to\infty}\frac{1}{R}\log\left(\mathcal{C}(R)\right) = \sqrt{|\mathcal{K}|}, \tag{56}$$

so

$$h'_q(\Phi) = (q-1)^{-1}\left\{1-\exp\left[(1-q)\sqrt{|\mathcal{K}|}\right]\right\}. \tag{57}$$

The result that two geodesics apart exponentially is valid in geodesic spaces (not necessarily manifolds) negatively curved in the sense of Gromov. A good program, we think, could be trying to work with flows in this kind of spaces in order to obtain bounds for the q-entropies. As a first step let, for example, $\Sigma = H^2/\Gamma$ where H^2 is the hyperbolic plane and Γ is the triangle group, i.e. the orientation preserving subgroup of the group generated by reflections in the sides of a triangle in H^2 with angles π/p, π/q, π/r with $1/p+1/q+1/r<1$. This group is usually denoted by $\Delta(p,q,r)$. Σ is an orbifold with three singularities which are cone points with angles $2\pi/p$, $2\pi/q$, $2\pi/r$, i.e. points with neighborhoods respectively homeomorphic to $\mathbf{R}^2/\mathbf{Z}_p$, $\mathbf{R}^2/\mathbf{Z}_q$, $\mathbf{R}^2/\mathbf{Z}_r$; and it is a base of a Seifert bundle. Besides, Σ has a hyperbolic structure except in the cone points. For a very nice account in this matter see, for example, Ref. [22]. The geodesic flow, in the generalized sphere tangent bundle $S(\Sigma)$, is Anosov and $S(\Sigma)$ is a manifold[6].

One interesting consequence of these results is the relation between q-entropy and a limit which involves the number of words in a presentation of $\pi_1(M)$ with a given length.

Let F a compact fundamental domain in M and let $\mathcal{S} = \{\gamma : \gamma \in \pi_1(M),\ \gamma(F)\bigcap F \neq \emptyset\}$ $\mathcal{S}$ is a system of generators for $\pi_1(M)$. We call $\mathcal{B}_{\mathcal{S}}(R) = Card\left\{\gamma : \gamma \in \pi_1(M),\ \ell(\gamma) \leq R\right\}$ where $\ell(\gamma)$ is the minimal n such that γ can be expressed as a word $\gamma = \gamma_1.\gamma_2\ldots.\gamma_n$ with $\gamma_i \in \mathcal{S}$.

The limit $W = \lim_{R\to\infty} \frac{1}{R} \log (\mathcal{B}_S(R))$exists by J. Milnor[16]. If $\ell(\gamma) \leq \frac{R}{D} - 1$ ($D = diam(F)$) it follows that $\tilde{d}(x, \gamma y) \leq R$ for $y \in F$, so $\gamma(F) \subset \mathcal{B}_R(x)$. Thus. $L \geq W$ and

$$h'_q(\Phi) \geq (q-1)^{-1} \{1 - \exp[(1-q) W]\} . \tag{58}$$

Therefore if $W \neq 0$ we have a genuine bound for the q-entropy in terms of the limit W.

For any group Γ finitely generated by a subset $\mathcal{S}$ the limit $W_S(\Gamma)$ is invariant by quasi-isometries respect the word-metric, which is defined as $d_S(\gamma_1, \gamma_2) = \ell(\gamma_1^{-1} . \gamma_2)$[7]

Remark 3.21. $W_S(\Gamma)$ does not depend on the system of generators , because if $\mathcal{S}'$ is another finite system of generators for Γ,we have $\mathcal{B}_{S'}(R) \leq \mathcal{B}_S(kR)$, where $k = \max\left\{\ell(\gamma') : \gamma' \in \mathcal{S}'\right\}$.

Remark 3.22. *If M* has negative curvature $\Gamma = \pi_1(M)$, verifies $\mathcal{B}_S(R) \geq ab^R$. for some constants $a, b. W_S(\Gamma)$ can be ∞, it happens for example when $\Gamma = F^m$ (a free group in m generators with $m \geq 2$).Because $W_S(\mathbf{Z}^n) = n$, $\mathbf{Z}^n$ and F^m cannot be quasy-isometrics.

Recall that a map f is a quasy-isometry between two metric spaces (X_1, d_1) and (X_2, d_2) if there exits two constants C_1(estrictily positive) and $C_2 \geq 0$, such that

$$\frac{1}{C_1} d_1(x,y) - C_2 \leq d_2(f(x), f(y)) \leq C_1 d_1(x,y) + C_2. \tag{59}$$

Therefore to get a bound for $h'_q(\Phi)$ can be used the limit W corresponding to any group quasy-isometric to $\pi_1(M)$.

We finish this Subsection with an approach for working in a more general context: Let X a geodesic space, i.e. a metric space such that for any $x, y \in X$, there exists an isometry $g : [0, a] \to X$, with $a = dist(x,y)$ and $g(0) = a, g(a) = y$. Complete Riemannian manifolds are a such spaces.

We consider a group Γ,which acts discretelly and isometrically on X, in such way that X/Γ is compact, let $\pi : X \to X/\Gamma$ the natural projection . A metric in X/Γ can be given setting

$$d(p,q) = \sup\left\{\tilde{d}(x,y) : x \in \pi^{-1}(p), y \in \pi^{-1}(q)\right\}. \tag{60}$$

Where $\tilde{d}$is the metric in X. Let $D = \{d(p,q) : p, q \in X/\Gamma\}$, D is finite because X/Γ is compact. As a fundamental domain can be taken $F = \mathcal{B}_D(x), x \in X$.

If X is simply connected Γ and $\pi_1(X/\Gamma)$ are isomorphic, Γ is finetelly generated and it is quasy-isometric to X. (Γ with the word metric).

For the case of a closed constant negativeley curved surface Σ it is presented as $\Sigma = X/\Gamma$, where $X = E^2, S^2$ or H^2,and $\Gamma \subset$Isom(X) which acts discretelly.

3.5. **q-entropy and symbolic dynamics** The q-entropy can be related with the "complexity " of a set of symbols. In this subchapter we present a brief description of the method for discretizing an hyperbolic flow in order to obtain these symbols, and the relationship between the q-entropy of the original flow and the q-entropy of the space built from bi-infinite sequences of symbols endowed with a topology.

Let $\Phi = \{\varphi_t\}_{t\in\mathbf{R}}$ a flow in compact differentiable manifold M , which has a Riemannian stucture.

We recall standard concepts:

Definition. A set $\Theta \subset M$ is a basic set if:

i) Is φ_t-invariant and closed.

ii) $\varphi_t \mid_\Theta$is topologically transitive.

iii)The set of periodic points contained in Θ is dense in Θ , and it does not have fixed points

iv) There esists an open set $U \supset \Theta$, such that $\Theta = \bigcap\limits_{t\geq 0} \varphi_t(U)$.

v) The tangent bundle to M restricted to Θ can be splitted as a Whitney sum of three φ_t-invariant sub-bundles:

$$TM \mid_\Theta = E^0 \oplus E^u \oplus E^s.$$

such that

$$\|\mathcal{D}\varphi_t(v)\| \leq Ae^{-Bt}\|v\|, \text{ for } v \in E^s, t \geq 0;\ A, B > 0 \tag{61}$$

and

$$\|\mathcal{D}\varphi_{-t}(v)\| \leq Ae^{-Bt}\|v\|, \text{ for } v \in E^u, t \geq 0;\ A, B > 0. \tag{62}$$

A set Θ which satisfies condition v) is usually called *hyperbolic* and the flow restricted to it ,is called an *hyperbolic attractor*.

Notice that if $\Theta = M$, and conditions i)-iv) are not assumed we have the definition of Anosov flow .

The *non − wandering* set Ω of the flow φ_t is defined by

$$\Omega = \{x \in M : if\ ,\ for\ U\ a\ neigborhood\ of\ x\ and\ t > 0,\ there\ exits\ a\ t_0 > t.$$

$$such\ that\ \varphi_{t_0}(U) \cap U \neq \emptyset\}$$

The flow φ_t satisfies *Smale's Axiom A* if Ω can be written as a disjoint union of a hyperbolic set which fulfills condition iii) and a finite number of hyperbolic fixed points.

In [4]Bowen construed bi-infinite sequences of symbols for Axiom A flows. In fact the flow is taken restricted to a basic set. A Markov system $\sum_A$ is achieved , by finding a suitable partition of Θ (a Markov partition) by "rectangles".

Then a semi-conjugacy between the flow $\varphi_t : \Theta \to \Theta$ and the suspension flow of $\sum_A$ is obtained. We recall breifly the main facts:

Let $\mathcal{A}=\{R_1, R_2, , , , Rn\}$ a Markov partition , where each R_i is a closed rectangle contained in Θ, let

$$\psi : \sum\nolimits_A \to \mathcal{A}$$

$$\psi(x) =. \bigcap_{n \in Z} f^{-1}(R_{x_n})$$

Where f is the first return time map for the rectangles (, they are cross-sections of the flow). ψ is continuos and the intersection is non empty by Markov condition and contains exactely a point by hyperbolicity, so let $\{y\} = \bigcap_{n \in Z} f^{-1}(R_{x_n})$.

Now another first return time map can be defined:

$$\tau : \sum\nolimits_A \to \mathbf{R}$$

$$\tau(x) = \inf\{t > 0 : \varphi_t(y) \in \mathcal{A}\}$$

Let $\Lambda = \{(x,t) \in \sum_A \times \mathbf{R} : t \in [0, \tau(x)]\} / \sim$, where $\sim$ does the identifications $(x, \tau(x)) \sim (\sigma(x), 0)$, and σ is the shift corresponding to the Markov system $\sum_A$.

We refresh the definition of the suspension flow a map $f : M \to M$, it is a flow on the space :$Susp(f, M) \equiv M \times I / \sim$, with I a real interval and the identifications are $(x, 1) \sim (f(x), 0)$. Now the suspension flow of f is defined as :

$$S_t : Susp(f, M) \to Susp(f, M) \tag{63}$$

$$S_t(x, \theta) = (f^{[t+\theta]}(x), t + \theta - [t + \theta])$$

where $[a]$ is the integer part of a. Then we take the suspension flow of the shift σ on $\Lambda = \Lambda(\sigma, \tau)$ and defined in this case by:

$$S_t : \Lambda \to \Lambda$$

$$S_t(\langle x,s\rangle) = \left\langle \sigma^l(x), y \right\rangle$$

where l is such that $y = t + s - \sum_{i=0}^{l-1} \tau(\sigma^i(x))$ satisfies : $0 \leq y \leq \tau(\sigma^l(x))$.

φ_t and S_t are semi-topologically conjugated i.e. there exits a continuos and surjective map $h : \Lambda \to \Theta$ such that the following diagram conmutes :

$$\begin{array}{ccc} \Lambda & \xrightarrow{S_t} & \Lambda \\ h\downarrow & & \downarrow h \\ \Theta & \xrightarrow{\varphi_t} & \Theta \end{array} \tag{64}$$

If the partition is Markov $\sum_A$ is a subshiht of finite type and S_t is hyperbolic.

Therefore a bound for the q-entropy of the flow $\Phi = \{\varphi_t\}_{t\in\mathbf{R}}$, restricted to a basic set, can be obtained in terms of the q-entropy of the flow $\Xi = (S_t)_{t\in\mathbf{R}}$.

Remark 3.23. More or less at the same time of Bowen, Rattner constructed Markov partitions for Anosov flows, which became a particular case.

In the particular case of having diffeomorphisms which satisfie Smale's Axiom A, a semiconjugacy with a Markov system is obtained [2]. Among the pioneers works in this matter we can mention: Sinai's construction of Markov partitions [25]and Smale's horseshoe.[26]

We follow now with some considerations about measure-theoretic $q-$entropy. Let $\mathcal{M}(X,\Phi)$ the set of φ_t- invariant measures (Borel normalized). If $.\mathcal{M}(X)$ is the set of measures defined over the σ-algebra of Borel sets in a metric space, it can be endowed with the weak topology i.e. the smallest topology making the map $f \longmapsto \int_X f d\mu$ continue. ($f \in C(X,\mathbf{R})$).

The filter of neigborhoods of μ_0 is

$$V_{\mu_0}(f_i,\epsilon) = \left\{\mu \in \mathcal{M}(X) : \left|\int_X f_i d\mu - \int_X f d\mu_0\right| < \epsilon,\ i = 1,2,...,k,\ f_i \in C(X,\mathbf{R})\right\} \tag{65}$$

With this topology, $\mathcal{M}(X)$ is compact[19], and a metric which induces the weak topology is

$$d(\mu_1,\mu_2) = \sum_{n=1}^{\infty} \frac{|\int f_n d\mu_1 - \int f_n d\mu_2|}{2^n \, \|f_n\|},$$

where $\{f_n\}$ is a dense set in $C(X,\mathbf{R})$.

The map k which makes the semi-conjugacty between the flows $\{\varphi_t\}_{t\in\mathcal{R}}$ and the discretizated flow $\{S_t\}_{t\in\mathbf{R}}$ induces a map

$$\hat{k} : \mathcal{M}(\Lambda,\Xi) \to \mathcal{M}(X,\Phi)$$

$$\hat{k}(\mu)(E) = \mu(h^{-1}(E))$$

where $\hat{k}$ is an isomorphism in the measure theoretic sense between the dynamical systems $(\Theta, \{\varphi_t\}, \mu_\Phi)$ and $(\Lambda, \{S_t\}, \mu_\Xi)$. Here μ_Φ is the Bowen's measure and μ_Ξ is the corresponding measure for the discretizated flow. Respect of μ_Φ the periodic orbits of Φ are uniformly distribuited[4]. In [18]W. Parry gave a weigthed version of Bowen's result. So by topological invariance of the measure-theoretic $q-$entropy:

$$h_q(\{\varphi_t\}, \mu_\Phi) = h_q(\{S_t\}, \mu_\Xi). \tag{66}$$

In the case of Axiom A diffeomorphisms f the measure is defined as a pull-back of the Parry's measure μ_σ , specifically $\mu_f(E) = \hat{\rho}(\ \mu_\sigma)(E) = \mu_\sigma(\rho^{-1}(E))$, where ρ is the semiconjugacy between Φ and the Markov system $\sum_A$. Then, in this case, (f, μ_f) is measure-theoretically isomorphic to $(\sum_A, \mu_\sigma)$[2] and therefore

$$h_q(f, \mu_f) = h_q(\sigma, \mu_\sigma). \tag{67}$$

The periodic points of f are distribuited according to μ_f whereas the periodic points of the shift are distribuited according the Parry's measure.

The measures μ_Φ and μ_f have interesting interpretations for particular cases; for example let consider a flow $\{\varphi_t : M \to M\}$ in a manifold M such that M is a quotient $M = \Gamma/K/\Gamma_0$, where Γ is a Lie group, K is a compact subset of Γ and Γ_0 is a discret subgroup of Γ,and let define the flow as:

$$\varphi_t(\Gamma_0 \gamma K) = \Gamma_0(\exp \gamma t_\rho) K$$

with ρ is the element in the Lie algebra of Γ such that $\exp(t_\rho) K = K \exp(t_\rho)$, for each t. A such flow is called algebraic. In this case μ_Φ is the Haar measure

Respect to the particular situation of considering Axiom A diffeomorphisms $f : M \to M$, with M a nilmanifold i.e. $M = \Gamma/D$, with M a nilpotent, simply connected, connected Lie group and D a discret subgroup of Γ. Assume that f is induced by a continous automorphism $g : \Gamma \to \Gamma$, such that $g(D) = D$ and the non wandering set of f is hyperbolic, in this circumstance f is called an hyperbolic automorphism. Under these conditions μ_f is the Haar measure.

Within a more general context: let $f : X \to X$ a continuos map, X locally compact; an approach to the problem of getting symbolic dynamics can be given by Conley's index theory . We sketch here the main aspects of it. For details can see[27]and references therein.

Given a set E, the problem we adress is to find a partition $\{E_1, E_{2, \ldots.}, E_n\}$, such that $f^j(x) \in E_{i_j}$ $(\{i_j\} \subset \{1, 2, \ldots., n\})$ for each $x \in E$. The approach to the solution is:

i) E is an isolated invariant set i.e. there exist a compact set $K \subset E$, such that

$$E = \{x \in K \ / \ \exists \ (x_k)_{k \in \mathbf{Z}} : x_0 = x, f(x_i) = x_{i+1} \ \forall \ i \ \in \mathbf{Z}\}$$

ii)Let $M_s = \left\{x \in K/\exists (x_k)_{k \in \mathbf{Z}} : \forall i \in \mathbf{Z}, f^j(x_i) \in E_{j_i}, i = 0, 1, \ldots, s-1 \text{ and } f^s(x_i) = x_{i+1}\right\}$

M_s has a defined *Conley index*, and if it is non trivial M_s is non empty (see below); therefore the possibility of getting a sequence of symbolic dynamics, and so the knowledege of the dynamical complexity is close to the study of the Conley index.

The Conley index is defined as a class of objects isomorphic in a cathegory whose objects are pairs $((X, x_0), [f])$, where (X, x_0) is a pointed topological space and $[f]$ are homotopy classes of preserving base point continuos map. Let $\mathcal{N}$ the class of objects isomorphic to the pair $(X, c)$where X is a single point-pointed space and $c : X \longrightarrow X$ is the constant map. A set A has a non-trivial Conley's index if it is different from $\mathcal{N}$; and this implies that $A \neq \emptyset$.

Then we have a continuos surjective map between (E, f^r)(for some positive integer r)and a Markov system of k symbols (but with infinites sequences), and therefore a bound for the $q-$entropy of f^r.

4. GENERATORS.

We begin this section by recalling the concept of generator for partitions.

Definition. Let $\mathcal{G} = \{G_1, G_2, \ldots\ldots, G_k\}$ a partition of a space X, for a transformation T. $\mathcal{G}$ is a generator if $x, y \in X$ $(x \neq y)$, then x and y have different bi-infinite names respect T.

Alternatively we can give the following definition:

Let $\{i_l\}_{l \in \mathbf{Z}}$, a bi-infinite string of numbers , with $i_l \in \{1, 2, \ldots\ldots, k\}$, $\mathcal{G} = \{G_1, G_2, \ldots\ldots, G_k\}$ is a generator for T if $\left\{\bigcap\limits_{l \in \mathbf{Z}} T^{-l}(G_{i_l}) : G_{i_l} \in \mathcal{G}\right\}$contains at most a point of X. (The intersection of these sets is of course empty).

Notice that a finite substring of $\{i_l\}_{l \in \mathbf{Z}}$,$(i_0, i_1, \ldots\ldots, i_{n-1})$,gives the name of x of length n.

This definition is strengthened to coverings as follows:

*Definition.*Let X a compact metric space and $T : X \longrightarrow X$ an homeomorphis. A finite cover $\mathcal{G}$ is a generator for T if, for every bi-infinite string $\{i_l\}_{l \in \mathbf{Z}}$, the set $\left\{\bigcap\limits_{l \in \mathbf{Z}} T^{-l}\left(\bar{G}_{i_l}\right) : G_{i_l} \in \mathcal{G}\right\}$ contains at most a point of X.

A basic concept for the matter of this section is expansiveness:

Definition. Let $T : X \longrightarrow X$ an homeomorphism, X compact metrizable. T is expansive if there exits a constant $\lambda > 0$, such that for every $x, y \in X, (x \neq y)$, exits an integer k with $d\left(T^k x, T^k y\right) > \lambda$

Generators and expansiveness are related by mean the important following result:

Theorem 4.1 (cf [21],[10]): T is an homeomorphism expansive if and only if it has a generator.

Remark 4.2. The concept of generator does not depend on the metric

Another important result is :

Theorem 4.3. ([29]):Let T an homeomorphism from a compact matric space $(X,d)$to itself . Let $\mathcal{G}$ a generator for T, then for each $\epsilon > 0$, there is a $m > 0$. such that each member of $\left\{ \bigcap\limits_{i=-m}^{m} T^{-i}(G_{\alpha_i}) : G_{\alpha_i} \in \mathcal{G} \right\}$ has diameter less than ϵ.

In this subject our main result is

Theorem 4.4. Let $T : X \to X$ an homeomorphism expansive. If $\mathcal{G}$ is a generator for T ($\mathcal{G}$ exits by **Theor 4.3.**), we have :

i) $h_q(T,\mathcal{G}) = h_q(T)$ for $q \leq 1$

ii) $h_q(T,\mathcal{G}) \geq h_q(T)$ for $q > 1$

Proof. Let $\mathcal{U}$ an arbitrary open covering of X, and δ a Lebesgue number for $\mathcal{U}$. Let $m > 0$ choosen in the way of the above theorem i.e. if

$$\mathcal{G}^{2m} = \left\{ \bigcap_{i=-m}^{m} T^{-i}(G_{\alpha_i}) : G_{\alpha_i} \in \mathcal{U} \right\},$$

then each member of $\mathcal{G}^{2m}$ has diameter less than δ. So each member of $\mathcal{G}^{2m}$ is contained in some set of $\mathcal{U}$; thus $\mathcal{G}^{2m}$ is a refinament et of $\mathcal{U}$ and therefore $N(\mathcal{G}^{2m}) \geq N(\mathcal{U})$.

Thus for $q < 1$:

$$\tilde{h}_q(\mathcal{U},T) \leq \tilde{h}_q(\mathcal{G}^{2m},T) =$$

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \left\{ \log \left[1 + (1-q) H_q(\left(\mathcal{G}^{2m} \right)^r) \right] \right\} \tag{68}$$

with

$$\left(\mathcal{G}^{2m} \right)^r = \left\{ \tilde{G}_{\alpha_{i_0}} \bigcap T^{-1}\left(\tilde{G}_{\alpha_{i_1}}\right) \bigcap \ldots\ldots \bigcap T^{-r-1}\left(\tilde{G}_{\alpha_{i_{r-1}}}\right) : \tilde{G}_{\alpha_{i_j}} \in \mathcal{G}^{2m} \right\}. \tag{69}$$

So,

$$\tilde{h}_q(\mathcal{U},T) \leq \overline{\lim_{r\to\infty}} \frac{1}{r} \left\{ \log \left[1 + (1-q) H_q(\bigcap_{j=-m}^{m+r-1} T^{-j}\left(G_{\alpha_{i_j}}\right)) \right] \right\}, G_{\alpha_{i_j}} \in \mathcal{G}. \tag{70}$$

The r.h.s. of Eq.(70) is equal to

$$\overline{\lim_{r\to\infty}}\frac{1}{r}\left\{\log\left[1+(1-q)\,H_q\left(\bigcap_{j=0}^{2m+r-1}T^{-j}\left(G_{\alpha_{i_j}}\right)\right)\right]\right\}=$$

$$\overline{\lim_{r\to\infty}}\frac{2m+r-1}{r}\frac{1}{2m+r-1}\left\{\log\left[1+(1-q)\,H_q\left(\bigcap_{j=0}^{2m+r-1}T^{-j}\left(G_{\alpha_{i_j}}\right)\right)\right]\right\}=\tilde{h}_q(\mathcal{G},T). \tag{71}$$

Therefore, for any open covering $\mathcal{U}$ and for $q<1$, we have

$$\tilde{h}_q(\mathcal{U},T)\le\tilde{h}_q(\mathcal{G},T) \tag{72}$$

and

$$\tilde{h}_q(T)\le\tilde{h}_q(\mathcal{G},T);$$

$$h_q(T)=h_q(T,\mathcal{G}). \tag{73}$$

For $q>1$, we have now $\left[N\left(\mathcal{G}^{2m}\right)\right]^{1-q}\le\left[N\left(\mathcal{U}\right)\right]^{1-q}$ and similarly

$$\tilde{h}_q(\mathcal{G},T)\le\tilde{h}_q(\mathcal{U},T)\le\tilde{h}_q\left(T\right). \tag{74}$$

and hence for $q>1$

$$h_q(T)\le h_q(T,\mathcal{G}) \tag{75}$$

.□

Respect to definition using span and separated sets exits a particular value for which the $q-$entropy is attained. We consider the balls $\mathcal{B}_R(x_i), i=1,2,......,k$, where x_i and R are choosen such that

$$\left\{\mathcal{B}_{\delta/2-2\delta_0}(x_i):i=1,2,.....,k\right\}\text{ covers }X$$

with $\delta_0<\delta/4$, where δ is the constant of expansiveness.So $2\delta_0$ is a Lebesgue number for the covering $\mathcal{G}=\left\{\mathcal{B}_{\delta/2}(x_i):i=1,2,.....,k\right\}$, and $\mathcal{G}$ is a generator. By Lemma **3.10:**

$$N\left(\mathcal{G}^n\right)\le\alpha_{n,\delta_0}\le\beta_{n,\delta_0} \tag{76}$$

and thus, for $q<1$,

$$\tilde{h}_q(T)=\tilde{h}_q(\mathcal{G},T)\le\widetilde{h'_q}(\delta_0,T)\le\widetilde{h'_q}(T). \tag{77}$$

Recall that for, $q<1$, $\tilde{h}_q(T)=\widetilde{h'_q}(T)$, therefore $\widetilde{h'_q}(T)=\widetilde{h'_q}(\delta_0,T)$.

We are now in condition of justifying the results of the subsection **3.1:** if $\mathcal{G} = \{U_i : i = 0, 1, \ldots k$ $\{x : x_0 = i\}$, is the canonical generator for the shift, then

$$[h_q(\sigma)]_{BS(0,1,\ldots k-1)} = (q-1)^{-1}\left[1 - k^{1-q}\right], \qquad for\ q < 1 \tag{78}$$

and

$$[h_q(\sigma)]_{BS(0,1,\ldots k-1)} \leq (q-1)^{-1}\left[1 - k^{1-q}\right], \qquad for\ q > 1 \tag{79}$$

with the same justification for Markov systems.

5. Topological q-pressure

In same spirit as along this paper , now we introduce one-parameter family of topological q-pressures, associated to a dynamical system (X, T), which allows us to recover, in a particular case, the families of entropies defined.

Let $f \in C(X, \mathbf{R}), \mathcal{U}$ a covering of X and $\mathcal{V}$ a finite subcovering of $\mathcal{U}$.For each member $V_i \in \mathcal{V}$, we consider: $\inf_{y \in V_i}\left\{\exp\left(\sum_{i=0}^{n-1} f\left(T^i(y)\right)\right)\right\}$, then we take

$$\sum_{i=1}^{Card\mathcal{V}} \inf_{y \in V_i}\left\{\exp\left(\sum_{i=0}^{n-1} f\left(T^i(y)\right)\right)\right\} \tag{80}$$

and then set :

$$\tilde{S}_n(T, f, \mathcal{U}) = \inf\left\{\sum_{i=1}^{Card\mathcal{V}} \inf_{y \in V_i}\left\{\exp\left(\sum_{i=0}^{n-1} f\left(T^i(y)\right)\right)\right\}\right\}, \tag{81}$$

where the infimun is taken over all the finite subcoverings of $\mathcal{V}^n$.

Let now :

$$S_{n,q,f} = (q-1)^{-1}\left[1 - \left(\tilde{S}_n(T, f, \mathcal{U})\right)^{1-q}\right] \tag{82}$$

and, for $q \neq 1$,

$$\tilde{h}_q(T, f, \mathcal{U}) = \overline{\lim_{n\to\infty}} \frac{1}{n}\left\{\log\left[1 + (1-q) S_{n,q,f}\right]\right\} \tag{83}$$

$\left(\text{for } q = 1, \log\left(\tilde{S}_n(T, f, \mathcal{U})\right) \text{ is considered}\right)$. Then we call

$$\tilde{h}_q(T, f) = \sup_{\mathcal{U}}\left\{\tilde{h}_q(T, f, \mathcal{U})\right\}. \tag{84}$$

Finally we put:

$$P_q(T, f) = (q-1)^{-1}\left[1 - \exp\left(\tilde{h}_q(T, f)\right)\right]. \tag{85}$$

$P_q(T,f)$ is called the *topological* $q-$*pressure* of f and the map

$$P_q(T,f) : C(X,\mathbf{R}) \to \mathbf{R} \cup \{\infty\}$$

$$f \longmapsto P_q(T,f)$$

is called the *topological* $q-$*pressure* associated to the dynamical system (X,T). Notice that if $f \equiv 0$ we have $P_q(T,f) = h_q\,(T)$.

For recovering the entropy $h_q'(T)$ we consider $\sum_{i=0}^{n-1} f\left(T^i(x)\right)$ and for $\epsilon > 0, n \geq 1$ we take

$$S_{n,\epsilon,f} = \inf\left\{\sum_{x\in F}\exp\left(\sum_{i=0}^{n-1} f\left(T^i(x)\right)\right)\right\}, \tag{86}$$

where the infimun is taken now over the $(n,\epsilon)-$ *span sets.* For X, analogously as before, let

$$\tilde{S}_{q,n,\epsilon,f} = (q-1)^{-1}\left[1-(S_{n,\epsilon,f})^{1-q}\right] \tag{87}$$

and, for $q \neq 1$

$$S_{q,\epsilon,f} = \overline{\lim_{n\to\infty}}\frac{1}{n}\left\{\log\left[1+(1-q)\,\tilde{S}_{q,n,\epsilon,f}\right]\right\}. \tag{88}$$

Thus

$$\widetilde{h_q'}(T,f) = \lim_{\epsilon\to 0} S_{q,\epsilon,f} \tag{89}$$

and, in the same way as above, we obtain

$$P_q'(T,f) = (q-1)^{-1}\left[1-\exp\left(\widetilde{h_q'}(T,f)\right)\right], \tag{90}$$

so, for $f \equiv 0$, we have $P_q'(T,f) = h_q'\,(T)$.

Now we work in the level of symbolic dynamics, i.e. in the case of a Markov system $(\sum_A,\sigma)$. We recall from Subsection 3.1 that the $q-$entropy (definition by coverings), can be computed using the distribution of periodic points of the shift σ. As before we call $P_n\,(\sigma) = Card\,\{x : \sigma^n x = x\} = Card(Fix\,(\sigma^n)) = Tr\,(A^n)$.

With the canonical generator $\mathcal{G} = \{U_i : i = 0,1,...k-1\}\,, U_i = \{x : x_{0=}i\}$, the calculation becomes:

$$\tilde{h}_q(\mathcal{G},\sigma) = \log\left(E^{1-q}\right),$$

where E is the eigenvalue of A given by Perron-Frobenius theorem, and we know that:

$$h_q(\sigma) = (q-1)^{-1}\left[1-E^{1-q}\right], \qquad for\ q < 1$$

and

$$h_q(\sigma) \leq (q-1)^{-1}\left[1-E^{1-q}\right], \qquad for\ q > 1.$$

We do not know if for an expansive operator, the value of the topological q-pressure is obtained with a generator , like as in the particular case of the entropy. We do the calculations for the canonical gnerator $\mathcal{G}$, but it will be omitted in the formulae.

Let $f \in C(\sum_A, \mathbf{R})$ (we can impose stronger condition of differentiability on f) , f is called an *observable.* In this case the Eq. (82) yields, for $q \neq 1$,

$$S_{n,q,f} = (q-1)^{-1}\left[1 - (Z_n(f))^{1-q}\right] \tag{91}$$

For $q = 1$ we consider $\log(Z_n(f))$.

Where $Z_n(f)$ is the partition function (in the context of Statiscal Machanics) for the space of configurations $x = (x_n)_{n\in\mathbf{Z}}$ which are allowed by the matrix A.For example, when the configurations are selected with periodic boundary conditions, we have

$$Z_n(f) = \sum_{x\in Fix(\sigma^n)} \left\{\exp\left(\sum_{i=0}^{n-1} f\left(\sigma^i(x)\right)\right)\right\}. \tag{92}$$

For $f \equiv 0$, $Z_n(0) = P_n(\sigma) = Tr(A^n)$. Then the q-entropy is recovered, in a particular case, as the trace of the potents of the tranfer matrix A when these configurations are considered.

For a general observable $f \in C^k(\sum_A)$ with finite range, we have $Z_n(f) = Tr\left[(\mathcal{L}f)^n\right]$ for a suitable trace-class operator $\mathcal{L} : C^k(\sum_A) \to C^k(\sum_A)$. $\mathcal{L}$ is known as the *Ruelle operator.* It is defined by

$$\mathcal{L}f[g](x) = \sum_{x_i\in\{0,1,\ldots k-1\}} A_{x_i x_0} \exp[f(\bar{x})]\, g(\bar{x}) \tag{93}$$

where $\bar{x} = (\bar{x}_n)_{n\in\mathbf{Z}}$ with $\bar{x}_0 = x_i$ and $\bar{x}_i = x_{i-1}$ for $i \geq 1$.

Similarly to the particular case when we consider periodic conditions: $\tilde{h}_q(\sigma, f) = \log(\Lambda^{1-q})$ where Λ is the leading eigenvalue of $\mathcal{L}$ as given by Perron-Frobenius theorem. So, if $q \neq 1$,

$$P_q(\sigma, f) = (q-1)^{-1}\left[1 - \Lambda^{1-q}\right]. \tag{94}$$

for the canonical generator $\mathcal{G}$.

Remark 5.1. For infinite range observables f, $\mathcal{L}$ can no longer be finite dimensional, which is the case, for example of the Kac model.

6. Relationship between measure theoretic and topological q-entropies

We do an attemp of relating both two families of entropies. Let fix a continous transformation $T : X \to X$ with X a compact metric space. We denote $\mathcal{B}(X)$ the

$\sigma-$algebra of Borel subsets of X, i.e. the smallest one containing the open and the closed sets of X.

We recall that $\mathcal{M}(X)$ is the set whose members are probability measures in $(X, \mathcal{B}(X))$, equipped with the weak topology, i.e. those one for which the filter of neighborhoods is given by Eq.(65). A relevant subset of $\mathcal{M}(X)$ is $\mathcal{M}(X, T) = \{\mu : \mu \text{ is invariant by } T\}$. $\mathcal{M}(X,T)$ is a compact subset in the weak topology of $\mathcal{M}(X)$ and also it is convex.

We introduce the $q-$entropy map:

$$\mathcal{M}(X,T) \longrightarrow \mathbf{R}$$

$$\mu \mapsto h_q(\mu, T)$$

Unfortunately for $q \neq 1$, in general, is not true that

$$h_q(\lambda_1\mu_1 + (1-\lambda_1)\mu_2) = \lambda_1 h_q(\mu_1) + (1-\lambda_1)_1 h_q(\mu_2).$$

When this condition is fullfilled, it says that the function is affine. In fact for $B.S$ the $q-$entropy is not affine. Neither this map is in general continuos.

Each member of $\mathcal{M}(X,T)$ can be represented in terms of ergodic members of it. Let

$$\mathcal{E}(X,T) = \{\mu : \mu \text{ is } T\text{-ergodic}\}.$$

For each measure of $\mathcal{M}(X,T)$, there is only one measure ω defined on $\mathcal{B}(\mathcal{M}(X,T))$such that $\omega(\mathcal{E}(X,T)) = 1$ and :$\mu = \int_{\mathcal{E}(X,T)} \tau d\omega_\tau$ by Choquet's representation [20].

Therefore if $\varphi : \mathcal{M}(X,T) \to \mathbf{R}$ is an affine semi-continuos map, we have $\varphi(\mu) = \int_{\mathcal{E}(X,T)} \varphi(\tau)\, d\omega_\tau$, so we can not use this representation for the $q-$entropy map, i.e. we can not expect in general that
$h_q(\mu, T) = \int_{\mathcal{E}(X,T)} h_q(\tau) d\omega_\tau$.

We try to get any conclusion about the relationship between measure theoretic and topological q-entropies. We begin with an elementary fact:

Let $\phi(x) = x^q$, $0 < q < 1$, then we have $(x > 0)$

$$\phi\left(\sum_{i=0}^{k-1} \lambda_i x_i\right) \geq \sum_{i=0}^{k-1} \lambda_i \phi(x_i)$$

with $\lambda_i > 0$ and $\sum_{i=0}^{k-1} \lambda_i = 1$. So if $\lambda_i = 1/k$ and $x_{i=}\mu(A_i)$ for each i and for a partition $\{A_0, A_1, ..., A_{k-1}\}$ then $\left(\sum_{i=0}^{k-1} \frac{1}{k}\mu(A_i)\right)^q \geq \sum_{i=0}^{k-1} \frac{1}{k}(\mu(A_i))^q$ and thus $\left(\frac{1}{k}\right)^q \geq \sum_{i=0}^{k-1} \frac{1}{k}(\mu(A_i))^q$, therefore:

$$H_q(\mu, \mathcal{A}) = \sum_{i=0}^{k-1}(\mu(A_i))^q \leq k^{1-q}$$

and finally

$$H_q(\mu, \mathcal{A}^n) \leq (N(\mathcal{A}^n))^{1-q} \tag{95}$$

We consider Bernoulli Schemes. Let as ever the canonical generator $\mathcal{G} = \{U_i : i = 0, 1, ...k-1\}$ $\{x : x_0 = i\}$. For each $\mu \in \mathcal{M}(X, \sigma)$ (X of course the phase space of the $B.S.$) . we have :

$$\tilde{h}_q(\mu, \sigma) = \tilde{h}_q(\mu, \mathcal{G}, \sigma) \tag{96}$$

and, because the shift is expansive, if $q < 1$

$$\tilde{h}_q(\sigma) = \tilde{h}_q(\mathcal{G}, \sigma). \tag{97}$$

Then, for $0 < q < 1$,

$$\tilde{h}_q(\mu, \sigma) = \overline{\lim_{n\to\infty}} \frac{1}{n} \{\log [H_q(\mu, \mathcal{G}, \sigma)]\} \leq$$

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \left\{\log [N(\mathcal{G}^n, \sigma)]^{1-q}\right\} = \tilde{h}_q(\mathcal{G}^n, \sigma) = \tilde{h}_q(\sigma) \tag{98}$$

Therefore, for $0 < q < 1$,

$$\tilde{h}_q(\mu, \sigma) \leq \tilde{h}_q(\sigma) \text{ and } h_q(\mu, \sigma) \leq h_q(\sigma) \tag{99}$$

So, at least for the Bernoulli Schemes, and for some values of the parameter q. the topological q-entropy bounds the measure theoretic one.

We show now a result in some special kind of spaces, which are called : Spaces with covering dimension k.

Definition. A metric space (X, d)has covering dimension k, if for any covering $\mathcal{U}$ of X, exits a refinament $\mathcal{V}$, such that each point of $x \in X$, belongs at most to $k+1$ members of $\mathcal{V}$.

This condition is fulfilled for any $k-$dimensional manifold.

We begin by considering a covering of (X, d), with X compact $\mathcal{B}_\epsilon = \{B_1, B_2, ..., B_l\}$. by balls of radius $\epsilon/2$, such that each point of X is an at most $k+1$ members of $\mathcal{B}_\epsilon$, and $T : X \to X$ a continuos map. We take also a measure $\mu \in \mathcal{M}(X, T)$ (notice that in a metric space every invariant measure is regular).

Let $\mathcal{U}_\epsilon = \{U_1, U_2, ..., U_l\}$ a partition of X , with $\bar{U}_i \subset B_i$.For each $x \in X$, there exists a neighborhood V_x of it, such that V_x intersects at most $k+1$ members of $\mathcal{U}_\epsilon$. So $\mathcal{V} = (V_x)_x$ is a covering of(X,d), and let $\{V_{x_i}\}_{i=1,2,...,m}$ a finite subcovering of $\mathcal{V}$.

Let now R a $(n,\delta) - span\ set$ of X for T,with δ a Lebesgue number for $\mathcal{V}$. Let $z \in R$, for $i = 0,1,.....,n-1$, we have $T^i(z) \in V_{x_i}(z) \in \mathcal{V}$, where $V_{x_i}(z)$ is the member of the covering which contains the ball $\overline{B}_\delta(T^i(z))$.

Then we call H_n the set of the strings $(j_0,j_1,......,j_{n-1})$, such that U_{j_i} , intersects $V_{x_i}(z)$, with $z \in R$, $i = 0,1,.....,n-1$. Now if $x \in \bigcap_{i=0}^{n-1} T^{-i}(U_{j_i})$,and we choose $z \in R$ in order that $d(T^i(x), T^i(z)) \leq \delta$, hence $T^i(x) \in V_{x_i}(z) \cap U_{j_i}$,and so $(j_0,j_1,......,j_{n-1})$belongs to H_n .Let $J_n = \left\{(j_0,j_1,......,j_{n-1}) : \bigcap_{i=0}^{n-1} T^{-i}(U_{j_i}) \neq \emptyset\right\}$, therefore $J_n \subset H_n$. By construction $V_{x_i}(z)$ can intersect at most $k+1$ members of $\mathcal{U}_\epsilon$, then

$$Card(H_n) \leq (k+1)^n Card(R)$$

and, in particular for the $(n,\delta) - span\ set$ of smallest cardinality,

$$Card(J_n) \leq (k+1)^n \alpha_{n,\delta}.$$

We have, by the previous calculations,

$$H_q(\mu, \mathcal{U}_\epsilon^n) \leq (Card(J_n))^{1-q},$$

because $Card(J_n) = Card\left\{(j_0,j_1,......,j_{n-1}) : \bigcap_{i=0}^{n-1} T^{-i}(U_{j_i}) \neq \emptyset\right\} = N(\mathcal{U}_\epsilon^n)$. So for, $0 < q < 1$,

$$H_q(\mu, \mathcal{U}_\epsilon^n) \leq (\alpha_{n,\delta})^{1-q} (k+1)^{n(1-q)}$$

and therefore

$$\widetilde{h'_q}(\mu, T) \leq \widetilde{h'_q}(T) + [\log(k+1)](1-q).$$

Then

$$h'_q(T) \leq (q-1)^{-1}\left\{1 - \exp\left(\widetilde{h'_q}(T)\right) + [\log(k+1)](1-q)\right\}. \tag{100}$$

So we have the topological $q-$entropy $(0 < q < 1)$ with a"perturbation"as a bound for the measure theoretic $q-$entropy, in a more general situation than $B.S.$

Using definition by coverings , the condition of covering dimension k over the space X, can be misleading . Let $\mathcal{A} = \{A_1, A_2, ...A_k\}$ a partition of X, and let $\mathcal{B} = \{B_0, B_1, ...B_k\}$ $(B_0 = X - \bigcup_{i=1}^{k} B_i)$ another partition with $B_i \subset A_i$. Because μ is regular, B_i can be chosen such that $\mu(A_i - B_i) < \epsilon$, for each $\epsilon > 0$. We define a covering $\mathcal{U} = \{U_0, U_1, ...U_k\}$ by $U_i = \left(X - \bigcup_{j \neq i} B_j\right) \cup B_0$.

Similarly to above, for $0 < q < 1$,

$$H_q(\mu, \mathcal{B}^n) \leq (N(\mathcal{U}^n))^{1-q} \times (2^n)^{1-q} \qquad (N(\mathcal{B}^n) \leq N(\mathcal{U}^n) \times 2^n)$$

and

$$\tilde{h}_q(\mu, T) \leq \tilde{h}_q(T) + (1-q)\log 2. \tag{101}$$

Thus

$$h_q(T) \leq (q-1)^{-1}\{1 - \exp(h_q(T)) + (1-q)\log 2\}. \tag{102}$$

In this case the bound contains a term $\log 2$ instead of $\log k$, for a predetermined k.

We finish with some considerations about measures for which the measure theoretic q—entropy equals the topological one. If μ is the $\left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}\right)$-measure product in $BS(p_0, p_1, \ldots, p_{k-1})$, we have, for $0 < q < 1$,

$$[h_q(\mu, \sigma)]_{BS\left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}\right)} = (q-1)^{-1}\left[1 - k^{1-q}\right] = h_q(\sigma). \tag{103}$$

So, for the $\left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}\right)$-measure product, topological and measure-theoretic q—entropies (with $0 < q < 1$) agree.

REFERENCES

[1] ADLER, R.,L., KONGHEIN, A.G. and McANDREW, M.H., Topological entropy. *Trans. Amer. Math. Soc.* **114** (1965), 309-319.

[2] BOWEN, R.E., Markov partition for Axiom A diffeomorphisms. *Am. J. Math.* **92** (1970), 725-747.

[3] BOWEN, R.E., Entropy for groups endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** (1971), 401-414.

[4] BOWEN, R.E.,Symbolic dynamics for hyperbolic flows. *Am. J. Math.* **95** (1973), 429-460.

[5] DINABURG, E.I., The relation between topological entropy and metric entropy. *Sov. Math.* **11** (1970), 13-16.

[6] FENLEY, S., Anosov flows in 3-manifolds. *Ann. Math.* **139** (1994), 79-115.

[7] GHYS, E. and DE LA HARPE, P., *Sur les Groups Hiperboliques d'après Mikhael Gromov*, (Progress in Math, vol 83, Birkhäser, Zurich, 1990).

[8] HEWITT, E. and ROSS, K.A., *Abstract of Harmonic Analysis,* Vol 1, (Springer-Verlag, 1963).

[9] KEANE, M.S., in *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, T. Bedford, M. Keane and C. Series, Eds., (Oxford University Press, Oxford, 1991).

[10] KEYNES, H.B. and ROBERTSON , J.B., Generators for topological entropy and expansiveness, *Math. Systems. Theory,* **3** (1969), 51-59.

[11] KOLMOGOROV, A.N., A new metric invariant of transitive dynamic systems and automorphism in Lebesgue spaces. *Dokl. Akad. Nauk., SSSR* **119** (1958), 861-864.

[12] MANNING, A., Topological entropy for geodesic flows. *Ann. Math.* **110** (1979), 567-573.

[13] MARGULIS, G., Certain applications of ergodic theory to the investigation of manifolds of negative curvature. *Func. Anal. Appl.* **3** (1969), 89-90.

[14] MELSHALKIN, L.D., A case of isomorphism of Bernoulli schemes. Dokl. Akad. Nauk., SSSR **147** (1962), 797-800.

[15] MESON, A.M and VERICAT, F., Invariant of dynamical systems: a generalized entropy. *J. Math. Phys.* **37** (1996), 4480-4483.

[16] MILNOR, J., A note on curvature an the fundamental group. *J. Diff. Geom.* **2** (1968), 1-7.

[17] ORNSTEIN, Bernoulli shift with the same entropy are isomorphic. D., Adv. in Math. **4** (1970),337-352.

[18] PARRY W, Synchronisation of canonical measures for hyperbolic attractors,*Commun. Math. Phys.* **106** (1986), 267-285.

[19] PARTASARATHY, K.R. , *Probability measures on metric spaces, (*Acad. Press, New York, 1967).

[20] PHELPS, R., *Lectures on Choquet's Theorem*, (Van Nostrand, Princeton,1966).

[21] REDDY,W.L., Lifting expansive homeomorphism of symbolic flows. *Math. Systems Theory* **2** (1968), 91-92.

[22] SCOTT, P., The geometry of 2 and 3-manifolds. *Bull. London. Math. Soc.* (1983), 401-487.

[23] SHANNON, C.E. in *The Mathematical Theory of Communication,* C.E. Shannon and W.Weaver. Eds. (University of Illinois, Urbana,1949).

[24] SINAI, Y., On the concept of entropy of a dynamical system. Dokl. Akad. Nauk., SSSR **124** (1959), 768-771.

[25] SINAI, Y., Construction of Markov partitions,*Func. Anal. Appl.* **2** (1968), 70-80.

[26] SMALE S. Differentiable dynamical systems, *Bull A.M.S.* **73** (1967), 747-817.

[27] SZYMCZAK A., The Conley index and symbolic dynamics. *Topology* **35** (1996), 287-299.

[28] TSALLIS, C., Possible generalization of Boltzmann-Gibbs statistics. *J. Statistical Phys.* **52** (1988), 479–487.

[29] WALTERS, P., *An Introduction to Ergodic Theory* (Springer-Verlag, New-York, 1982).

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