Notas de Matemática

A Posteriori Error Estimator for a Mixed Finite Element Method for Reissner-Mindlin Plate

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1999

UNIVERSIDAD NACIONAL DE LA PLATA
Departamento de Matemática - Facultad de Ciencias Exactas

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Abstract

We present an a posteriori error estimator for a mixed finite element method for the Reissner-Mindlin plate model. The finite element method we deal with was analyzed in [16] and can also be seen as a particular example of the general family analyzed in [13]. The estimator is based on the evaluation of the residual of the finite element solution. We show that the estimator yields locally lower and globally upper bounds of the error in the numerical solution in a natural norm for the problem. which includes the H^1 norms of the terms corresponding to the deflection and the rotation and a dual norm for the shearing force. The estimates are valid uniformly with respect to the plate thickness.

1 Introduction

In the implementation of numerical methods for approximation of partial differential equations, the definition of a posteriori error estimators is the basic tool for adaptive mesh-refinement techniques, necessary when we are in presence of local singularities of the solution.

In this paper we present an a posteriori error estimator for the finite element approximation of the Reissner-Mindlin plate model. which describes the displacement of a plate with moderate thickness subject to a transverse load. The definition of the estimator is based on the evaluation of the residual of the finite element solution.

Several a posteriori error estimators have been defined for different linear and nonlinear elliptic problems by using the residual equations (see for example [3, 4. 5, 15. 20, 21])

For a fixed plate thickness the Reissner Mindlin plate model is a linear elliptic problem. But for small thickness the ellipticity constant deteriorates and makes difficult the treatment of the problem. In particular in the definition of an estimator the main difficulty is the attainment of equivalence with an error norm, independently of the plate thickness.

For the numerical solution of the Reissner-Mindlin equations there are several mixed finite element methods which present good approximation of the solutions [2, 6, 7, 9, 10. 11, 16, 18] and are free from locking [8, 12, 13. 16, 18].

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We present an a posteriori error estimator for a method analyzed in [16] which can also be seen as a particular example of the general family analyzed in [13].

We define the error estimator for the H^1 norm of the deflection and the rotation, and for a sum of norms for the shear force which includes the $\mathbf{H}_0(rot)'$ norm, and show that it yields locally lower and globally upper bounds of the error in the numerical solution, valid uniformly with respect to the plate thickness. It must be remarked that even though these norms are natural for the problem: in particular the $inf - sup$ condition holds for the $\mathbf{H}_0(rot)'$ norm [12] and. when $t \to 0$, $\mathbf{H}_0(rot)'$ becomes the appropiatre space for the shear, convergence for the shear force in this dual norm has not been proved, as far as we know

The rest of the paper is organized as follows. In section 2 we introduce the Reissner-Mindlin model and we analyze its approximation with the finite element method. We also give an aditional a priori estimate related with the L^2 norm of the error in the rotor of the shear force. In section 3 we define a weak norm for the error in the rotation and in the shear force and obtain estimates for this norm. Finally in section 4, we define the estimator for the whole error and show the corresponding relations beetwen the estimator and the natural error norm.

2 The Reissner-Mindlin equations and mixed finite element approximation

Let $\Omega \times [-t/2, t/2]$ be the region occupied by the undeformed elastic plate of thickness $0 < t < 1$, where $\Omega \subset R^2$ is a simply connected polygon.

Let us denote by w and $\boldsymbol{\beta}$ the transverse displacement of the midsection of the plate and the rotation of fibers normals to it, respectively. Then, assuming for simplicity, that the plate is clamped along the boundary of Ω, the Reissner-Mindlin problem is:

Find $w \in H_0^1(\Omega)$ and $\boldsymbol{\beta} \in \mathbf{H}_0^1(\Omega)$, such that

$$t^3 a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \lambda t(\nabla w - \boldsymbol{\beta}, \nabla\zeta - \boldsymbol{\eta}) = (g, \zeta) \quad \forall \boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega)\ , \forall \zeta \in H_0^1(\Omega) \tag{2.1}$$

where (,) denotes the scalar product in either $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$, and $a(\boldsymbol{\beta}, \boldsymbol{\eta})$ is a coercive and continuous bilinear form, defined by

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) = \frac{E}{12(1-\nu^2)} \int_\Omega D\Xi(\boldsymbol{\beta}) : \Xi(\boldsymbol{\eta})$$

where $\Xi(\boldsymbol{\eta})$ is the symmetric part of the gradient of $\boldsymbol{\eta}$, D is defined by

$$D\Upsilon = [(1-\nu)\Upsilon + \nu tr(\Upsilon)I]$$

E is the Young modulus, ν is the Poisson ratio, $\lambda = Ek/2(1+\nu)$, where k is the shear correction factor, and g represents the transverse load.

To analyze the problem for small values of t, g is scaled in the form $g = t^3 f$, so that the solution tends to a nonzero limit as t tends to zero [12]. Taking, for the sake of simplicity $\lambda = 1$, and introducing,

$$\gamma = t^{-2}(\nabla w - \boldsymbol{\beta}) \tag{2.2}$$

the equation (2.1) can be written equivalently as,

$$\begin{cases} a(\boldsymbol{\beta},\boldsymbol{\eta}) + (\boldsymbol{\gamma}, \nabla\zeta - \boldsymbol{\eta}) = (f,\zeta) & \forall \boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega)\ , \forall\zeta \in H_0^1(\Omega) \\ t^2(\boldsymbol{\gamma},\chi) - (\nabla w - \boldsymbol{\beta}, \chi) = 0 & \forall\chi \in \mathbf{L}^2(\Omega) \end{cases} \tag{2.3}$$

which in the limit $t \longrightarrow 0$ takes the form of a saddle point problem.

Let

$$\mathbf{H}_0(rot,\Omega) = \{\chi \in \mathbf{L}^2(\Omega)\ : rot(\chi) \in L^2(\Omega)\ and\ \chi.\boldsymbol{\tau} = 0\ on\ \partial\Omega\}$$

where $\partial\Omega$ denotes the boundary of Ω and

$$\|\chi\|_{\mathbf{H}_0(rot,\Omega)} := \|\chi\|_0 + \|rot\chi\|_0$$

The following Proposition, which is proved in [12], gives a descomposition for any $\chi \in \mathbf{H}_0(rot,\Omega)$, showing also that $\boldsymbol{\gamma} \in \mathbf{H}_0(rot,\Omega)$.

Proposition 2.1 *Let B defined on $\mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$ by:*

$$B : (\boldsymbol{\eta},\zeta) \longrightarrow (\nabla\zeta - \boldsymbol{\eta})$$

The mapping B is surjective onto the space $\mathbf{H}_0(rot,\Omega)$ and for every $\chi \in \mathbf{H}_0(rot,\boldsymbol{\Omega})$ there exists $(\boldsymbol{\eta},\zeta) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\chi = \nabla\zeta - \boldsymbol{\eta}$$

and

$$\|\nabla\zeta\|_0 + \|\boldsymbol{\eta}\|_1 \leq C\{\|\chi\|_0 + \|rot\chi\|_0\}$$

with C independent of χ. □

Let

$$\boldsymbol{\Gamma} = \mathbf{H}_0(rot,\boldsymbol{\Omega})' = \{\chi \in \mathbf{H}^{-1}(\boldsymbol{\Omega})/div\chi \in \mathbf{H}^{-1}(\boldsymbol{\Omega})\}$$

with the definition of the norm

$$\|\chi\|_{\boldsymbol{\Gamma}}^2 = \|\chi\|_{-1}^2 + \|div\chi\|_{-1}^2$$

which is equivalent to the dual norm.
From this follows inmediately that the following $inf - sup$ condition holds:

$$\sup_{\substack{(\eta,\zeta) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega) \\ (\eta,\zeta) \neq (0,0)}} \frac{(\nabla\zeta - \boldsymbol{\eta},\chi)}{\|\boldsymbol{\eta}\|_1 + \|\zeta\|_1} \geq \|\chi\|_{\boldsymbol{\Gamma}} \qquad \forall\chi \in \boldsymbol{\Gamma} \tag{2.4}$$

Let $\{\mathcal{T}_h\}_{0<h<1}$ be a regular family of triangulations of Ω , where h stands for the maximum diameter of the elements in the triangulation $\mathcal{T}_h$. In order to define a mixed finite element approximation we have to give finite element spaces for the rotations, the transverse displacement and the shear strain. Also we have to define an operator, usually some kind of interpolation, in order to relax the discrete equation corresponding to (2.2).

We use the standard notation $\mathcal{P}_m$ for the space of polynomials of degree less than or equal to m and set $\mathbf{P}_m = \mathcal{P}_m \times \mathcal{P}_m$.

Given an element T, let $\{\lambda_i\}_{1\leq i\leq 3}$ be its barycentric coordinates and $\boldsymbol{\tau}_i$ be the tangential vector to the edge ∂T_i where $\lambda_i = 0$. We define,

$$\boldsymbol{\phi}_1 = \lambda_2\lambda_3\boldsymbol{\tau}_1\ , \quad \boldsymbol{\phi}_2 = \lambda_3\lambda_1\boldsymbol{\tau}_2 \quad and, \quad \boldsymbol{\phi}_3 = \lambda_1\lambda_2\boldsymbol{\tau}_3$$

then, the finite element spaces for the method , $\mathbf{H}_h \subset \mathbf{H}_0^1(\Omega)$ for the rotations. $W_h \subset H_0^1(\Omega)$ for the transverse displacement and, $\boldsymbol{\Gamma}_h \subset \mathbf{L}^2(\Omega)$ for the shear strain. are defined as follows,

$$\mathbf{H}_h = \{\boldsymbol{\eta}_h \in \mathbf{H}_0^1(\Omega) \ : \boldsymbol{\eta}_h|_T \in \mathbf{P}_1 \oplus span\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3\}, \forall\, T \in \mathcal{T}_h\}$$

$$W_h = \{\zeta_h \in H_0^1(\Omega) \ : \zeta_h|_T \in \mathcal{P}_1, \forall\, T \in \mathcal{T}_h\}$$

and $\boldsymbol{\Gamma}_h$ is a rotation of the lowest order Raviart-Thomas space [12],

$$\boldsymbol{\Gamma}_h = \{\boldsymbol{\eta}_h \in \mathbf{H}_0(rot, \Omega) \ : \boldsymbol{\eta}_h|_T \in \mathbf{P}_0 \oplus (x_2, -x_1)\mathcal{P}_0, \forall\, T \in \mathcal{T}_h\}$$

In particular the inclusion,

$$\nabla W_h \subset \boldsymbol{\Gamma}_h \tag{2.5}$$

holds.

We define the interpolation operator $\boldsymbol{\Pi}$ for this method by $\boldsymbol{\Pi}\boldsymbol{\eta}|_T = \boldsymbol{\eta}_I$ where $\boldsymbol{\eta}_I$ is such that,

$$\int_{\partial T_i} \boldsymbol{\eta}_I.\boldsymbol{\tau}_i = \int_{\partial T_i} \boldsymbol{\eta}.\boldsymbol{\tau}_i \qquad i = 1, 2, 3 \tag{2.6}$$

and which satisfies

$$||\boldsymbol{\eta} - \Pi\boldsymbol{\eta}||_0 \leq Ch||\boldsymbol{\eta}||_1 \quad \forall \boldsymbol{\eta} \in \mathbf{H}_0^1(\Omega) \tag{2.7}$$

Therefore the approximate solution $(\boldsymbol{\beta}_h, w_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_h \times W_h \times \boldsymbol{\Gamma}_h$ is defined by.

$$\begin{cases} a(\boldsymbol{\beta}_h, \boldsymbol{\eta}_h) + (\boldsymbol{\gamma}_h, \nabla\zeta_h - \boldsymbol{\Pi}\boldsymbol{\eta}_h) = (f, \zeta_h), & \forall \boldsymbol{\eta}_h \in \mathbf{H}_h, \ \ \forall \zeta_h \in W_h. \\ \\ \boldsymbol{\gamma}_h = t^{-2}(\nabla w_h - \boldsymbol{\Pi}\boldsymbol{\beta}_h) \end{cases} \tag{2.8}$$

Also the discrete $inf - sup$ condition holds for $\|\ .\ \|_{\boldsymbol{\Gamma}_h}$ defined in an appropiate way [13].

For this method, it is known [13], [16] that when Ω is a convex polygon,

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|w - w_h\|_1 \leq Ch\{\|\boldsymbol{\beta}\|_2 + t\|\boldsymbol{\gamma}\|_1 + \|\boldsymbol{\gamma}\|_0\}, \tag{2.9}$$

and also [13],

$$\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{-1} \leq Ch\{\|\boldsymbol{\beta}\|_2 + t\|\boldsymbol{\gamma}\|_1 + \|\boldsymbol{\gamma}\|_0\} \tag{2.10}$$

with

$$\|\boldsymbol{\beta}\|_2 + t\|\boldsymbol{\gamma}\|_1 + \|\boldsymbol{\gamma}\|_0 \leq C\|f\|_0 \tag{2.11}$$

In [17] it is also proved that

$$\|\beta - \beta_h\|_0 \leq Ch^2\|f\|_0 \tag{2.12}$$

Here and hereafter C denotes a constant which could depend on the minimum angle of the triangulation but is independent of the thickness t and the meshsize h, and the symbol $\|.\|$ denotes a norm over the region Ω, if no explicit reference to the region is made

We add to this a priori estimates an estimate related with $\|rot(\gamma - \gamma_h)\|_0$.

Lemma 2.1 *Let Ω be a convex polygon, then*

$$t^2\|rot(\gamma - \gamma_h)\|_0 \leq Ch\|f\|_0 \tag{2.13}$$

PROOF. From de definition of γ and γ_h it follows that

$$t^2 rot(\gamma - \gamma_h) = -rot(\beta - \Pi\beta_h)$$

Then,

$$t^2\|rot(\gamma - \gamma_h)\|_0 \leq \|rot(\beta - \Pi\beta)\|_0 + \|rot(\Pi\beta - \Pi\beta_h)\|_0 \tag{2.14}$$

It is known [12] that for $\eta \in \mathbf{H}_0(rot, \Omega)$,

$$rot(\Pi\eta) = Prot(\eta) \tag{2.15}$$

where P denotes the L^2 proyection operator into $Q_h := rot(\Gamma_h)$ and

$$\|rot(\eta - \Pi\eta)\|_0 \leq Ch\|\eta\|_2 \tag{2.16}$$

From (2.14), (2.15) and (2.16) we obtain

$$t^2\|rot(\gamma - \gamma_h)\|_0 \leq C\{h\|\beta\|_2 + \|\beta - \beta_h\|_1\} \leq Ch\|f\|_0 \tag{2.17}$$

where the last inequality follows from the a priori estimates in (2.9) and (2.11), so (2.13) is proved. □

3 Preliminary Error Estimates

Our first estimates are for the errors in the rotation and the shear force.

Let k be a fixed integer, $k \geq 1$. The estimator is defined for any $T \in \mathcal{T}_h$ as:

$$\begin{aligned}\varepsilon_T = {} & \|P^k f\|_{0,T}|T|^{1/2} + \tfrac{1}{2}\textstyle\sum_{\partial T_i \subset \partial T} \|[\gamma_h \mathbf{n}_i]_J\|_{0,\partial T_i}|\partial T_i|^{1/2} \\ & + \|\mathbf{div} D\Xi(\beta_h) + \gamma_h\|_{0,T}|T|^{1/2} + \tfrac{1}{2}\textstyle\sum_{\partial T_i \subset \partial T} \|[D\Xi(\beta_h)\mathbf{n}_i]_J\|_{0,\partial T_i}|\partial T_i|^{1/2}\end{aligned} \tag{3.1}$$

where P^k is the L^2 projection onto $\mathcal{P}_k$, $|T|$ and $|\partial T_i|$ are the area of T and the length of ∂T_i, $\mathbf{n}_i$ is the normal vector to the edge ∂T_i and $[.]_J$ denote de jump of the corresponding function across ∂T_i.

Next we define a weak norm for the error in β and γ, as the dual norm in $\mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$ of the operator $a(\beta - \beta_h, \eta) + (\gamma - \gamma_h, \nabla\zeta - \eta)$, that is:

$$\|(\boldsymbol{\beta}-\boldsymbol{\beta}_h),(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h)\|_{*,\boldsymbol{\Omega}} = \sup_{\substack{(\eta,\zeta)\in \mathbf{H}_0^1(\Omega)\times H_0^1(\Omega)\\ (\eta,\zeta)\neq(\mathbf{0},0)}} \frac{|a(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\eta})+(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\nabla\zeta-\boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_1+\|\zeta\|_1} \tag{3.2}$$

For each $T\in\mathcal{T}_h$ let

$$\omega_T=\{\cup\,\tilde{T}\in\mathcal{T}_h\ :\ T\cap\tilde{T}\neq\emptyset\}$$

Theorem 3.1 *There exist two constants C_1 and C_2, depending on the minimum angle of the mesh such that*

$$\|(\boldsymbol{\beta}-\boldsymbol{\beta}_h),(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h)\|_* \le C_1\sum_{T\in\mathcal{T}_h}\{\varepsilon_T+\|f-P^kf\|_{0,T}|T|^{1/2}\} \tag{3.3}$$

$$\varepsilon_T\le C_2\{\|(\boldsymbol{\beta}-\boldsymbol{\beta}_h),(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h)\|_{*,\omega_T}+\sum_{\tilde{T}\in\omega_T}|\tilde{T}|^{1/2}\|f-P^kf\|_{0,\tilde{T}}\} \tag{3.4}$$

PROOF. From (2.3) we have

$$a(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\eta})+(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\nabla\zeta-\boldsymbol{\eta})=(f,\zeta)-a(\boldsymbol{\beta}_h,\boldsymbol{\eta})-(\boldsymbol{\gamma}_h,\nabla\zeta-\boldsymbol{\eta}) \tag{3.5}$$

For $\psi\in H_0^1(\Omega)$ or $\mathbf{H}_0^1(\Omega)$ we denote by $\psi_I\in H_0^1(\Omega)$ or $\mathbf{H}_0^1(\Omega)$ respectively, a piecewise linear average interpolant as defined in [14, 19], satisfying

$$\|\psi-\psi_I\|_0\le Ch\|\psi\|_1 \tag{3.6}$$

and

$$\|\psi_I\|_1\le C\|\psi\|_1 \tag{3.7}$$

Taking $\boldsymbol{\eta}_h=\boldsymbol{\eta}_I$ and $\zeta_h=\zeta_I$ in (2.8), and subtracting it from (3.5) we get

$$\begin{aligned}
&a(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\eta})+(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\nabla\zeta-\boldsymbol{\eta})=\\
&(f,\zeta-\zeta_I)-a(\boldsymbol{\beta}_h,\boldsymbol{\eta}-\boldsymbol{\eta}_I)-(\boldsymbol{\gamma}_h,(\nabla\zeta-\boldsymbol{\eta})-(\nabla\zeta_I-\boldsymbol{\eta}_I)+(\boldsymbol{\gamma}_h,\boldsymbol{\eta}_I-\boldsymbol{\Pi}\boldsymbol{\eta}_I)\\
&=\textstyle\sum_{T\in\mathcal{T}_h}\{(f,\zeta-\zeta_I)_T\ -\ \frac{1}{2}\sum_{\partial T_i\subset\partial T}\int_{\partial T_i}[\boldsymbol{\gamma}_h\mathbf{n}_i]_J(\zeta-\zeta_I)\ +\ (\mathbf{div}D\Xi(\boldsymbol{\beta}_h)+\boldsymbol{\gamma}_h,\boldsymbol{\eta}-\boldsymbol{\eta}_I)_T\\
&\textstyle-\frac{1}{2}\sum_{\partial T_i\subset\partial T}\int_{\partial T_i}[D\Xi(\boldsymbol{\beta}_h)\mathbf{n}_i]_J(\boldsymbol{\eta}-\boldsymbol{\eta}_I)\}\ +\ (\boldsymbol{\gamma}_h,\boldsymbol{\eta}_I-\boldsymbol{\Pi}\boldsymbol{\eta}_I)\\
&\le\ \textstyle\sum_{T\in\mathcal{T}_h}\{\|f\|_{0,T}\|\zeta-\zeta_I\|_{0,T}\ +\ \frac{1}{2}\sum_{\partial T_i\subset\partial T}\ \|[\boldsymbol{\gamma}_h\mathbf{n}_i]_J\|_{0,\partial T_i}\|\zeta-\zeta_I\|_{0,\partial T_i}\\
&\textstyle+\|\mathbf{div}D\Xi(\boldsymbol{\beta}_h)+\boldsymbol{\gamma}_h\|_{0,T}\|\boldsymbol{\eta}-\boldsymbol{\eta}_I\|_{0,T}\ +\ \frac{1}{2}\sum_{\partial T_i\subset\partial T}\|[D\Xi(\boldsymbol{\beta}_h)\mathbf{n}_i]_J\|_{0,\partial T_i}\|\boldsymbol{\eta}-\boldsymbol{\eta}_I\|_{0,\partial T_i}\}\\
&+(\boldsymbol{\gamma}_h,\boldsymbol{\eta}_I-\boldsymbol{\Pi}\boldsymbol{\eta}_I)\\
&\le C\textstyle\sum_{T\in\mathcal{T}_h}\{\|f\|_{0,T}|T|^{1/2}+\frac{1}{2}\sum_{\partial T_i\subset\partial T}\|[\boldsymbol{\gamma}_h\mathbf{n}]_J\|_{0,\partial T_i}|\partial T_i|^{1/2}+\ \|\mathbf{div}D\Xi(\boldsymbol{\beta}_h)+\boldsymbol{\gamma}_h\|_{0,T}|T|^{1/2}\\
&\textstyle+\frac{1}{2}\sum_{\partial T_i\subset\partial T}\|[D\Xi(\boldsymbol{\beta}_h)\mathbf{n}_i]_J\|_{0,\partial T_i}|\partial T_i|^{1/2}]\}\{\|\boldsymbol{\eta}\|_1+\|\zeta\|_1\}\ +\ (\boldsymbol{\gamma}_h,\boldsymbol{\eta}_I-\boldsymbol{\Pi}\boldsymbol{\eta}_I)
\end{aligned} \tag{3.8}$$

We are going now to bound the term $(\boldsymbol{\gamma}_h, \boldsymbol{\eta}_I - \boldsymbol{\Pi}\boldsymbol{\eta}_I)$.

It is known ([17], Lemma 3.3) that for $\boldsymbol{\eta}_I$ as defined above, there exists $\phi \in H_0^1(\Omega)$ such that $\phi|_T \in \mathcal{P}_2$ and

$$\nabla\phi = \boldsymbol{\eta}_I - \boldsymbol{\Pi}\boldsymbol{\eta}_I$$

In [17] it is also proved that ϕ vanishes at all the nodes of the triangulation. Let ϕ_I the Lagrange interpolant of ϕ. Then,

$$\begin{aligned}
(\boldsymbol{\gamma}_h, \boldsymbol{\eta}_I - \boldsymbol{\Pi}\boldsymbol{\eta}_I) &= (\boldsymbol{\gamma}_h, \nabla\phi) = \textstyle\sum_{T\in\mathcal{T}_h} \frac{1}{2}\sum_{\partial \mathrm{T_i}\subset\partial \mathrm{T}} \int_{\partial \mathrm{T_i}} [\boldsymbol{\gamma}_h \mathbf{n}_i]_J(\phi - \phi_I) \\
&\le \textstyle\sum_{T\in\mathcal{T}_h} \frac{1}{2}\sum_{\partial \mathrm{T_i}\subset\partial \mathrm{T}} \|[\boldsymbol{\gamma}_h \mathbf{n}_i]_J\|_{0,\partial \mathrm{T_i}} \|\phi - \phi_I\|_{0,\partial \mathrm{T_i}} \\
&\le C\textstyle\sum_{T\in\mathcal{T}_h} \{\frac{1}{2}\sum_{\partial \mathrm{T_i}\subset\partial \mathrm{T}} \|[\boldsymbol{\gamma}_h \mathbf{n}_i]_J\|_{0,\partial \mathrm{T_i}} |\partial T_i|^{1/2}\} \|\nabla\phi\|_{0,T} \\
&\le C\textstyle\sum_{T\in\mathcal{T}_h} \{\frac{1}{2}\sum_{\partial \mathrm{T_i}\subset\partial \mathrm{T}} \|[\boldsymbol{\gamma}_h \mathbf{n}_i]_J\|_{0,\partial \mathrm{T_i}} |\partial T_i|^{1/2} |T|^{1/2} \|\boldsymbol{\eta}\|_{1,T}\}
\end{aligned} \tag{3.9}$$

where we have used (2.7) and (3.7) to obtain the last inequality.

This shows that the last inner product in (3.8), can be bounded by the previous terms of the same expression.

From (3.8) and (3.9) we obtain:

$$\frac{|a(\boldsymbol{\beta} - \boldsymbol{\beta}_h, \boldsymbol{\eta}) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \nabla\zeta - \boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_1 + \|\zeta\|_1} \le C \sum_{T\in\mathcal{T}_h} \{\varepsilon_T + \|f - P^k f\|_{0,T} |T|^{1/2}\} \tag{3.10}$$

from which it follows (3.3).

In order to proof the inequality (3.4) we need the following lemma:

Lemma 3.1 *Let $T \in \mathcal{T}_h$. Given $\mathbf{q} \in \mathbf{L}^2(T)$, $\mathbf{p} \in \mathbf{L}^2(\partial T)$, there exists $\hat{\boldsymbol{\eta}}_T \in \mathbf{P}_{k+3}$ such that*

$$\begin{cases}
(\hat{\boldsymbol{\eta}}_T, \mathbf{r})_T = (\mathbf{q}, \mathbf{r})_T & \forall \mathbf{r} \in \mathbf{P}_k(T) \\
\int_{\partial \mathrm{T_i}} \hat{\boldsymbol{\eta}}_T \mathbf{s} = \int_{\partial \mathrm{T_i}} \mathbf{p}\mathbf{s} & \forall \mathbf{s} \in \mathbf{P}_{k+1}(\partial T) \\
\hat{\boldsymbol{\eta}}_T = \mathbf{0} & \text{at the vertices of } T.
\end{cases} \tag{3.11}$$

and

$$\|\hat{\boldsymbol{\eta}}_T\|_{0,T} \le C\{\|\mathbf{q}\|_{0,T} + \sum_{\partial \mathrm{T_i}\subset\partial \mathrm{T}} |\partial T_i|^{1/2} \|\mathbf{p}\|_{0,\partial \mathrm{T_i}}\} \tag{3.12}$$

In particular if $\mathbf{p} = \mathbf{0}$ then $\hat{\boldsymbol{\eta}}_T|_{\partial \mathrm{T}} = \mathbf{0}$.

PROOF. The proof follows with arguments similar to those given in [1]. □

In particular the previous result is also valid for scalar functions, that is:

Lemma 3.2 *Let $T \in \mathcal{T}_h$. Given $q \in L^2(T)$, $p \in L^2(\partial T)$, there exists $\hat{\zeta}_T \in \mathcal{P}_{k+3}$ such that*

$$\begin{cases} (\hat{\zeta}_T, r)_T = (q, r)_T & \forall r \in \mathcal{P}_k(T) \\ \int_{\partial T_i} \hat{\zeta}_T \, s = \int_{\partial T_i} p \, s & \forall s \in \mathcal{P}_{k+1}(\partial T) \\ \hat{\zeta}_T = 0 \quad \text{at the vertices of } T. \end{cases} \tag{3.13}$$

and

$$\|\hat{\zeta}_T\|_{0,T} \leq C\{\|q\|_{0,T} + \sum_{\partial T_i \subset \partial T} |\partial T_i|^{1/2} \|p\|_{0,\partial T_i}\} \tag{3.14}$$

In particular if $p = 0$ then $\hat{\zeta}_T|_{\partial T} = 0$. □

Now for fixed $T \in \mathcal{T}_h$ we take

$$\begin{aligned} q &= P^k f |T| \in \mathcal{P}_k(T) \\ p|_{\partial T_i} &= \tfrac{1}{4} |\partial T_i| [\gamma_h \mathbf{n}_i]_J, \quad p \in \mathcal{P}_1(\partial T) \end{aligned} \tag{3.15}$$

and we take the corresponding $\hat{\zeta}_T$ defined in Lemma 3.2, making appropriate modifications when T intercepts $\partial\Omega$; whereas for each $\tilde{T} \in \omega_T$, $\tilde{T} \neq T$, $\hat{\zeta}|_{\tilde{T}}$ is defined by the same Lemma taking now

$$\begin{aligned} q &= 0 \\ p|_{\partial T_i} &= \begin{cases} 0 \ \text{if} \ \partial \tilde{T}_i \cap \partial T = \emptyset \\ \text{the same as in (3.15) if } \partial \tilde{T}_i \cap \partial T \neq \emptyset \end{cases} \end{aligned} \tag{3.16}$$

Let $\hat{\zeta}$ defined such that $\hat{\zeta}|_T = \hat{\zeta}_T$ if $T \in \omega_T$ and 0 outside of ω_T. From its definition we see that $\hat{\zeta} \in H_0^1(\Omega)$. and

$$\begin{aligned} &\|P^k f\|_{0,T}^2 |T| + \tfrac{1}{4} \textstyle\sum_{\partial T_i \subset \partial T} \|[\gamma_h \mathbf{n}_i]_J\|_{0,\partial T_i}^2 \| |\partial T_i| \\ &= \textstyle\sum_{\tilde{T} \in \omega_T} \{(\gamma_h, \nabla\hat{\zeta})_{\tilde{T}} + (P^k f, \hat{\zeta})_{\tilde{T}}\} = (\gamma_h - \gamma, \nabla\hat{\zeta})_{\omega_T} + \textstyle\sum_{\tilde{T} \in \omega_T} (P^k f - f, \hat{\zeta})_{\tilde{T}} \end{aligned} \tag{3.17}$$

For the same fixed T we proceed in the same way and determine $\hat{\boldsymbol{\eta}}_T$ applying (3.11) for

$$\begin{aligned} \mathbf{q} &= -(\mathbf{div} D\Xi(\boldsymbol{\beta}_h) + \boldsymbol{\gamma}_h)|T| \quad \in \mathbf{P}_1(T) \\ \mathbf{p}|_{\partial T_i} &= \tfrac{1}{4} [D\Xi(\boldsymbol{\beta}_h)\mathbf{n}_i]_J |\partial T_i|, \quad \mathbf{p} \in \mathbf{P}_1(\partial T) \end{aligned} \tag{3.18}$$

and $\hat{\boldsymbol{\eta}}_{\tilde{T}}$ for $\tilde{T} \in \omega_T$. $\tilde{T} \neq T$, making the corresponding modifications as in (3.16).

Let $\hat{\boldsymbol{\eta}} \in \mathbf{H}_0^1(\Omega)$ defined as $\hat{\boldsymbol{\eta}}|_T = \hat{\boldsymbol{\eta}}_T$ if $T \in \omega_T$ and $\mathbf{0}$ outside of ω_T. Then.

$$\begin{aligned} &\|\mathbf{div} D\Xi(\boldsymbol{\beta}_h) + \boldsymbol{\gamma}_h\|_{0,T}^2 |T| \ + \tfrac{1}{4} \textstyle\sum_{\partial T_i \subset \partial T} \|[D\Xi(\boldsymbol{\beta}_h)\mathbf{n}_i]_J\|_{0,\partial T_i}^2 |\partial T_i| \\ &= \textstyle\sum_{\tilde{T} \in \omega_T} \{a(\boldsymbol{\beta}_h, \hat{\boldsymbol{\eta}})_{\tilde{T}} - (\boldsymbol{\gamma}_h, \hat{\boldsymbol{\eta}})_{\tilde{T}}\} = a(\boldsymbol{\beta}_h - \boldsymbol{\beta}, \hat{\boldsymbol{\eta}})_{\omega_T} + (\boldsymbol{\gamma}_h - \boldsymbol{\gamma}. -\hat{\boldsymbol{\eta}})_{\omega_T} \end{aligned} \tag{3.19}$$

Adding (3.17) and (3.19) we obtain

$$\frac{\varepsilon_T^2}{\|\hat{\zeta}\|_{1,\omega_T}+\|\hat{\boldsymbol{\eta}}\|_{1,\omega_T}} \le C\left\{\frac{|a(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\hat{\boldsymbol{\eta}})_{\omega_T}+(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\nabla\hat{\zeta}-\hat{\boldsymbol{\eta}})_{\omega_T}|}{\|\hat{\zeta}\|_{1,\omega_T}+\|\hat{\boldsymbol{\eta}}\|_{1,\omega_T}}+\frac{\sum_{\tilde{T}\in\omega_T}(P^kf-f,\hat{\zeta})_{\tilde{T}}}{\|\hat{\zeta}\|_{1,\omega_T}+\|\hat{\boldsymbol{\eta}}\|_{1,\omega_T}}\right\}$$

$$\le C\left\{\|(\boldsymbol{\beta}-\boldsymbol{\beta}_h),(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h)\|_{*,\omega_T}+\frac{\sum_{\tilde{T}\in\omega_T}\|P^kf-f\|_{0,\tilde{T}}\|\hat{\zeta}\|_{0,\tilde{T}}}{\|\hat{\zeta}\|_{1,\omega_T}+\|\hat{\boldsymbol{\eta}}\|_{1,\omega_T}}\right\} \tag{3.20}$$

Replacing (3.15) in (3.14) and (3.18) in (3.12), we get the following bound

$$\|\hat{\zeta}\|_{0,\tilde{T}}+\|\hat{\boldsymbol{\eta}}\|_{0,\tilde{T}} \le C|\tilde{T}|^{1/2}\varepsilon_T, \quad \textit{for } \tilde{T}\in\omega_T$$

and by standard scaling arguments we also get,

$$\|\hat{\zeta}\|_{1,\omega_T}+\|\hat{\boldsymbol{\eta}}\|_{1,\omega_T} \le C\varepsilon_T$$

Using these bounds in (3.20) it follows that

$$\varepsilon_T \le C\{\|(\boldsymbol{\beta}-\boldsymbol{\beta}_h),(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h)\|_{*,\omega_T}+\sum_{\tilde{T}\in\omega_T}|\tilde{T}|^{1/2}\|P^kf-f\|_{0,\tilde{T}}\}$$

so the Theorem is proved. □

4 Error estimator

Now we are able to define an estimator for the whole error. For any $T\in\mathcal{T}_h$ we define it as:

$$\eta_T=\varepsilon_T+\|\boldsymbol{\beta}_h-\boldsymbol{\Pi}\boldsymbol{\beta}_h\|_{0,T}+\|rot(\boldsymbol{\beta}_h-\boldsymbol{\Pi}\boldsymbol{\beta}_h)\|_{0,T} \tag{4.1}$$

Proposition 4.1 *There exists a constant C, such that*

$$\begin{aligned}&\|\nabla w-\nabla w_h\|_0+\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_1+t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0+t^2\|rot(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h)\|_0\\ &\le C\ \textstyle\sum_{T\in\mathcal{T}_h}\{\eta_T+\|f-P^kf\|_{0,T}|T|^{1/2}\}\end{aligned} \tag{4.2}$$

PROOF. Consider the expression

$$\frac{|a(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\eta})+(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\nabla\zeta-\boldsymbol{\eta})|}{\|\boldsymbol{\eta}\|_1+\|\zeta\|_1} \tag{4.3}$$

If we replace in (4.3) $\zeta=w-w_h$ and $\boldsymbol{\eta}=\boldsymbol{\beta}-\boldsymbol{\beta}_h$, we obtain

$$\frac{|a(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)+t^2\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0^2+(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\beta}_h-\boldsymbol{\Pi}\boldsymbol{\beta}_h)|}{\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_1+\|\nabla w-\nabla w_h\|_0} \le \|(\boldsymbol{\beta}-\boldsymbol{\beta}_h),(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h)\|_* \tag{4.4}$$

Taking into account that $\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h \in \mathbf{H}_0(rot, \Omega)$, and according to Proposition 2.1, there exist $\boldsymbol{\phi} \in \mathbf{H}_0^1(\Omega)$ and $\psi \in H_0^1(\Omega)$ such that

$$\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h = \nabla\psi - \boldsymbol{\phi} \tag{4.5}$$

with

$$\|\nabla\psi\|_0 + \|\boldsymbol{\phi}\|_1 \leq C\{\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0\} \tag{4.6}$$

Replacing again in (4.3) $\zeta = \psi$ and $\boldsymbol{\eta} = \boldsymbol{\phi}$, and using (4.6) we get

$$\frac{|a(\boldsymbol{\beta} - \boldsymbol{\beta}_h, \boldsymbol{\phi}) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)|}{\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0} \leq C\frac{|a(\boldsymbol{\beta} - \boldsymbol{\beta}_h, \boldsymbol{\phi}) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \nabla\psi - \boldsymbol{\phi})|}{\|\nabla\psi\|_0 + \|\boldsymbol{\phi}\|_1} \tag{4.7}$$

$$\leq C\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_*$$

Then,

$$\begin{aligned}
&|(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)| \leq \\
&C\,\{\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_*[\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0] + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1\|\boldsymbol{\phi}\|_1\} \\
&\leq C\{\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_*[\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0] \\
&+ [\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0]\ \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1\} \\
&\leq C\{[\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_* + \|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0] \\
&[\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|\nabla w - \nabla w_h\|_0 + \|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0]\}
\end{aligned} \tag{4.8}$$

where we have used continuity of a(,) to obtain the first inequality, and (4.6) to bound $\|\boldsymbol{\phi}\|_1$ in the second inequality.

Returning to (4.4) we can see that

$$\begin{aligned}
&a(\boldsymbol{\beta} - \boldsymbol{\beta}_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0^2 \leq \\
&\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_*\{\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|\nabla w - \nabla w_h\|_0 + \|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0\} \\
&+ |(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)|
\end{aligned} \tag{4.9}$$

so, using coercivity of $a(,)$ and (4.8) we also have

$$\begin{aligned}
&\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0^2 \\
&\leq C\{\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_* + \|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0\} \\
&\{\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|\nabla w - \nabla w_h\|_0 + \|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0\}
\end{aligned} \tag{4.10}$$

From the definition of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_h$ we have the following identity:

$$\nabla w - \nabla w_h = t^2(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h) + (\boldsymbol{\beta} - \boldsymbol{\beta}_h) + (\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h) \tag{4.11}$$

from which it follows that

$$\|\nabla w - \nabla w_h\|_0 \le t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 \tag{4.12}$$

Adding $\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0^2 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0^2$ to both members in (4.10) and making use of (4.12) we arrive to

$$\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0^2 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0^2 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1^2 + t^2\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0^2$$

$$\le C\{\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_* + \|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0\} \tag{4.13}$$

$$\{\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0\}$$

from which we obtain

$$\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0$$

$$\le C\{\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_* + \|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0\} \tag{4.14}$$

Also from (4.11) we have

$$t^2 rot(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h) = -rot(\boldsymbol{\beta} - \boldsymbol{\beta}_h) - rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h) \tag{4.15}$$

from which

$$t^2\|rot(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0 \le \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0 \tag{4.16}$$

From (4.12) and (4.16) we see that

$$t^2\|rot(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0 + \|\nabla w - \nabla w_h\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0$$

$$\le C\{\|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0\} \tag{4.17}$$

From this inequality and (4.14) we get

$$t^2\|rot(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0 + \|\nabla w - \nabla w_h\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0$$

$$\le C\{\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h), (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_* + \|\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h\|_0 + \|rot(\boldsymbol{\beta}_h - \boldsymbol{\Pi\beta}_h)\|_0\} \tag{4.18}$$

Finally (4.2) follows easily from (4.18), using (3.3) and the definition of η_T. □

Theorem 4.1 *There exist two constants C_1 and C_2 depending on the minimum angle of the mesh such that*

$$\|\nabla w - \nabla w_h\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + t^2\|rot(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_\Gamma$$

$$\le C_1 \sum_{T\in\mathcal{T}_h}\{\eta_T + \|f - P^k f\|_{0,T}|T|^{1/2}\} \tag{4.19}$$

and

$$\eta_T \le C_2\{\|(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{1,\omega_T} + \|\nabla w - \nabla w_h\|_{0,\omega_T} + t^2\|rot(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_{0,\omega_T} + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\omega_T}$$

$$+ \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{\Gamma,\omega_T} + \sum_{\tilde{T}\in\omega_T}|\tilde{T}|^{1/2}\|f - P^k f\|_{0,\tilde{T}}\} \tag{4.20}$$

PROOF. For the proof of the first inequality, taking into account Proposition 4.1. we have to bound only $\|\gamma - \gamma_h\|_\Gamma$. For fixed η

$$\frac{a(\beta - \beta_h, \eta)}{\|\eta\|_1} + \frac{(\gamma - \gamma_h, -\eta)}{\|\eta\|_1} \leq \|(\beta - \beta_h), (\gamma - \gamma_h)\|_*$$

From this

$$\|\gamma - \gamma_h\|_{-1} \leq C\{\|\beta - \beta_h\|_1 + \|(\beta - \beta_h), (\gamma - \gamma_h)\|_*\} \tag{4.21}$$

Also

$$\|div(\gamma - \gamma_h)\|_{-1} \leq \|(\beta - \beta_h), (\gamma - \gamma_h)\|_* \tag{4.22}$$

Combining (4.21) ,(4.22) and the results in Proposition 4.1 and Theorem 3.1. we arrive to (4.19).

To obtain (4.20), we consider the bound for ε_T from Theorem 3.1. and the following inequality:

$$\|(\beta - \beta_h), (\gamma - \gamma_h)\|_{*,\omega_T} \leq \|\beta - \beta_h\|_{1,\omega_T} + \|\gamma - \gamma_h\|_{-1,,\omega_T} + \|div(\gamma - \gamma_h)\|_{-1,\omega_T} \tag{4.23}$$

On the other hand we obtain from (4.11)

$$\|\beta_h - \Pi\beta_h\|_{0,T} \leq \|\nabla w - \nabla w_h\|_{0,T} + \|\beta - \beta_h\|_{1,T} + t\|\gamma - \gamma_h\|_{0,T} \tag{4.24}$$

and from (4.15)

$$\|rot(\beta_h - \Pi\beta_h)\|_{0,T} \leq t^2\|rot(\gamma - \gamma_h)\|_{0,T} + \|\beta - \beta_h\|_{1,T} \tag{4.25}$$

The proof is completed by adding (3.4), (4.24) and (4.25) and making use of (4.23). □

Acknowledgments

I want to thank Dr. R. Durán for his helpful comments.

References

[1] D.N. ARNOLD AND F. BREZZI, *Mixed and non conforming finite element methods implementation, postprocessing and error estimates*, R.A.I.R.O., Modèl. Math. Anal. Numer. 19, 1985, pp. 7-32.

[2] D. N. ARNOLD AND R. S. FALK, *A uniformly accurate finite element method for the Reissner-Mindlin plate*, SIAM J. Numer. Anal. **26** (1989), 1276-1290.

[3] I. BABUŠKA AND A.MILLER, *A feedback finite element method with a posteriori error estimation. Part I: The finite element method and some basic properties of the a posteriori error estimator*, Comp. Meth. Appl. Mech. Eng. **61**(1987), 1-40.

[4] I. BABUŠKA AND W. C. RHEINBOLDT, *A posteriori error estimators in the finite element method*, Int. J. Num er. Methods Eng.**12**(1978), 1587-1615.

[5] R. E. BANK AND A. WEISER, *Some a posteriori error estimators for elliptic partial differential equations*, Math. Comp.**44**, (1985), 283-301.

[6] K. J. BATHE AND F. BREZZI, *On the convergence of a four-node plate bending element based on Mindlin-Reissner plate theory and a mixed interpolation*, MAFELAP V (J. R. Witheman,ed.), London, 1985, pp. 491-503.

[7] K. J. BATHE AND F. BREZZI, ,*A simplified analysis of two plate bending elements - the MITC4 and MITC9 elements*, NUMETA 87(G. N. Pande and J.Middleton. eds.), Numerical Techniques for Engineering Analysis and Design, vol. 1, Martinus Nijhoff, Dordrecht, 1987.

[8] K.J. BATHE, F. BREZZI AND M. FORTIN, *Mixed-interpolated elements for Reissner-Mindlin plates*, Int. J. Numer. Methods Eng.**28**(1989), 1787-1801.

[9] K. J. BATHE AND E. N. DVORKIN, *A four-node plate bending element based on Mindlin Reissner plate theory and a mixed interpolation*, J. Numer. Methods Engrg. **21**(1985),367-383.

[10] F. BREZZI, K. J. BATHE, AND M. FORTIN, *Mixed interpolated elements for Reissner-Mindlin plates*, Internat. J. Numer. Methods Eng.**28**(1989), 1787-1801.

[11] F. BREZZI AND M. FORTIN, *Numerical approximation of Mindlin-Reissner plates*. Math. Comp. **47** (1986), 151-158.

[12] F. BREZZI AND M. FORTIN. *Hybrid and mixed finite element methods*.Springer-Verlag New York (1991).

[13] F. BREZZI. M. FORTIN AND R. STENBERG, *Quasi-optimal error bounds for approximation of shear stresses in Mindlin-Reissner plate models*, Math. Models Methods Appl. Sci. **1** (1991), 125-151.

[14] P. CLEMENT. *Approximation by finite element functions using local regularization*. RAIRO Anal. Numér., **9** (1975) 77-84.

[15] E. DARI, R. DURÁN, C. PADRA, V. VAMPA *A posteriori error estimators for nonconforming finite element methods*, M2AN(RAIRO)bf 30(5) (1996), 385-400.

[16] R. DURÁN AND E. LIBERMAN. *On mixed finite element methods for the Reissner-Mindlin plate model*, Math. Comp.**58**, Num. 198(1992), 561-573.

[17] R.DURÁN,L.HERVELLA-NIETO.E.LIBERMAN,R.RODRIGUEZ AND J.SOLOMIN. *Approximation of the vibration modes of a plate by Reissner- Mindlin equations* to appear in Math. Comp.

[18] P. PEISKER AND D. BRAESS, *Uniform Convergence of Mixed Interpolated Elements for Reissner-Mindlin Plates* M2AN(RAIRO)bf 26(5) (1992), 557-574.

[19] L.R. SCOTT AND S. ZHANG, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990) 483-493.

[20] R. VERFÜRTH, *A posteriori error estimators for the Stokes Equations* Numer. Math. 55 (1989), 309-325.

[21] R. VERFÜRTH, *A posteriori error estimates for nonlinear problems. finite element discretizations of elliptic equations* Math. Comp.**62**, Num. 206(1994), 445-475.