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Witten invariant and surgery diagrams

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Abstract

The calculation of the Witten invariant is done by mean of technics which combine mainly the theory of Temperley-Lieb algebras for evaluating diagrams and Dehn surgery on links. The treatment is essentially combinatorial and is applied to calculate the invariant associated to lens spaces.

1 Introduction

Witten invariant can be considered in the context of Topological Quantum Field Theory as a version of the Feymann partition function in the area of Statistical Mechanics. For a phase space X (the configuration space in the language of Statistical Mechanics), let be S a set of symbols $\{s_1, s_2, \ldots, s_n\}$; a "coloration" is simply a map $x \longmapsto s(x)$, let $\mathcal{M}(X, S) = \{x \longmapsto s(x)\}$ set of all colorations: for this model and for another ones in Statistical Mechanics an assignment $s \longmapsto E(s)$ is considered, depending on the model studied and the partition function or Feymann path integral is:

$$Z(X) = \int_{\mathcal{M}(X,S)} E(s)\, \mu(s) \quad \text{, with } \mu \text{ some measure defined in } \mathcal{M}(X, S).$$

In the context of Topological Quantum Field Theory the phase space is a compact 3-manifold M and each one of the colorations is a $1-$form connection A for the principal bundle $P \equiv M \times G$; thus A is valuated in the Lie algebra of G and is locally defined respect a base of the frame bundle (the gauges in the language of Field Theory). Usually the structure group G is reduced to $SU(2)$ and so all the connections trivialize. Now the invariant is defined by :

$$Z_n(M) = \int_{\mathcal{A}} e^{2\pi i L(A)} \mu(A)$$

where $L(A) = \frac{1}{\pi} \int_M Tr\left(A \wedge dA \wedge \frac{2}{3} A^3\right)$ is the Chern-Simons Lagrangian and $\mathcal{A}$ is the set of connections (or $1-$form connections). Two connections A_1 and A_2 are equivalent if the exists a transformation $\varphi : M \longrightarrow SU(2)$ such that $A_1 = \varphi A \varphi^{-1} + \varphi^{-1} d\varphi$. The group $\mathcal{G}(M, G)$ (the gauge group) acts over $\mathcal{A}$.

After the pioneering work of Witten[1] much effort was done in order to calculate this kind of invariants : in [2]numerical calculations by using surgery diagrams and arguments in Topological Quantum Field Theory for lens spaces are tabulated. Analytic expressions for a system of invariants called the Chern-Simons-Witten-Jones invariant , also for lens spaces and achieved by mean of Gaussian sums , are reported in [3]. Another meaningful contributions are the works of Kirby and Melvin [4]and L. Jeffrey [5].

The goal of this article is to derive expressions for the Witten invariant by using combinatorial arguments over surgery diagrams. The main technics to be used were developed by Kauffman and Lins and are a combination between two subjects : the theory of Temperley-Lieb algebras and the link surgery diagrams (Kirby calculus and Lickorish theorem). Temperley-Lieb algebra is generated by "tangles"(more general than the braids used by Jones in the definition of his invariant polynomial), the Kauffman-Lins polynomial is, constructed in a similar way than Jones did, i.e. by mean of a functional :

$$Tr : T_n \rightarrow \mathbf{Z}\left[X, X^{-1}\right],$$

with T_n an algebra of tangles. For a $2-$cell special complex an invariant is defined (the Turaev-Viro one), which agrees with a Kauffman-Lins invariant obtained from the original polynomial. It can be seen as a partition function associated to vertex model complex. When the aim to construct invariants associated to $3-$manifolds is when the Dehn surgery on knots enters in scene. by the Lickorish theorem every closed $3-$manifold can be obtained by surgery of a link. The surgery diagram provided by the Kirby calculus extended to rational surgery will play the role of the special complex and the partition function with an adequate normalization is the Witten invariant.

In Section II we give a more or less detailed description of the elements needed, in the next Section calculations for lens spaces are performed and we finish with some considerations about Brieskorn spheres $\sum(2,3,6n\pm1)$, which are obtained by surgery over the trefoil knot and with the perspectives of using this kind of invariant for the construction of exotic spaces.

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2 Temperley-Lieb algebras. Diagrams.

We begin by reviewing the main topics that will be used , for more details can see [6]. Let K a knot , i.e. an embedding $K: S^1 \hookrightarrow S^3$; as usual a planar representation is considered. A link is understood as a finite union of knots. In each crossing labels X and X^{-1} are attached, schematically acording to fig.1

Taking a smoothing , a state in the nomenclature of Statistical Mechanics, in each crossing the planar representation diagram converts in a finite number of Jordan curves; we assign a label X or X^{-1} in each crossing accordingly the following diagrammatic scheme shown in fig.2:

Let now $M_S(K)$ the product of the labels X or X^{-1} attached in each crossing for each state S and let be $N_S(K)$ the number of the Jordan curves resulting by the smoothing S. Then the Kauffman-Lins polynomial is defined by :

$$P_K=\sum_S M_S(K)\,t^{N_S(K)} \tag{1}$$

where $t=-X^2-X^{-2}$, and the sum is extended over the all states S.

One remarkable recurrence property is :

$$P_K=XP_{K^+}+X^{-1}P_{K^-} \tag{2}$$

where K, K^+ and $K^-$are three planar knot diagrams which differ only in a crossing in such a way that in this crossing K is like in figure 1, and K^+ ,K^- are like left and right hand in figure 2, respectively.

P_K is an invariant of regular isotopy and normalized as : $Q_K=\dfrac{(-X^3)^{Wr(K)}P_K}{P_{K^{(0)}}}$ becomes an invariant of ambient isotopy ($K^{(0)}$ is the diagram corresponding to an unknotted circle 0−framed and $Wr(K)$ is the writhing of the knot K). Another important fact is the relationship with the Jones polynomial V_K., indeed : $V_K(t) = Q_K\left(t^{\frac{1}{4}}\right)$.

The Temperley-Lieb algebra is generated by the elementary tangles depicted in fig. 3:

which satisfy the relations :

$$R : \begin{cases} u_i^2 = tu_i \\ u_i u_{i+1} u_i = u_i \\ u_i u_j = uju_i \quad \text{for } |i-j| \geq 2 \end{cases} \tag{3}$$

where t represents the loop O and the product is the concatenation of the $i-th$ input strand of one tangle with the $i-th$ output strand of the other . Now the Temperley-Lieb algebra T_n is an algebra over $\widehat{\mathbf{Z}}[X, X^{-1}] = \left\{\frac{P}{Q} : P, Q \in \mathbf{Z}[X, X^{-1}]\right\}$ with a presentation in terms of generators and relators : $T_n = \langle u_1, u_2, \ldots, u_n, R\rangle$.

If σ is an element belonging to the Temperley-Lieb algebra , a knot (or link) can be obtained simply by gluing $i-th$ input with the $i-th$ output of itself. The knot got in this way will be denoted by $\widehat{\sigma}$.

Temperley-Lieb algebra is a particular case of the " tangle algebras". i.e. algebras with a presentation : $\Omega_n = \langle S, R\rangle$ where S is a set of generators (tangles) and R are "multiplicative relations". For this kind of algebras the following functional can be defined :

$$\begin{gathered} Tr : \Omega_n \rightarrow \mathbf{Z}[X, X^{-1}] \\ \sigma \longmapsto P_{\widehat{\sigma}} \end{gathered} \tag{4}$$

for the special case of the Temperley-Leib algebras the element σ is a word in the generators $u_1, u_2, \ldots, u_n$,i.e. $\sigma = u_1^{k_1} u_2^{k_2} \ldots u_n^{k_n}$ and $Tr(\sigma) = P_{\widehat{\sigma}} = t^{N_S(\widehat{\sigma})}$.

Thus the Kauffman-Lins polynomial appears defined from a representation of a tangle algebra as well as the Jones polynomial is defined from the braid group B_n. Notice because the relation :

$$K = K^+X \quad + K^-X^{-1} \quad = Xu_1 + X^{-1}I_2$$

any element of the Artin group B_n can be expanded in terms of elements of T_n,so the braids are special cases of tangles . Here by an abuse of notation we denote the crossing where the knots differ with the same symbol of the knots. A representation of B_n in T_n is obtained by :

$$\begin{aligned} \rho(b_i) &= Xu_1 + X^{-1}I_n \\ \rho\left(b_i^{-1}\right) &= X^{-1}u_1 + XI_n \end{aligned} \tag{5}$$

where b_i, b_i^{-1} are the generators of the braid group B_n, i.e b_i is the braid such that its $i-th$ input is connected with the $(i+1)-th$ output and the $(i+1)-th$ input is connected with the $i-th$ output. The Jones trace is . $Tr(b) = P_{\widehat{\rho(b)}}$

The Jones-Wenzl projector is defined as follows: let be f_i given inductively as :

$$\begin{aligned} f_0 &= I_n \\ f_{i+1} &= f_i + s_{i+1} f_i u_i f_i \end{aligned}$$

with : $s_0 = t^{-1}, s_{i+1} = (t - s_i)^{-1}$,so that f_i is an element of T_n By a result reported in[6]: $Tr(f_i) = \phi_n(-X^2)$, where ϕ_n is the Tchebischev function : $\phi_n(X) = \dfrac{X^{n+1} - X^{-n-1}}{X - X^{-1}}$.

The Jones-Wenzl projector is element diagrammatically denoted by the draw of the fig. a, and is developed by the formal sum:

$\dfrac{1}{[n]} \sum_{\sigma \in S_n} (X^{-3})^{sg(\sigma)} \quad \in T_n$ (Jones-Wenzl projector with "n-strands")

where $[n] = \prod_{k=1}^{n} \left(\dfrac{1 - X^{-4k}}{1 - X^{-4}} \right)$, S_n is the permutation group of n elements and $\hat{\sigma}$ is the closure of the braid associated to the permutation σ is such a way that any transposition (i, j) is thought as the braid which connects the $i-th$ input with the $j-th$ output. This object is very useful for evaluating diagrams as we shall see. The term projector is due that a concatenation with itself gives the same (see fig.4):

We shall stand for a diagram a pair $\mathcal{D} = (\Sigma, \alpha)$. with $\sum$ a 2−cell complex such that in each 0−cells incide four 1−cells and in each 1−cell incide three 2−cells and α is a map :

$$\alpha : \Sigma^{i} \rightarrow \{0, 1, ...s-2\} \quad (\Sigma^{i} = i- \text{ cells of } \textstyle\sum):$$

a coloration of an $i-$ cell the image by α of it. In each i−cell ($i = 0, 1$) a 3−valent diagram is assigned by this rule displayed by the fig.5:

Three-valent diagrams can be evaluated by a formalism due to Kauffman and Lins which essentially consist in placing a Jones-Wenzl projector in each $1-$cell with a number of strands equal to to the color of the cell ; the scheme is shown in fig. 6:

with : $i = \frac{a+b-c}{2}$, $j = \frac{a+c-b}{2}$, $k = \frac{b+c-a}{2}$.

Each projector is developed in the form of the formula (6) and in each one term the Kauffman-Lins polynomial is applied. The final result,which we shall call also the Kaufrfman -Lins polynomial i.e. the sum of the values of the polynomial corresponding, for each of the projector placed in each one of the $1-$cells is called the evaluation of the diagram. The resulting formulae are indeed large [6], but in the special case of links diagrams are simpler and in the special case of lens spaces are considerably less complicated.

Thus for a special $2-$cell complex :

$0 - cell$	$\longleftrightarrow$	$Tet \begin{bmatrix} i & b & l \\ j & a & k \end{bmatrix}$
$1 - cell$	$\longleftrightarrow$	$\theta(x, y, z)$
$2 - cell$	$\longleftrightarrow$	ϕ_i (Tchevischev function)

accordingly the earlier diagram with the corresponding colorations. Tet and θ mean the evaluations by the above formalism for the diagrams of the Fig.6.

Then a partition function (or Turaev-Viro invariant) is defined :

$$\Psi(X, \alpha) = \prod_{v \in X^0} \widetilde{Tet}(v) \times \prod_{e \in X^1} \theta(e)^{-1} \times \prod_{f \in X^2} \varphi_f \tag{6}$$

with

$$\widetilde{Tet}(v) = \overline{A}_k^{i,a} A_j^{l,a} Tet \begin{bmatrix} i & b & l \\ j & a & k \end{bmatrix}$$

$$\left(A_t^{r,s} = (-1)^{\frac{r+s-t}{r}} X^{\frac{r'+s'-t'}{2}}, a' = \frac{a(a+2)}{2}, \overline{A} = \frac{1}{A} \right)$$

The evaluations will be taken for $t = -q - q^{-1}. q = e^{i\pi/n}$ (so $X^2 = q$). and one coloration will be "admissible" if al evaluations for these values of q

are non-zero. Now let be :

$$\Psi\left(\sum\right) = \sum_{\alpha \text{ adm}} \Psi\left(\sum, \alpha\right) \tag{7}$$

For the special of a link diagram the unbounded region is colored with 0 and we have the meaningful result [6]: If $P_K(b_1, b_2, ..., b_r)$ is the Kauffman-Lins polynomial obtained by the evaluation of the diagram with a projector placed in each component of K , then :

$$P_K(b_1, b_2, ..., b_r) = \Psi(K, b_1, b_2, ..., b_r) \tag{8}$$

in such a way that in the calculation of $\Psi(K, b_1, b_2, ..., b_r)$ each component is colored with $b_1, b_2, ..., b_r$:

To get finally an invariant of 3−manifolds Dehn surgery over links and Kirby calculus are used, let recall the main aspects of this matter : let be M^3 a 3− compact manifold with boundary topologically a torus T^2, if $[r]$ is an isotopy class in ∂M, the manifold $M(r)$ is defined as that obtained by attaching a solid torus $\mathcal{J}$ to ∂M such that r bounds a disk in $\mathcal{J}$. $M(r)$ is called the manifold obtained from M by surgery along r. We denote $M(r)$ $= M \bigsqcup_{\partial} \mathcal{J}$. For $M = S^3 - int(N(K))$, let

$$\begin{cases} f : \partial M \to \partial \mathcal{J} \\ f(m) - pm + ql \\ f(l) = rm + sl \end{cases}$$

where $\{l, m\}$ is the "longitude-meridian" basis in ∂M; $M(K)$ $M\left(K\left(\frac{p}{q}\right)\right)$ $= M \bigsqcup_{\partial} \mathcal{J}$ is the manifold obtained by $\frac{p}{q}$−surgery over K. Now if f_* is the map induced over the integral homology group $\mathbf{Z^2}$, respect the basis $\{l, m\}$ it has the matrix :

$$f_* \equiv \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL_2(\mathbf{Z})$$

so any rational surgery can be characterized by an 2×2 integral matrix: for the special case of integral surgery :

$M(p) = M \bigsqcup_{l \equiv m} \mathcal{J}$.

The following two theorems are basic in this theory :

Theorem (Lickorish) : Every compact orientable 3−manifold can be obtained by rational surgery over a link.

Theorem (Kirby) : $M(K)$ and $M(K')$ are homeomorphic if and only if K and K' can be transformed one in other by a sequence of elementary transformations of their planar representation diagrams.

Once $P_K(b_1, b_2, ..., b_r)$ is defined , for q a root of the unity $\left(q = e^{i\pi/n}\right)$, we set :

$$P_K = \sum_{b_1, b_2, ..., b_r = 0}^{n-2} \phi_{b_1}\phi_{b_2}.....\phi_{br} P_K(b_1, b_2, ..., b_r) \qquad (9)$$

P_K with an adequate normalization is the Witten's invariant [6].

Remark : If K is a knot : $P_K = \sum_{b=0}^{n-2} \phi_b P_K(b)$.

Now an invariant for a link K is defined by following the route $K \rightarrow M^3(K) \rightarrow Z_n(M^3(K))$, where Z_n is the polynomial P_K with a normalization factor. Currently the normalization is defined in such a way that $Z(S^3) = Z\left(M^3\left(K^{(1)}\right)\right) = 1$ or $Z(S^2 \times S^1) = Z\left(M^3\left(K^{(0)}\right)\right) = 1$ (with integral surgery). By $K^{(i)}$ we denote the diagram consistent in an unknotted circle i−framed. We have :

$$P_{K^{(1)}}(b) = \sum_{b=0}^{n-2} \phi\, We(b)$$

For evaluating unknotted circles with a framing this property of the Jones-Wenzl projector is useful: is we have a projector like in the fig. a. but with a curl in the upper j−strands (fig. b), which means the concatenation of the projector with a twisted cable with b_j strands. Its development is equal to the fig. a multiplied by a factor $(-1)^j X^{-b_j(b_j+2)}$. We could scheme this by putting :

$Fig.\ b = (-1)^j X^{-b_j(b_j+2)} Fig.\ d$

Then we denote :

$$P_{K^{(1)}}(b_j) = (-1)^{a_j} X^{-b_j(b_j+2)}\phi_{b_j}\ ,\ X = e^{\pi i/2n}$$

If it is considered a concatenation with aj−cables with b_j strands (fig. c), with the above conventions, inductively :

$Fig.\ c = (-1)^{a_j b_j} X^{-a_j b_j(b_j+2)} Fig.d$

In order to result $Z(S^2 \times S^1) = 1$. let be $A_{i,j}$ the linking matrix of K. i.e. :

$$A = A_{i,j} = \begin{cases} Lk(K_i, K_j) & \text{if } i \neq j \\ Wr(K_i) & \text{if } i = j \end{cases}$$

$K_1, K_2, \ldots\ldots$ are the components of K.

The signature of A is the signature of the intersection form :

$$\langle \quad , \quad \rangle : H_2\left(W^4\right) \times H_2\left(W^4\right) \to \mathbf{Z}$$

where $W^4 = W^4(K)$ is the four-manifold obtained by integral surgery along K. i.e. :

$$W^4 = D^4 \coprod_{\partial} \left(D^2 \times D^2\right)$$

the 2−handle is attached by an embedding $S^1 \times D^2 \hookrightarrow S^3$. and $W^4 = \partial M^3$.

Let recall : $S^3 = M^3\left(K^{(1)}\right)$ and $S^2 \times S^1 = M^3\left(K^{(0)}\right)$; we denote :

$$\begin{gathered}
\left[P_{K^{(0)}}\right]_n = \sum_{b=0}^{n-2} \phi_b^2 = \frac{-2n}{(X^2 - X^{-2})^2} \\
\left[P_{K^{(1)}}\right]_n = \sum_{b=0}^{n-2} \phi_b P\ (b) = \sum_{b=0}^{n-2} \phi_b^2 (-1)^b X^{b(b+2)} = \\
\beta^{-1} (-i)^{n-2} e^{i\pi\left[\frac{3(n-2)}{4n}\right]},
\end{gathered} \tag{10}$$

with $X = e^{\pi i/2n}$ and $\beta = \sqrt{\frac{2}{n}} \sin\left(\frac{\pi}{n}\right)$. The results are obtained by development of Gauss sums.

Let $\gamma = \beta\left[P_{K^{(1)}}\right]_n$. and let be $\sigma(K) = \sigma(A)$ (the signature of A) : if the invariant is defined by :

$$Z_n(M) = Z_n(M(K)) = \left[P_{K^{(1)}}\right]_n^{-\sigma(K)} P_K. \tag{11}$$

then $Z_n(S^3) = Z_n\left(M\left(K^{(1)}\right)\right) = 1$. If the definition is :

$$Z_n(M) = \beta^{N(K)-1}\gamma^{-\sigma(K)} P_K \text{ (} N(K) \text{ the number of components of } K\text{)}. \tag{12}$$

then $Z(S^2 \times S^1) = Z\left(M^3\left(K^{(0)}\right)\right) = 1$ and $Z_n(S^3) = \beta$.

3 Lens spaces. Calculations.

Let M the manifold obtained by $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_r}{q_r}$ surgeries over an unknotted circle in $S^2 \times S^1$, i.e.:

$$M = M\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_r}{q_r}\right) = S^2 \times S^1 - int\left(\bigcup_{i=1}^{r} N\left(D_i^2 \times S_i^1\right)\right) \bigsqcup_{\partial} (\mathcal{J}_1 \cup \ldots \cup \mathcal{J}_r) \tag{13}$$

it means that r solid torus are removed and the reglued after twistings given by the surgery matrices : $\begin{pmatrix} p_i & r_i \\ q_i & s_i \end{pmatrix} \in SL_2(\mathbf{Z})$, $i = 1, 2, \ldots, r$ and respect the basis $\{l_i, m_i\}$ with $m_i = \partial D_i \times pt$, $l_i = pt \times S^1$. The surgery diagram is a big circle with small meridian circles linked, which represent the rational surgeries $\frac{p_i}{q_i}$ (see fig. 7)

M is a Seifert manifold , the total space of a Seifert bundle. The lens spaces are defined as the special case : $L(p, q) = M\left(\frac{q}{p}\right)$.

One result about surgery diagrams is that if there is a meridian circle to a component it can be "blew -up" by changing the coefficient attached to the component and the new diagram (without the circle) is equivalent to the old one. By equivalent it stand for that the manifolds obtained by surgery over them are homeomorphic(see fig. 8, 11)

Thus the lens spaces can considered as obtained by $-\frac{p}{q}$ surgery over a $0-$framed circle. At this stage we are needing a diagram for the lens spaces $L(p, q)$, for this we consider the continued fraction for $-\frac{p}{q}$:

$$-\frac{p}{q} = (a_1, a_2, \ldots, a_r) \equiv a_r - \cfrac{1}{a_{r-1} - \ldots}$$

By the earlier property of diagrams, we obtain the equivalences between diagrams depicted in the fig. 9:

We compute now the partition function for a diagram similar to the Fig. 12 but with circles $0-$framed, where each circle colorated with $b_1, b_2, \ldots, b_r$.

the unbounded region is labelled by 0 and the faces have the labels : $i_1, i_2, \ldots, j_1, j_2 \ldots$(see fig.10)

In each crossing (0–cells) , accordingly the assignment of Section 2 and the colorations considered, these evaluations must be performed :

$$\overline{A}_0^{i_l,b_l} A_{i_{l+1}}^{j_l,b_l} Tet \begin{bmatrix} i_l & b_{l+1} & 0 \\ i_{l+1} & b_l & j_l \end{bmatrix}$$
$$\overline{A}_{j_l}^{i_l,b_{l-1}} A_{i_{l+1}}^{0,b_{l-1}} Tet \begin{bmatrix} i_l & b_{l+1} & 0 \\ i_{l+1} & b_l & j_l \end{bmatrix}$$

And in each 1–cell the evaluations are :

$$\theta\left(i_l, b_{l+1}, j_l\right)$$
$$\theta\left(j_l, b_l, i_{l+1}\right)$$

The conditions of admissibility for q a root of the unity , $q = e^{i\pi/n}$ are given by :

1) $i + j + k \equiv 0_{\ mod\ 2}$

2) $i + j - k\ ,\ j + k - i,\ i + k = j \geq 0$

3) $i + j + k \leq 2n - 4$

It holds $\theta\left(i, j, k\right) = 0$ if and only if $i + j + k = 2n - 2$. We denote the set of the admissible triples $\{i, j, k\}$ by $Adm_q\left(i, j, k\right)$.

We have :

$$P_K\left(b_1, b_2, \ldots, b_r\right) = \sum_{Adm_q} \prod_{i=1}^{r-1} \overline{A}_0^{i_l,b_l} A_{i_{l+1}}^{j_l,b_l} Tet \begin{bmatrix} i_l & b_{l+1} & j_l \\ i_{l+1} & b_l & 0 \end{bmatrix} \times$$
$$\prod_{l=1}^{r-1} \overline{A}_{j_l}^{b_{l+1},i_l} A_{b_{l+1}}^{0,b_{l+1}} Tet \begin{bmatrix} i_l & b_l & 0 \\ i_{l+1} & b_{l+1} & j_l \end{bmatrix} \times \left[\theta\left(0, b_1, i_1\right)\right]^{-1} \times$$
$$\left[\theta\left(i_r, b_r, 0\right)\right]^{-1} \times \left[\prod_{l=1}^{r-1} \theta\left(i_l, j_l, b_{l+1}\right)\right]^{-1} \times \left[\prod_{l=1}^{r-1} \theta\left(j_l, b_l, i_{l+1}\right)\right]^{-1} \times$$
$$\prod_{l=1}^{r} \phi_{i_l} \times \prod_{l=1}^{r-1} \phi_{j_l} \qquad (14)$$

By admissibility must be : $i_l = b_l,\ l = 1, 2, \ldots, r$; now :

$$\theta\left(0, b_1, i_1\right) = \phi_{b_1}$$
$$\cdots\cdots \qquad (15)$$
$$\theta\left(i_r, b_r, 0\right) = \phi_{b_r}$$

and :

$$Tet\begin{bmatrix} i_l & b_{l+1} & j_l \\ i_{l+1} & b_l & 0 \end{bmatrix} = Tet\begin{bmatrix} b_l & b_{l+1} & j_l \\ b_{l+1} & b_l & 0 \end{bmatrix} = \theta\left(b_l, b_{l+1}, 0\right) \tag{16}$$

Then:

$$P_K\left(b_1, b_2, ..., b_r\right) = \sum_{\substack{j_1 \in Adm_q(b_1, b_2, j_1) \\ \vdots \\ j_{r-1} \in Adm_q(b_{r-1}, b_r, j_{r-1})}} \Phi_{j_1} ......\Phi_{j_{r-1}} \times \prod_{j=1}^{r-1}\left[(-1)^{b_l} \times A_{i_{l+1}}^{j_l, b_l} \times \overline{A}_{j_l}^{b_{l+1}, i_l}\right] \tag{17}$$

with : $\overline{A}_0^{i_l, b_l} = (-1)^{b_l} X^{-b_l'}$, $A_{i_{l+1}}^{j_l, b_l} = (-1)^{\frac{j_l + b_l - b_{l+1}}{2}} X^{\frac{j_l' + b_l' - b_{l+1}'}{2}}$, $\overline{A}_{j_l}^{b_{l+1}, i_l} = (-1)^{\frac{b_{l+1} + i_l - j_l}{2}} X^{\frac{j_l' - b_{l+1}' - b_l'}{2}}$

$\left(x' = x\left(x+2\right)\right)$.

Thus : $\overline{A}_0^{i_l, b_l} \times A_{j_{l+1}}^{j_l, b_l} \times \overline{A}_{j_l}^{b_{l+1}, i_l} = (-1)^{b_l + \frac{j_l + b_l - b_{l+1}}{2} + \frac{b_{l+1} + i_l - j_l}{2}} X^{-b_l' + \frac{j_l' + b_l' - b_{l+1}'}{2} + \frac{j_l' - b_{l+1}' - b_l'}{2}} = (-1)^{2b_l} \times X^{j_l' - b_l' - b_{l+1}'}$.

Now the Kauffman-Lins polynomial associated to a diagram consistent in r circles is given by :

$$P_K = \sum_{b_1, b_2, ..., b_r = 0}^{n-2} \Phi_{b_1} ......\Phi_{b_r} P_K\left(b_1, b_2, ..., b_r\right)$$

with :

$$P_K\left(b_1, b_2, ..., b_r\right) = \sum_{\substack{j_1 \in Adm_q(b_1, b_2, j_1) \\ \vdots \\ j_{r-1} \in Adm_q(b_{r-1}, b_r, j_{r-1})}} \Phi_{j_1} ......\Phi_{j_{r-1}} \times \prod_{j=1}^{r-1} X^{(j_l(j_l+1) - b_l(b_l+2) + b_{l+1}(b_{l+1}+2))/2}$$

and $X = e^{i\pi/2n}$, $q = e^{i\pi/n} = X^2$.

To compensate the framings $a_1, a_2, ..., a_r$ in each circle, we use the property of the Jones-Wenzl projector mentioned earlier; thus if $K^{(a_1,a_2,....,a_r)}$ is the diagram for lens spaces, we have :

$$P_K^{(a_1,a_2,....,a_r)} = \sum_{b_1,b_2,....,b_r=0}^{n-2} \phi_{b_1}......\phi_{b_r} (-1)^{\sum_{j=1}^{r} a_j b_j} X^{\sum_{j=1}^{r} a_j b_j (b_j+2)} \times \times P_K(b_1, b_2, ..., b_r) \tag{18}$$

Finally we state the main result of this article:

Theorem : The Witten invariant for the lens spaces $L(p,q)$ is given by :

$$Z_n(L(p,q)) = \left(\sqrt{\frac{2}{n}} \sin\frac{\pi}{n}\right)^{r+1} \gamma^{\sum sig(a_i)} P_K^{(a_1,a_2,....,a_r)}, \; n \geq 2 \tag{19}$$

where P_K is given by $Eq. (21)$, and as before : $\gamma = \beta \left[P_{K^{(1)}}\right]_n$. $a_1, a_2, ..., a_r$ are the terms of the continued fraction of $-\frac{p}{q}$ and the normalization factor is chosen such that : $Z(S^2 \times S^1) = 1$ and $Z(S^3) = \beta$. $\sum sig(a_i)$ is the signature of the linking matrix for the link corresponding to the lens spaces.

4 Prospects.

Similar treatment can be performed over spaces obtained by surgery over the trefoil knot. The $\pm\frac{1}{n}$–surgery $\left(Tref\left(\pm\frac{1}{n}\right)\right)$ over the trefoil knot gives the Brieskorn spheres : $\sum(2,3,6n \pm 1)$ ($+$ or $-$ accordingly the right or left handed trefoil). Recall that $\sum(a,b,c)$ can be represented as a the link of the zeros of the polynomial $: z_0^a + z_1^b + z_2^c$ $(z_i \in \mathbf{C})$ or by the homogeneous space : S^3/Γ, where Γ is the group with presentation :

$$\Gamma = \left\langle x, y, z : x^a = y^b = z^c = xyz = 1 \right\rangle.$$

With integral surgery it gets the famous Poincare sphere : $\sum(2,3,5)$; besides $\sum(2,3,6n \pm 1) \equiv M\left(\frac{2}{\pm 1}, \frac{3}{\mp 1}, \frac{6n \pm 1}{\mp n}\right)$, so $Tref\left(\frac{1}{n}\right)$ is a Seifert space.

We have : $M(-2,3) \approx S^3$, because by the " blowing-up property" (fig. 11).

Adding another meridian circle to the right diagram we obtain a surgery diagram for the trefoil, more precisely :by Kirby calculus $\frac{p}{q}$−surgery along this meridian is performed over the trefoil. The surgery results in $M\left(-2,3,\frac{p}{q}\right)$. now:

$$Tref\,(p) \equiv M\,(-2,3,p+6)\,.$$

The correspondent diagram surgery is displayed in fig. 12 :

In general : $Tref\left(\frac{p}{q}\right) \equiv M\left(-2,3,\frac{p}{q}+6\right)$, thus to compute the invariant for spaces derived from surgery over the trefoil knot we just add circles to the meridian labelled with $\frac{p}{q}+6$,from the continued fraction expansion. Then similar calculations can be done.

Another thing interesting to check is the periodicity of the Witten invariant :

$$Z_n\left(M\left(\frac{1}{q}\right)\right) = Z_n\left(M\left(\frac{1}{q+K(n)}\right)\right)$$

finding a good estimation for $K(n)$.

By the earlier discussion $\frac{1}{q}$−surgery is equivalent to $-q$−surgery along a meridian with coefficient 0 (fig. 13)

The construction of some " exotic " manifolds was done by surgery diagrams In [7] is constructed a four-dimensional manifold M^4 with $\partial M^4 = \partial N^4$, and such M and N are homeomorphic but non diffeomorphic. One of the key arguments for justifying this assertion is that if a manifold M contains a Brieskorn homology sphere $\sum(2,3,7)$ its Donaldson invariant is non-zero. The question is if invariants like-Witten in dimension four , with a normalization such that for instance $Z(S^4) = Z(W^4(Hopf\ link)) = 1$, for the diagram in the construction of such manifolds can be used for separating diffeomorphism classes. The ability of the Witten invariant to separate diffeomorphism classes was numerically pointed out in [2] of course in dimension three for lens spaces, indeed: $Z_n(L(7,q)) \neq Z_n\left(L\left(7,q'\right)\right)$ for $n \leq 125$ except for $n = 1, 4$.

Another "exotic" spaces are the "exotics $\mathbf{R^4}' s$ ", i.e. manifolds homeomorphic to $\mathbf{R^4}$ but non diffeomorphic to it. Constructions of such a spaces were performed in ,[8],[9], there were found classes of exotics $\mathbf{R^4}' s$ with the set of representatives in each class of diffeomorphism having the cardinality of the continuum in ZCF; again the for separating the representatives was used the Donaldson invariant. And again the question is the same that in the earlier paragraph.

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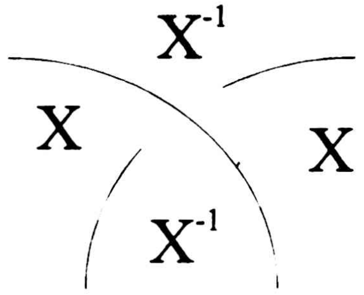
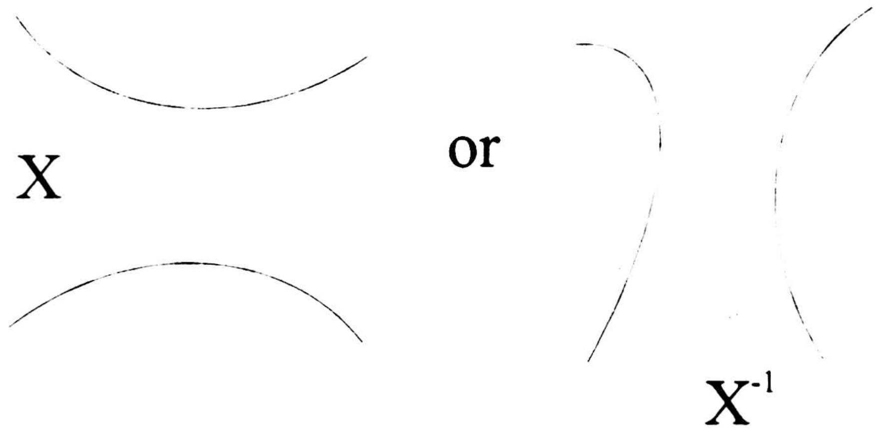


Fig. 1



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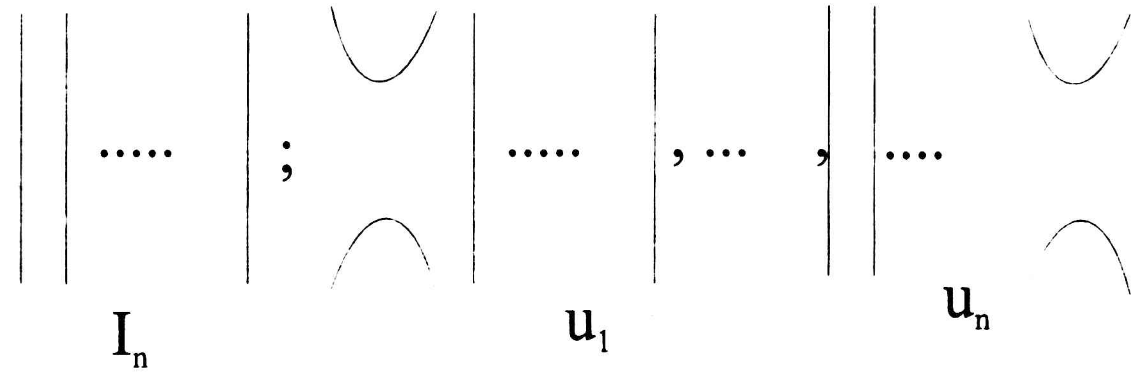

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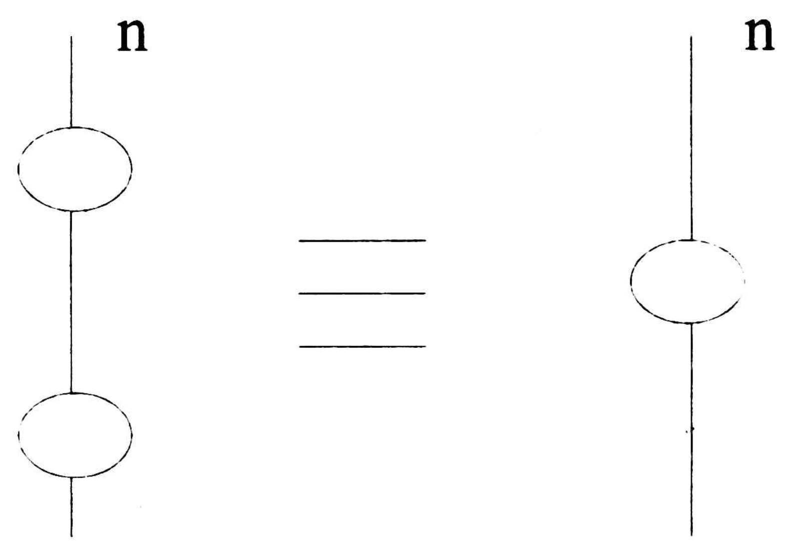

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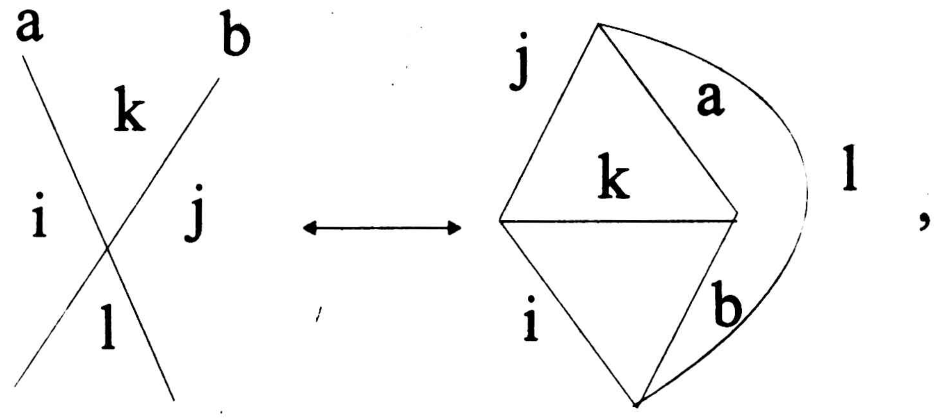

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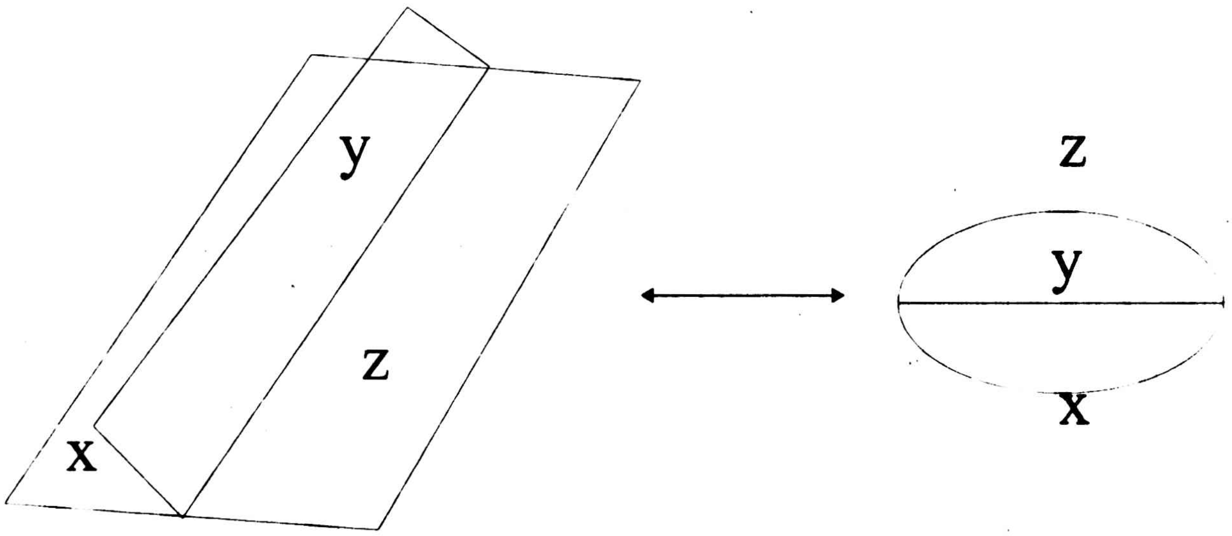

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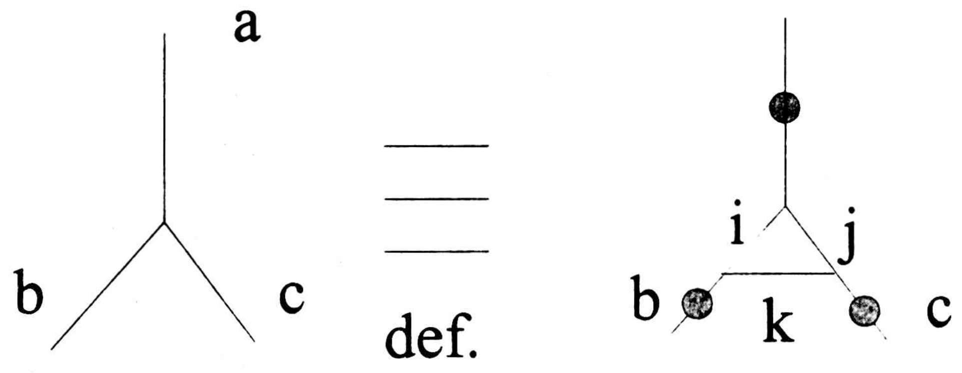

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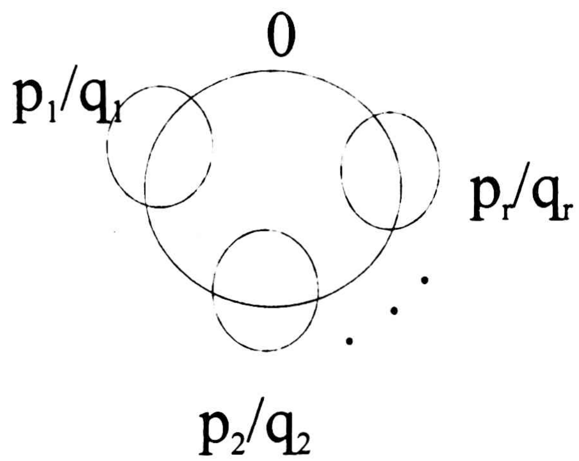

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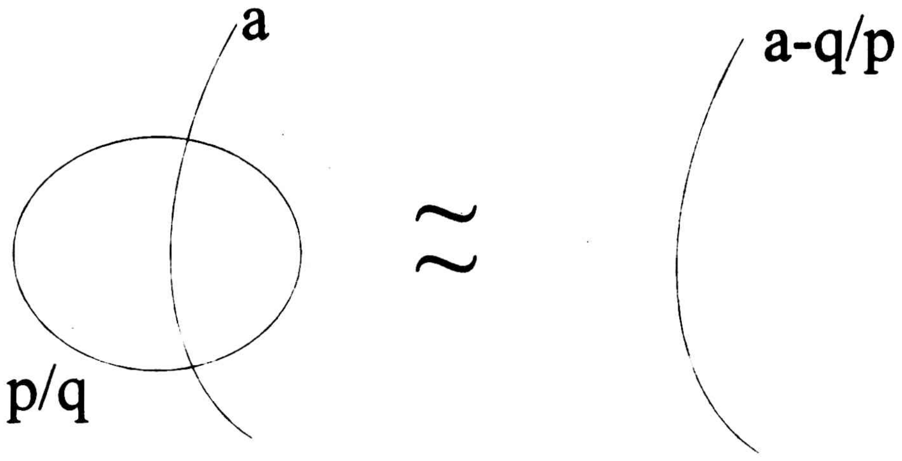

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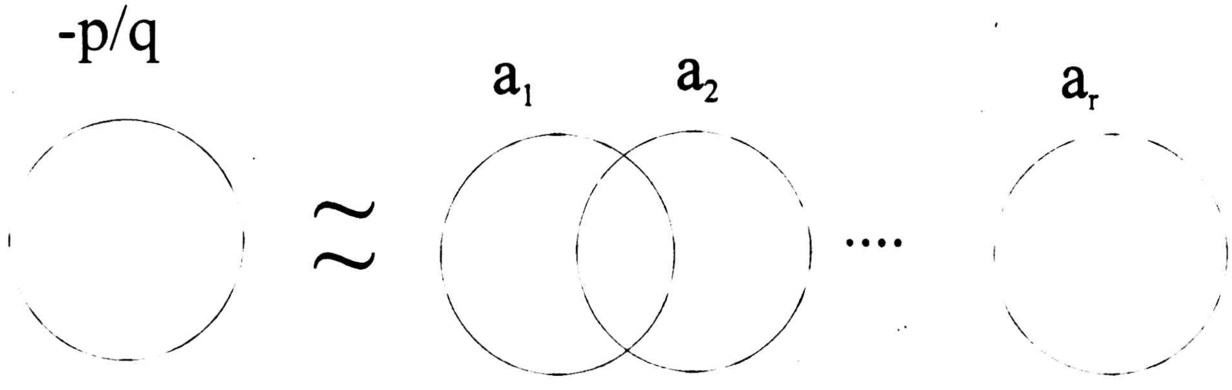


Fig. 9

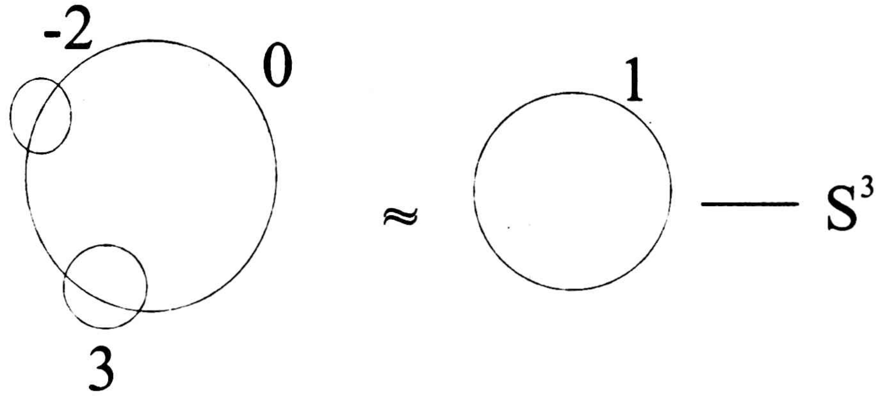
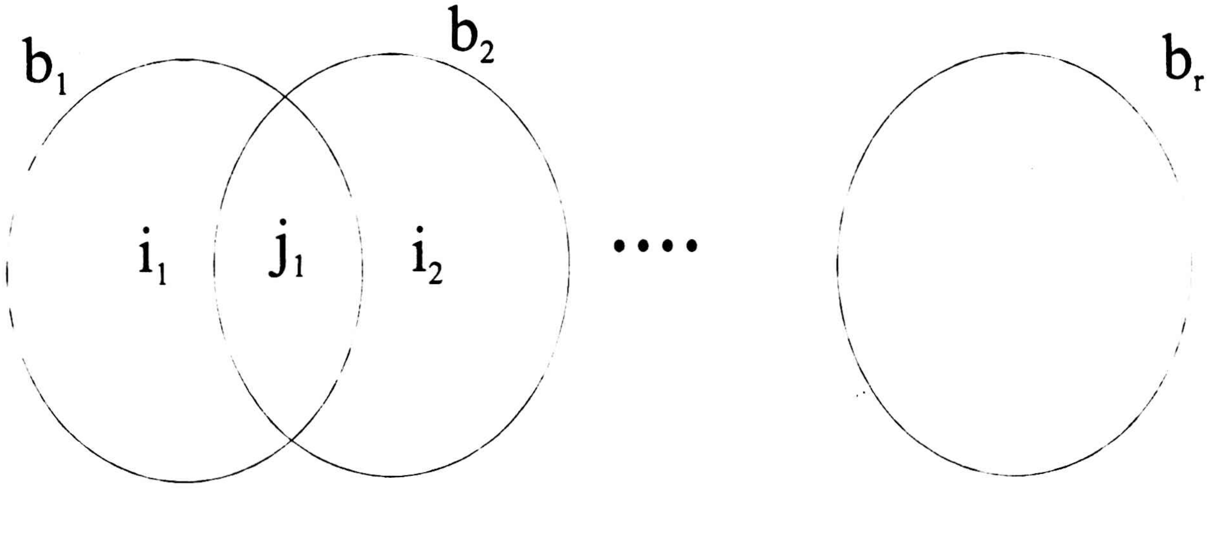


Fig. 11



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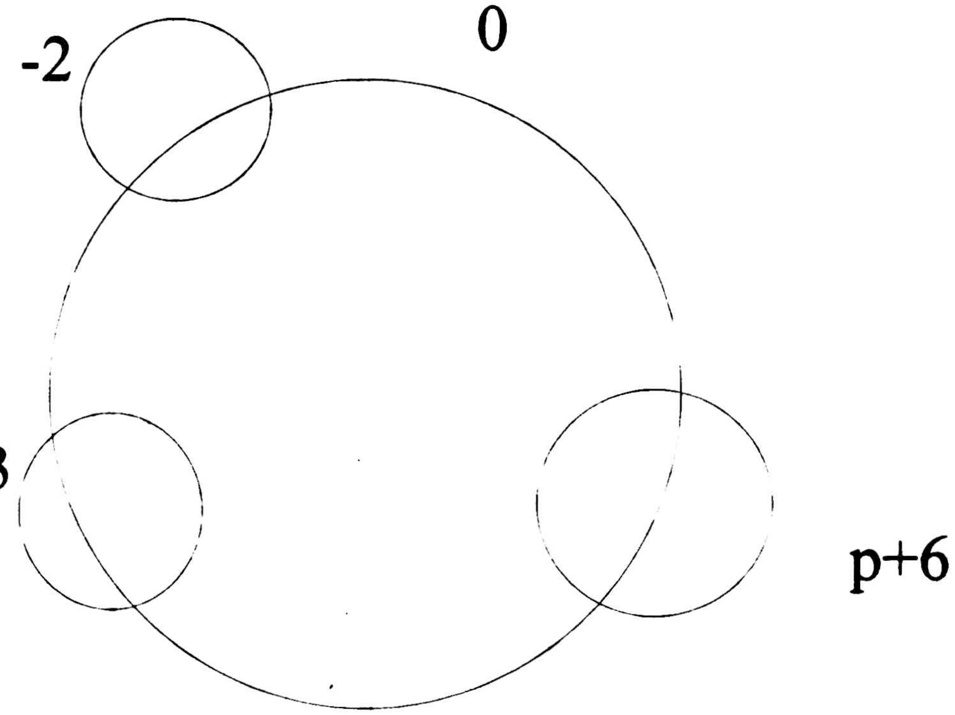


Fig. 12

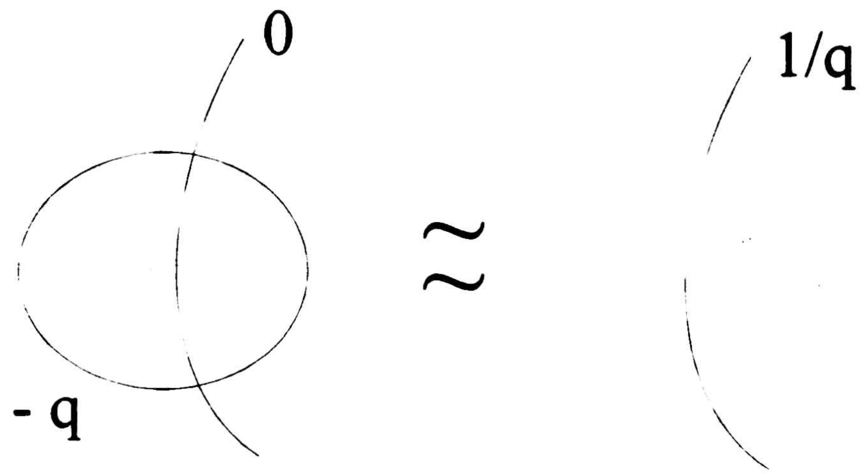


Fig. 13

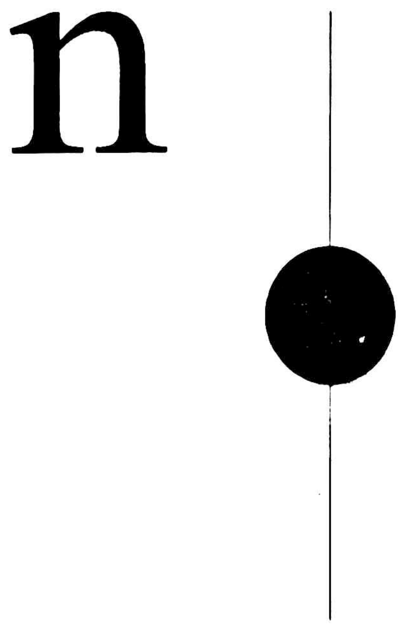
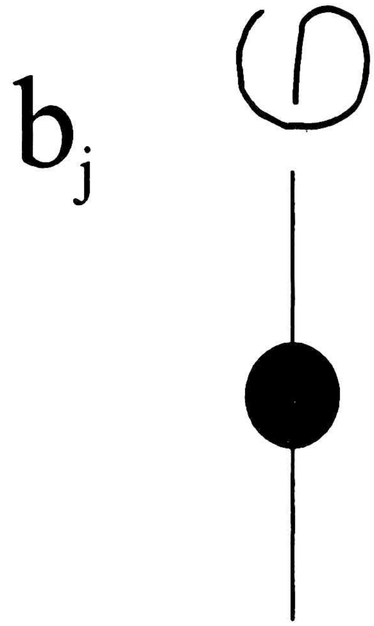


Fig. a



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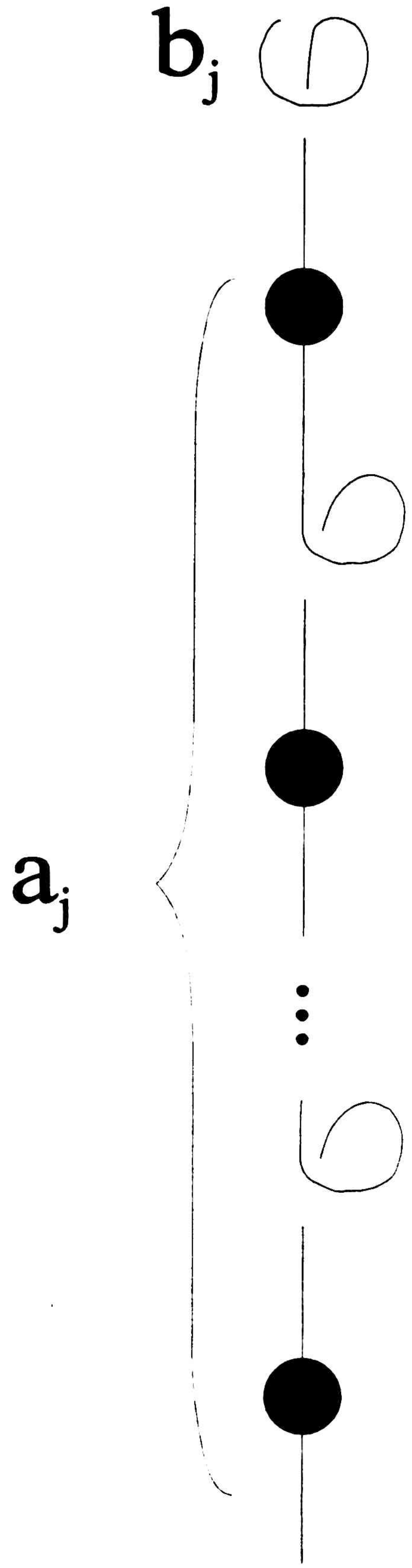

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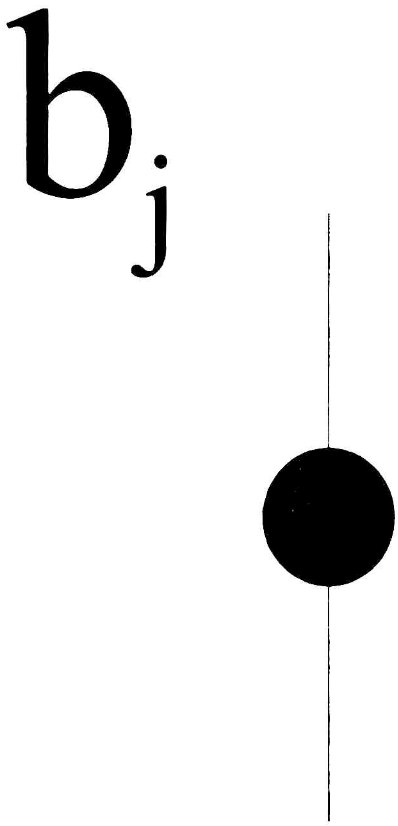

Fig. d