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On some aspects of the stirring rate of planetesimal velocities by a protoplanet

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ABSTRACT

We discuss some aspects of the evolution of the relative velocities of a swarm of planetesimals stirred by a protoplanet. We show that the prescriptions most commonly used in semi-analytical 'oligarchic growth' models overestimate the relative velocities of planetesimals by a non-negligible factor. We discuss the probable origin of this discrepancy, proposing a correction factor that provides good agreement between these prescriptions and the results of numerical experiments. The proposed correction factor can be easily implemented in semi-analytical accretion models.

Key words: celestial mechanics – planets and satellites: formation – Solar system: formation.

1 INTRODUCTION

In a protoplanetary disc, planets first grow in a mode termed runaway accretion (Greenberg 1980; Wetherill & Cox 1985; Wetherill & Stewart 1989; Ohtsuki & Ida 1990), in which the growth rate of the largest bodies is the fastest. In this regime, the self-interaction between planetesimals dominates the dynamical evolution of the system. The relative velocities among planetesimals are slow, in such a way that the gravitational cross-sections of the biggest planetesimals are the largest ones. The dynamical influence of the largest embryos quickly dominates the velocity evolution of the surrounding planetesimals, however, and the growth switches to a slower regime, in which the ratio of the masses of adjacent protoplanets tends to unity over time. This mode of accretion has been termed 'oligarchic growth' (Ida & Makino 1993; Kokubo & Ida 1998). In a standard environment such as the minimum-mass solar nebula (Hayashi, Nakazawa & Nakagawa 1985), the transition from runaway to oligarchic growth occurs when the masses of the protoplanets are still orders of magnitude smaller than one Earth mass. The relevant regime of accretion is therefore that of oligarchic growth, and, in fact, almost all published semi-analytic models of planetary formation include only this accretion mode (Thommes, Duncan & Levison 2003; Ida & Lin 2004a,b; Chambers 2006).

A crucial factor in these models is the prescription for the evolution of the relative velocities of the planetesimals, which regulates the accretion rate of the planetary embryos. A common feature of these kinds of models is that the self-interaction among planetesimals is ignored. Most authors (e.g. Thommes et al. 2003; Ida & Lin 2004a,b) assume that the planetesimal relative velocities achieve an equilibrium value for which the scattering by the embryos is compensated by the damping caused by the gas drag acting on small planetesimals. Chambers (2006) relaxed this hypothesis, adopting a velocity evolution given by the three-body theory of Ohtsuki, Stewart & Ida (2002), which is, at present, the set of analytical expressions providing the best agreement with numerical experiments. The accuracy of these analytical expressions has been tested with the results of numerical simulations in the case of low-mass planets (near the lower limit for oligarchic growth). Although oligarchic growth models starts with very small planetary embryos, the theoretical prescription for the velocity evolution is used during the whole accretion process, even when the planets achieve several Earth masses, which is the typical mass of the core of a giant planet. Here, by a series of numerical simulations of planet–planetesimal interactions, we show that the planetesimal relative velocities are slower than the ones predicted by the theory, especially when the protoplanets are bigger than 1 $M_{\oplus}$. We discuss the possible origin of this discrepancy, proposing an empirical correction factor that is able to reconcile the results obtained in numerical simulations with the velocity evolution predicted by the theory. An important characteristic of this correction factor is that it is very simple, and can be included in semi-analytic oligarchic growth models without loss of computational efficiency.

1.1 Stirring rate by a protoplanet

In the oligarchic model, the relative velocity of the planetesimals is completely regulated by the gravitational stirring of the protoplanet and the damping caused by gas drag. In this paper we focus our attention on the former effect. Ida & Makino (1993) have shown that, in the oligarchic regime, the self-interaction among planetesimals can be neglected, and therefore we also neglect this effect.

If we assume that (i) the planetesimal orbits have low eccentricity and inclination (e.g. e, $i \ll 1$), (ii) the semimajor axes of the planetesimal orbits are close to the semimajor axis of the protoplanet, and (iii) the masses of the planetesimals and the protoplanet mass

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are much smaller than that of the central star, the motions of the planetesimals are well described by Hill's form of the restricted three-body problem (Hill 1878). In Hill's problem, the Jacobi integral can be written as

$$E = \frac{1}{2}(e^2 + i^2) - \frac{3}{8}b^2 h_{\rm r}^2 + \frac{9}{2} \tag{1}$$

(Hasegawa & Nakazawa 1990), where b is the planetesimal–protoplanet distance in units of the protoplanet Hill's sphere, and

$$h_{\rm r} = \left(\frac{M}{3\,{\rm M}_\odot}\right)^{1/3},$$

M being the mass of the planet. In oligarchic growth models, the equations governing the evolution of the random velocities of the planetesimals are used within the feeding zone of the protoplanet, which is defined as the region in which nearby planetesimals can enter into the Hill's sphere of the planet. Hayashi, Nakazawa & Adachi (1977) have shown, through numerical simulations, that this region corresponds to particles with $E \geqslant 0$. In the frame of the restricted three-body problem, the feeding zone, as a fraction of the Hill radius of the planet, is given by

$$b \leqslant \sqrt{12 + \frac{4}{3}(\tilde{e}^2 + \tilde{i}^2)} \tag{2}$$

(Hayashi et al. 1977), where $\tilde{e}, \tilde{i}$ are the planetesimal eccentricity and inclination scaled as

$$\tilde{e} = \frac{e}{h_{\rm r}},$$
$$\tilde{i} = \frac{i}{h_{\rm r}}.$$

The distance $bR_{\rm H}$ from the planet is then called the feeding zone:

$$f = bR_{\rm H} = ba_{\rm p}h_{\rm r}, \tag{3}$$

where $a_{\rm P}$ is the protoplanet's semimajor axis. Although the feeding zone depends on e and i, which are being stirred by the protoplanet, in most oligarchic growth models (Thommes et al. 2003; Chambers 2006) it is defined as a fixed factor of the Hill's sphere of the planet.

The stirring rates of the rms eccentricity and inclination by the gravitational action of the protoplanet can be modelled as (Ohtsuki et al. 2002)

$$\frac{{\rm d}e^2}{{\rm d}t} = \left(\frac{M}{3b\,{\rm M}_\odot P}\right)P_e,$$
$$\frac{{\rm d}i^2}{{\rm d}t} = \left(\frac{M}{3b\,{\rm M}_\odot P}\right)Q_i, \tag{4}$$

where M is the mass of the protoplanet, P is its orbital period, and

$$\begin{aligned} P_e &= \left[\frac{73\tilde{e}^2}{10\Lambda^2}\right]\ln(1 + 10\Lambda^2/\tilde{e}^2) \\ &\quad + \left[\frac{72\ I_P(\beta)}{\pi\tilde{e}\tilde{i}}\right]\ln(1+\Lambda^2), \\ Q_i &= \left[\frac{4\tilde{i}^2 + 0.2\tilde{i}\tilde{e}^3}{10\Lambda^2\tilde{e}}\right]\ln(1 + 10\Lambda^2\tilde{e}^2) \\ &\quad + \left[\frac{72\ I_Q(\beta)}{\pi\tilde{e}\tilde{i}}\right]\ln(1+\Lambda^2), \end{aligned} \tag{5}$$

where $\Lambda = \tilde{i}(\tilde{e}^2 + \tilde{i}^2)/12$, P is the orbital period of the planet, and the functions I_P and I_Q are given by

$$\begin{aligned} I_P &= \int_0^1 \frac{5K(\theta) - 12(1-\lambda^2)E(\theta)/(1+3\lambda^2)}{\beta + (\beta^{-1} - \beta)\lambda^2}\,{\rm d}\lambda \\ I_Q &= \int_0^1 \frac{K(\theta) - 12\lambda^2 E(\theta)/(1+3\lambda^2)}{\beta + (\beta^{-1} - \beta)\lambda^2}\,{\rm d}\lambda, \end{aligned} \tag{6}$$

where $K(\theta)$ and $E(\theta)$ are complete elliptical integrals of the first and second kind, $\theta = \sqrt{(3 - 3\lambda^2)}/2$, and $\beta = i/e$. These expressions have been checked with the results of numerical experiments in the frame of the restricted three-body problem. However, even though their accuracy has been tested only for a limited set of planetary masses, representing a small fraction of the Earth's mass (see, for example, Ohtsuki et al. 2002), they are used in models of the oligarchic growth of giant planets up to masses of several Earth masses. In addition, these equations are only valid if e and i are small, a condition that is not always satisfied when large planets are responsible for the gravitational stirring of the planetesimal velocities. In the next section we will present a comparison of the velocity evolution predicted by these expressions and the results of numerical simulations, for a wide range of planetary masses.

2 NUMERICAL EXPERIMENTS

We have performed a set of numerical experiments, all of them in the frame of the restricted three-body problem (i.e. for exactly the model used to derive equation 4). The numerical experiments reported here are similar to the ones reported in almost all papers devoted to the study of the stirring rate of planetesimal velocities by a protoplanet, with the exception that here we cover a wide range of planetary masses.

We computed the dynamical evolution of a set of 3000 mass-less planetesimals initially placed within an annulus of $\pm\,6R_{\rm H}$ around the protoplanet, which was placed at 5 au orbiting around a 1 $\mathrm{M}_\odot$ star in a circular orbit. The perturbation from the protoplanet over the particles was fully taken into account. The planetesimals were initially placed in orbits of small e and i that were generated according to a Rayleigh distribution with $\langle e^2\rangle^{1/2} = 2\langle i^2\rangle^{1/2} = 10^{-3}$ (Ida & Makino 1993). The numerical integration was performed with the code EVORB (Fernández, Gallardo & Brunini 2002) with a step-size of 0.01 yr. The time-span of all our numerical experiments was 10 000 yr, which is long enough to show the relevant features of this problem. As mentioned above, when the drag is included the relative velocities tend to an equilibrium (Thommes et al. 2003) that is achieved on this time-scale. We performed simulations with $M = 0.5, 1, 2, 5$ and 10 $\mathrm{M}_\oplus$. As the theory was developed in the non-collisional case, we set the radii of the planets to an arbitrary small (50 km) value in all simulations. In this way, the number of collisions is small (∼3 per cent of the particles in each simulation end up colliding with the planet), and therefore close encounters with very small impact parameters are considered. The results of the integrations are shown in Fig. 1, in which the evolution of $\langle e^2\rangle^{1/2}$ given by equation (4) is also shown. As can be appreciated, the theory overestimates the relative velocities by a non-negligible factor. It is worth noting that this overestimation is also apparent in fig. 5 of Ida & Makino (1993). We obtained a very similar result for the inclinations. We repeated the simulations with planetary radii equal to their physical radii (adopting a density of 1.5 g cm^{-3}). Although the number of collisions increased by up to ∼25 per cent, the situation was the same as that shown in Fig. 1. It is clear that the discrepancy between theory and experiment increases with the planetary mass.

Part of the origin of the discrepancy lies in the definition of the feeding zone itself. From equation (2) it is clear that the feeding

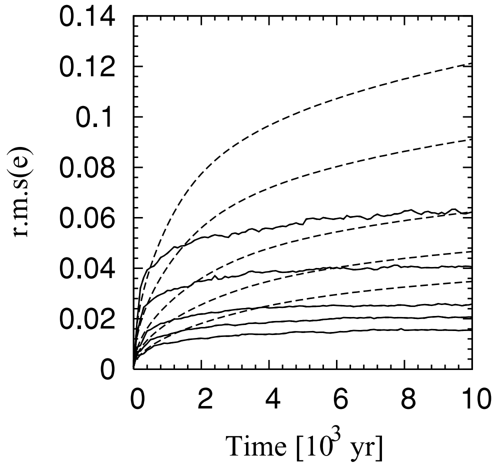

Figure 1. The rms evolution of e. Dashed lines: results from the stirring-rate equations (see text). Solid lines: results from the numerical simulations. The five lines (dashed or solid) are for masses of 0.5, 1, 2, 5, and 10 $M_\oplus$ from bottom to top.

zone expands as the eccentricities and inclinations are pumped up by the planet. Figs 2(a), (b) and (c) show the distribution of (a, e) for a simulation with a one-Earth-mass planet at three different times. The same behaviour was found in all the experiments: the particles whose eccentricities are pushed up are those lying inside the feeding zone *at the origin of the simulation* (i.e. computed with $e \sim 0$ in equation 2). More distant particles are affected by orbital resonances, but this effect acts on a much longer time-scale than the gravitational stirring acting inside the feeding zone, and these distant planetesimals would probably be maintained with low eccentricities by the action of the gas drag in reality. When we restrict the computation to the evolution of those planetesimals *originally inside the feeding zone*, we obtain a much better agreement, as shown in Fig. 3. There is, however, still some discrepancy between theory and experiment. The origin of this discrepancy could be that, because of the symmetry of the problem at small impact parameters, the rms eccentricities of planetesimals within a distance to the planet given by

$$\Delta \leqslant R_{\rm H}\sqrt{\frac{8}{2\tilde{e} + 2.5}} \tag{7}$$

(Hasegawa & Nakazawa 1990) are not stirred by the planet.

Figs 2(a), (b) and (c) show that the planetesimals that are initially within a distance Δ from the planet almost maintain their initial eccentricity, in agreement with the theoretical results. Fig. 4 shows the rms eccentricity of the planetesimals inside Δ.

The conservation of the Jacobi energy (see equation 1) implies that a change in e and i must be compensated by a change in b. The conservation of the Jacobi integral during close encounters leads to the relationship (Hasegawa & Nakazawa 1990)

$$\frac{{\rm d}b^2}{{\rm d}t} = \frac{4}{3}h_{\rm r}^{-2}\left(\frac{{\rm d}e^2}{{\rm d}t} + \frac{{\rm d}i^2}{{\rm d}t}\right). \tag{8}$$

Therefore, planetesimals with $a < a_{\rm p}$ reduce their semimajor axis, whereas those with $a > a_{\rm p}$ increase it (i.e. the planetesimals are pushed away from the planet). It is clear from equation (8) that the time-scale of increase of the feeding zone by this mechanism is of the same order as the stirring-rate time-scale. Our next step will be to model the effect of this expansion of the feeding zone on the velocity evolution of planetesimals. To do this, we use the expressions given in Ida (1990) for the stirring rate of planetesimals

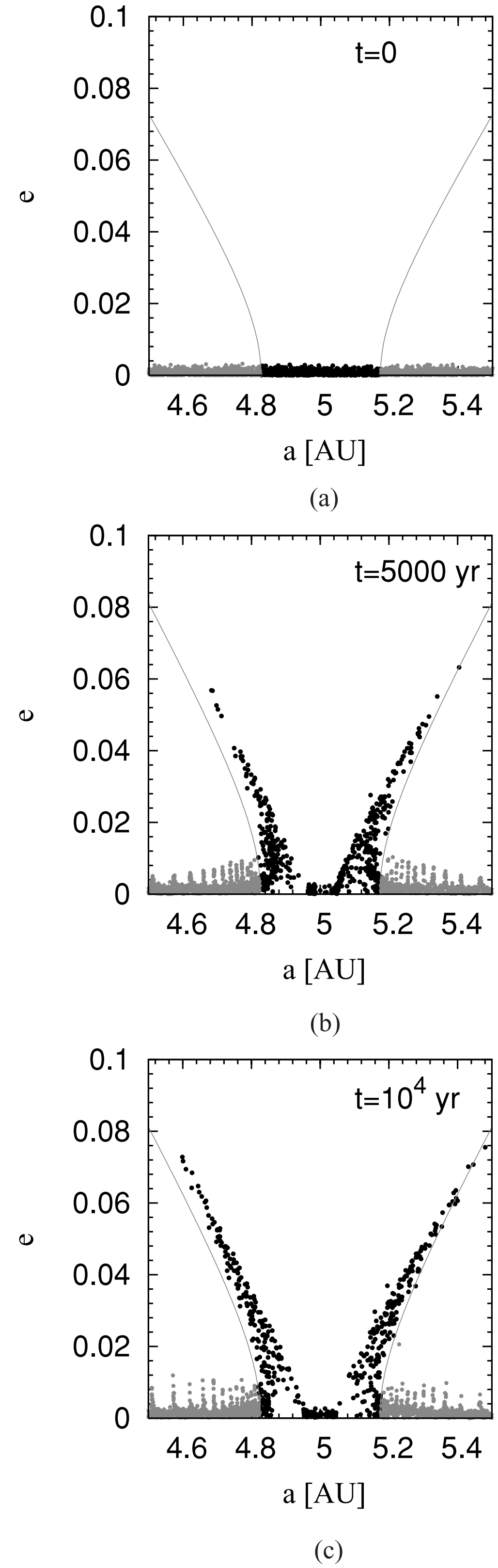

Figure 2. Orbital eccentricity versus semimajor axis for the simulation of a 1 $M_\oplus$ planet at three different times: (a) $t = 0$, (b) $t = 5000$ yr, and (c) $t = 10\,000$ yr. The black dots correspond to planetesimals that lie inside the feeding zone of the planet at the beginning of the simulation. The grey dots are those planetesimals outside the feeding zone. The region between the two lines is the feeding zone defined by equation (2) (see text).

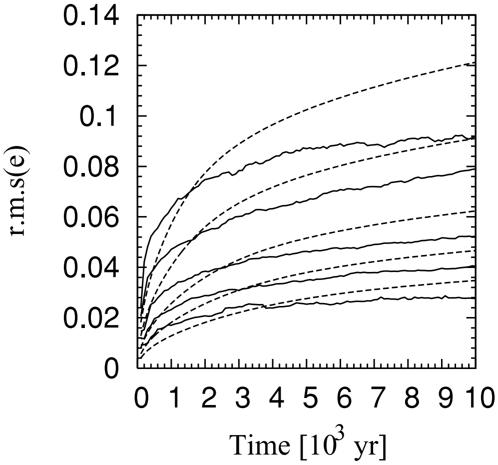

Figure 3. As Fig. 1, but the numerical simulation results were computed considering only those planetesimals that were initially within the feeding zone of the planet.

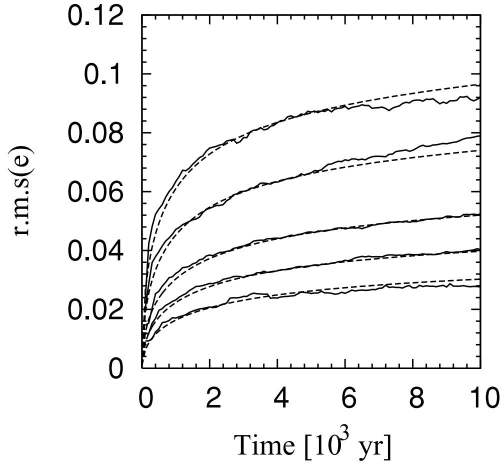

Figure 5. As Fig. 1, but the numerical simulation results were computed considering only those planetesimals that are initially within the feeding zone of the planet, and the correction factor given by equation (11) was applied (see text).

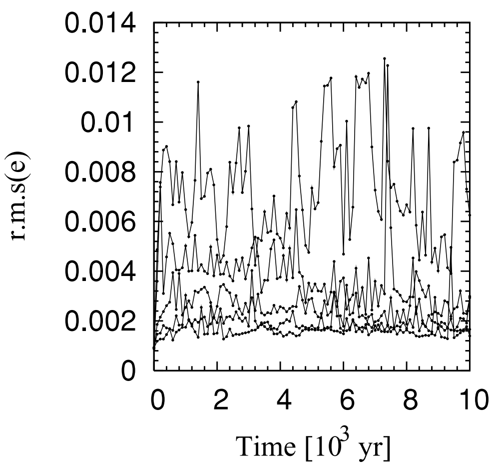

Figure 4. The rms evolution of e of those particles inside the distance to the planet where the stirring by it is negligible (see equation 7). This distance is $\sim$1.5–1.7$R_{\rm H}$. The different lines represent the rms (e) obtained in each one of the five simulations.

in the dispersion-dominated regime:

$$\frac{\mathrm{d}e^2}{\mathrm{d}t} = 80\pi M^2 a^2 n_{\rm M}/e^2 P, \tag{9}$$

where $n_{\rm M}$ is the surface number density of protoplanets. It was simply approximated by $n_{\rm M} \sim 1/2(2\pi a f)$ in equation (4), where f is the width of the feeding zone, taken as a constant value (typically $f = 5R_{\rm H}$ in most models of oligarchic accretion). However, f evolves according to equation (8). We have therefore introduced an empirical correction factor to equation (4), which we rewrite as

$$\frac{\mathrm{d}e^2}{\mathrm{d}t} = \left(\frac{M}{3b\,\mathrm{M}_\odot P}\right) P_e C_{\rm F},$$
$$\frac{\mathrm{d}i^2}{\mathrm{d}t} = \left(\frac{M}{3b\,\mathrm{M}_\odot P}\right) Q_i C_{\rm F}, \tag{10}$$

where

$$C_{\rm F} = 2.5\left(\frac{f(t=0)}{f(t)}\right)^4, \tag{11}$$

and $f(t)$ is evaluated according to equation (3). The results of applying this correction factor are shown in Fig. 5, from which we can appreciate that the agreement between theory and numerical experiment is much better than that without $C_{\rm F}$.

In an oligarchic growth model, the protoplanets accrete mass, and the feeding zone increases not only as a result of the eccentricity stirring, but also as a result of the increase in the Hill radius of the protoplanets. It can be shown that, if the mass of the planet increases at a rate of $\mathrm{d}M/\mathrm{d}t$, the feeding zone expands at a rate given by

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{3}\frac{\mathrm{d}M/\mathrm{d}t}{M}. \tag{12}$$

The accretion rate is much slower than the eccentricity excitation rate or the expansion of the feeding zone given by equation (8). To introduce the correction factor in oligarchic growth models in these conditions, $f(t = 0)$ must be computed with equation (3), as the feeding zone of the planet with its present mass (at the present time t) but with the original e, i of the planetesimals.

It is important to note that equation (4) gives the evolution of the rms (e) and rms (i) of those particles *initially* inside the feeding zone. Fresh planetesimals entering into the feeding zone because of the growth of the planet have lower eccentricities than those given by equation (4).

3 CONCLUSIONS

We have shown that the way that the analytical prescriptions for the dispersion of the relative velocities of planetesimals stirred by the dynamical influence of a planet are used in most models of oligarchic growth of planets overestimates the relative velocities of the planetesimals. This fact was interpreted as resulting from two independent factors.

(i) The feeding zone expands with the stirring of the eccentricities and inclinations of the planetisimals. Nevertheless, only planetesimals for which all of a, e, i satisfy equation (1) simultaneously can be accreted by the planet. This effect is neglected in most models of oligarchic growth that consider a fixed feeding zone (usually of 10$R_{\rm H}$). This results in an overestimation of the surface density of planetesimals, because not all the planetesimals can be accreted by the planet. Within this fixed feeding zone, only those planetesimals inside $\sim 3.5R_{\mathrm{H}}$ are able to be accreted by the protoplanet. A possible way to correct for this effect in accretion models is to consider that the planetesimal density decreases not only by the accretion of the planetary embryos but also by the expansion of the feeding zone, because it expands but the number of planetesimals involved in the accretion process remains the same (except for those planetesimals that are accreted).

(ii) Planetesimals are pushed away from the protoplanet, and, in addition, the region very near the protoplanet, where planetesimals are more prone to be accreted, is insensitive to the scattering by the protoplanet.

A simple empirical correction factor taking into account these effects has been proposed. The corrected expressions have been shown to provide good agreement with the results of numerical experiments for a wide range of planetary masses.

There are a number of effects not taken into account in the theory of the stirring rate of planetesimal velocities. Three of them are as follows.

(i) In a protoplanetary disc, during the planet formation phase, several protoplanets are growing simultaneously. Nearby protoplanets could send planetesimals within the feeding zone of adjacent protoplanets, contributing to increase the protoplanet–planetesimal relative velocity.

(ii) Planetesimals migrate as a result of gas drag. They are therefore continuously leaving and entering the feeding zone. If the feeding zones of adjacent protoplanets do not overlap, the rms $\langle e \rangle$ of the planetesimals that enter the feeding zone should be lower than that of those planetesimals leaving it.

(iii) If planetesimals are massive, the orbit of the protoplanet will shift in response to scattering.

A more realistic model should include these effects.

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