

SURFACES OF REVOLUTION IN a -PARALLEL COORDINATES

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Abstract

In this paper we present a method for visualizing surfaces of revolution in \mathbb{R}^3 using an extension of the parallel coordinate system: the a -parallel coordinate system. The representation of 0-flats and 1-flats in this new system is also given. In order to find the representation of surfaces of revolution in this new coordinate system, we define the extension of Eickemeyer's flats to a wider class of sets, named flats $_a$.

Keywords: Parallel coordinates, visualization of surfaces.

1 Introduction

It is well known that parallel coordinates provide a methodology for unambiguous visualization of multidimensional functions and, in turn, multivariate relations. In the Parallel Coordinate system, relations among the N real variables are mapped uniquely into subsets of 2D-space, thus enabling the visualization of the corresponding N -dimensional hypersurfaces. Using this system of coordinates, only certain kind of hypersurfaces can be represented by their planar images.

There are many areas the Parallel Coordinates System can be applied - in economy, for instance, hypersurfaces are used to model the economy of a country. Applications range from process control to conflict resolution for air traffic control, medicine, data mining on real data sets and many others (see, for example, [1, 3]).

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Hung and Inselberg ([4]) gave several results for developable and partial results for the representation in Parallel Coordinates of ruled surfaces using the Eickemeyer flats and the intersection of the surface with an adequate plane. In this paper, we extend this representation to 3-dimensional surfaces of revolution. The method of parallel coordinates defined by Inselberg and Dimsdale ([2]) fails when representing surfaces of revolution, as proved in this article. Therefore, it is not possible to use the Eickemeyer flats as defined in ([5]). Another class of flats that includes the other, is needed. In order to do this, we define a new parallel coordinate system and a class of flats which we named a -parallel coordinates and flats $_a$, respectively. Our approach uses a variation that allows the representation of developable, ruled and revolution surfaces, according to the value of the parameter a .

The paper is organized as follows: In Section 2 we summarize previous relevant results about parallel coordinates, we give some useful definitions and define the notation that will be used throughout the paper. In Section 3 we define the a -parallel coordinate system and give the representation of p -flats ($p = 0, 1$) in this system. In Section 4 we define the p -flats $_a$ and describe the method used to visualize surfaces of revolution in \mathbb{R}^3 in a system of a -parallel coordinates. In Section 5 we give an example of application. We end this paper in Section 6 by drawing the conclusions and outlining the future work.

2 Previous definitions and notation

The Euclidean N -space is referred to as \mathbb{R}^N . A p -flat in \mathbb{R}^N is a linear subspace of \mathbb{R}^N spanned by $p + 1$ linearly independent points. Thus, a 0-flat is a point, a 1-flat is a line, a $(N - 1)$ -flat in \mathbb{R}^N is a hyperplane. The terms *point*, and *line* are exclusively used for the objects that represent those subspaces, and that live in the parallel coordinate plane. Vectors in \mathbb{R}^N are denoted by lower case English letters in boldface, and their components by the corresponding letter in italic with subscripts. The object in the parallel coordinate plane are usually labeled with the same corresponding symbol that labels the object in \mathbb{R}^N they represent, overlined, and with adequate subscripts.

The projective plane on which we draw parallel axes and plot the representations is denoted P^2 . The points coordinates on this plane are usually written using homogeneous coordinates.

In the Euclidean plane \mathbb{R}^2 with XY -Cartesian coordinates, N copies of the real line

labeled $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_N$ are placed perpendicular to the X -axis. They are the axes of the *parallel coordinates system* for \mathbb{R}^N all having the same positive orientation as the Y -axis. The vector $\mathbf{d} \in \mathbb{R}^N$ denotes the vector whose components are the (signed) distances between the Y -axis and the parallel axes on the projective plane. Another vector of much significance is $(1, 1, \dots, 1)^T$. We shall call the 2-dimensional vector subspace spanned by this vector and \mathbf{d} , the Eickemeyer's 2-flat and denote it by π^E .

3 Representation of 0-flats and 1-flats in a system of a -parallel coordinates

Given a 0-flat $\pi = (\xi_1, \xi_2, \dots, \xi_N)^T \in \mathbb{R}^N$, we plot the points $(d_1, \frac{\xi_1}{a})$, (d_i, ξ_i) , $i = 2, \dots, N$, and call them $\bar{\pi}_1^a, \bar{\pi}_2^a, \dots, \bar{\pi}_N^a$. Then π is associated to the polygonal line joining $\bar{\pi}_i^a$ and $\bar{\pi}_{i+1}^a$, $i = 1, 2, \dots, N - 1$. Figure 1 shows the representation of the 0-flat $\pi = (4, 0, 1)$ in the system of a -parallel coordinates for $a = 2$.

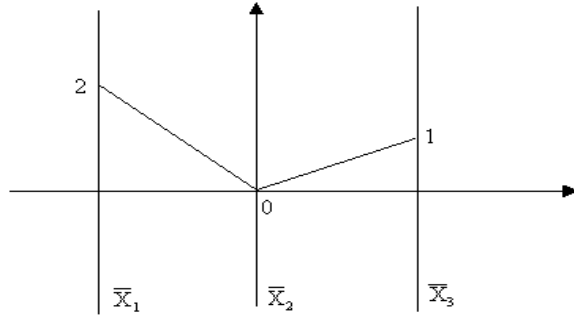


Figure 1: Representation of the point $(4, 0, 1)$ in 2-parallel coordinates.

The set of indexed points $\{\bar{\pi}_1^a, \bar{\pi}_2^a, \dots, \bar{\pi}_N^a\}$ uniquely identify π and hence it will be referred to as the (*a-parallel coordinate*) *representation of the (0-flat) π* .

Remark 3.1 *Let the parametric equation of the 1-flat π be $r(\lambda) = \mathbf{p} + \lambda\mathbf{h}$, where $\mathbf{p}, \mathbf{h} \in \mathbb{R}^3$ are constant vectors. Denote the points:*

$$\bar{\pi}_{1,i}^a = (ah_id_1 - h_1d_i, p_1h_i - p_ih_1, ah_i - h_1)$$

for all $1 < i \leq N$. Then,

$$\bar{\pi}_{i,j}^a = (h_i d_j - h_j d_i, p_j h_i - p_i h_j, h_i - h_j)$$

for all $1 < i \neq j \leq N$.

The points $\bar{\pi}_{i,j}^a$ ($1 \leq i < j \leq N$) uniquely identify the 1-flat π , but we do not need all these points to determine the 1-flat. In fact, if $N = 3$ this representation is stated in the following definition.

Definition 3.1 *Let π be a 1-flat in \mathbb{R}^3 and $\bar{\pi}_{i,j}^a$ the point in the a -parallel coordinate projective plane like in Remark 3.1. The a -parallel coordinate representation of the 1-flat π is given by any of the points $\{\bar{\pi}_{1,2}^a, \bar{\pi}_{2,3}^a\}$ or $\{\bar{\pi}_{1,3}^a, \bar{\pi}_{2,3}^a\}$.*

From Definition 3.1, follows that a 1-flat is represented by two points in the a -parallel coordinate system.

4 Representation of surfaces of revolution

Without loss of generality, we consider the surfaces of revolution generated by the rotation of a curve C in \mathbb{R}^3 around a line L of equation $\mathbf{P}(t) = t\mathbf{n}$, $t \in A \subseteq \mathbb{R}$, $\mathbf{n} = (n_1, n_2, n_3)$, $n_i \in \mathbb{R}$, $i = 1, 2, 3$. The curve C is even respect to L and is contained in a plane containing L .

In order to represent the surface in this system, we need two curves C_1 and C_2 . The first one is the image of the curve C and the second one the image of the curve \tilde{C} , obtained as the intersection of the surface with appropriate planes. In parallel these planes are the Eickemeyer planes π_1^E and π_2^E , respectively. The plane π_1^E is generated by the vectors \mathbf{u} and \mathbf{d} and the plane π_2^E is generated by \mathbf{u} and $\tilde{\mathbf{d}}$, where \mathbf{d} and $\tilde{\mathbf{d}}$ are appropriately chosen.

Lemma 4.1 *Let $\mathbf{n} = (n_1, n_2, n_3)$, $\mathbf{n} \neq \mathbf{0}$. If $n_1 = n_2 = n_3$ every Eickemeyer plane contains \mathbf{n} and some vector $\mathbf{d} = (d_1, d_2, d_3)$, $d_1 < d_2 < d_3$.*

Proof. The planes of the form $(d_3 - d_2)x + (d_1 - d_3)y + (d_2 - d_1)z = 0$ contains the vector \mathbf{n} and any vector $\mathbf{d} = (d_1, d_2, d_3)$, then we can choose it so that $d_1 < d_2 < d_3$. QED

Example 4.1 *The vectors $\mathbf{d} = (1, 2, 3)$ and $\mathbf{u} = (1, 1, 1)$ are contained in the Eickemeyer plane $x - 2y + z = 0$.*

Without loss of generality, in what follows we suppose that the vector $\mathbf{n} = (n_1, n_2, n_3)$ is not parallel to \mathbf{u} . Let, for example, $n_2 \neq n_3$, $n_2 < n_3$. The following results are valid.

Theorem 4.1 *There exists an Eickemeyer's plane that contains the vector $\mathbf{n} = (n_1, n_2, n_3)$ and a vector $\mathbf{d} = (d_1, d_2, d_3)$, $d_1 < d_2 < d_3$ if and only if $n_1 < n_2$.*

Proof. If $n_1 < n_2 < n_3$, it's enough to choose $\mathbf{d} = \mathbf{n}$.

Let π be the Eickemeyer's plane that contains the vector \mathbf{d} . Then:

$$d_1(n_2 - n_3) + d_2(n_3 - n_1) + d_3(n_1 - n_2) = 0.$$

Considering a fixed but arbitrary $d_3 \geq 0$, it follows that:

$$d_1 = \frac{n_3 - n_1}{n_3 - n_2} d_2 + \frac{n_1 - n_2}{n_3 - n_2} d_3.$$

Then we have found a linear relationship between d_1 and d_2 . In order to ensure that exists a vector \mathbf{d} satisfying the required conditions, it must be:

$$\frac{n_3 - n_1}{n_3 - n_2} > 1;$$

and then $n_1 < n_2$. QED

From this theorem follows only those surface of revolution with $n_1 < n_2 < n_3$ could be represented using Eickemeyer's planes. This motivates the following definition.

Definition 4.1 *The 2-flat $_a$ π_a is the 2-dimensional vector subspace spanned by \mathbf{d} and $\mathbf{v}_a = (a, 1, 1)$, $a \in \mathbb{R}$.*

In what follows, we suppose $a > 0$.

Remark 4.1 *The 2-flats $_1$ are the Eickemeyer's 2-flats.*

Remark 4.2 *If a 1-flat $_a$ lies on π_a then the points $\bar{\pi}_{1,2}^a$, $\bar{\pi}_{2,3}^a$ (repectively $\bar{\pi}_{1,3}^a$, $\bar{\pi}_{2,3}^a$) defined in Remark 3.1 are coincident. Therefore, the representation of the 1-flat $_a$ is a single point in the a -parallel coordinate system.*

Theorem 4.2 *If $n_2 < n_3$ then exists a 2-flat $_a$, $a < 1$, that contains \mathbf{n} .*

Proof: We can suppose, without loss of generality, that $d_3 > 0$.

- If $0 < n_3$, results that

$$n_3(1-a) + n_1 - n_2 < 0,$$

and then follows

$$K = \frac{an_2 - n_1}{n_3(1-a) + n_1 - n_2} > 1.$$

Then it suffices to choose $d_2 < Kd_3$. So taking into account that

$$d_1 = \frac{an_3 - n_1}{n_3 - n_2} d_2 + \frac{n_1 - an_2}{n_3 - n_2} d_3, \quad (1)$$

it follows

$$\begin{aligned} d_2 - d_1 &= d_2 \left(1 - \frac{an_3 - n_1}{n_3 - n_2}\right) - \frac{n_1 - an_2}{n_3 - n_2} d_3 \\ &= \frac{n_3 - n_2 - an_3 + n_1}{n_3 - n_2} d_2 - \frac{n_1 - an_2}{n_3 - n_2} d_3 \\ &= \frac{n_3(1-a) + n_1 - n_2}{n_3 - n_2} d_2 - \frac{n_1 - an_2}{n_3 - n_2} d_3. \end{aligned} \quad (2)$$

From the last member of Equations (2) we have the following inequality

$$\begin{aligned} d_2 - d_1 &> \left(\frac{n_3(1-a) + n_1 - n_2}{n_3 - n_2} - \frac{n_1 - an_2}{n_3 - n_2}\right) d_3 \\ &> \frac{(n_3 - n_2)(1-a)}{n_3 - n_2} d_3 \\ &= (1-a) d_3 > 0, \end{aligned}$$

and then it is possible to conclude that $d_1 < d_2$.

- If $n_3 < 0$ then we have

$$-K = -\frac{an_2 - n_1}{n_3(1-a) + n_1 - n_2} > 1$$

So, for any $d_2 < d_3$ it is enough to choose

$$d_1 = \frac{an_3 - n_1}{n_3 - n_2} d_2 + \frac{n_1 - an_2}{n_3 - n_2} d_3,$$

and then follows that $d_1 < d_2$.

- If $n_3 = 0$, we have

$$d_1 = \frac{n_1}{n_2} d_2 + \frac{an_2 - n_1}{-n_2} d_3.$$

As $n_1 > n_2$, with $n_2 < 0$ results that

$$\frac{n_1}{n_2} < 1.$$

Then

$$K = \frac{an_2 - n_1}{n_1 - n_2} < 1,$$

and we can conclude that $d_1 < d_2$ for any selection of $d_2 < d_3$. QED

The geometric interpretation of the proof of Theorem 4.2 is as follows.

Consider in the plane two perpendicular axes, and call them d_1 and d_2 , and d_3 any real number. We can think $d_1 = \frac{an_3-n_1}{n_3-n_2} d_2 + \frac{n_1-an_2}{n_3-n_2} d_3$ as a line L in this system coordinate. The intersection between L and the line $d_1 = d_2$ is the point of coordinates: (Kd_3, Kd_3) , where $K = \frac{an_2-n_1}{n_3(1-a)+n_1-n_2}$.

Case 1. If the slope of L is less than one it must be $K < 1$, in order to exist $d_2 \in (Kd_3, d_3)$ that satisfies the given condition (see Figure 2).

Case 2. If the slope of L is greater than 1, it must be $K > 1$. In this case any $d_2 < d_3$ satisfies the given requirement, as can be seen from Figure 3.

Case 3. If $a < 1$ the cases slope of L greater than one and $K < 1$ or slope of L less than one and $K > 1$ are not possible as is shown in Figures 4 and 5.

If $a > 1$ the cases mentioned in the last paragraph are possible. Then, choosing $d_2 < Kd_3$ for Case 1, the vector \mathbf{d} is determined. This shows that Theorem 4.2 gives a necessary but not a sufficient condition, as follows from Example 4.2.

Example 4.2 If $a = 2$, $\mathbf{n} = (2, 1, 3)$, $\mathbf{d} = (-2, -1, 1)$ and $\mathbf{v}_2 = (2, 1, 1)$ then \mathbf{n} is contained in the 2-flat₂ generated by \mathbf{v}_2 and \mathbf{d} .

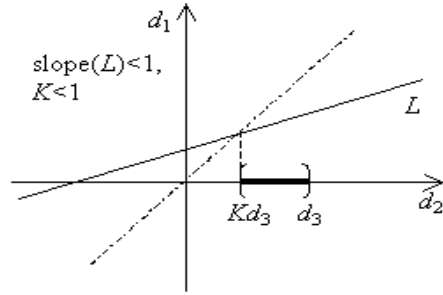


Figure 2: Case 1.

Theorem 4.3 If $n_3 > n_1 - n_2$ then exists a 2-flat_a, $a > 1$ that contains the vector \mathbf{n} .

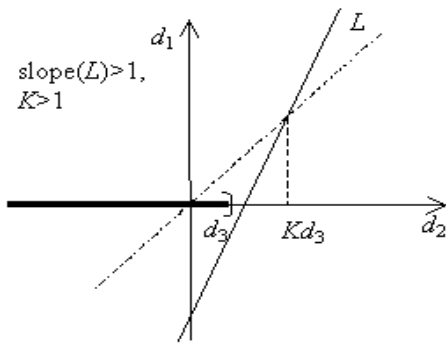


Figure 3: Case 2.

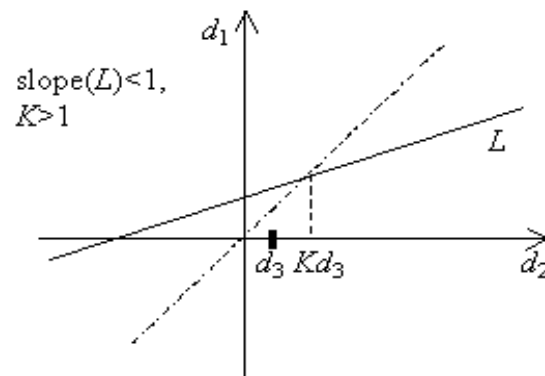


Figure 4: Case 3.

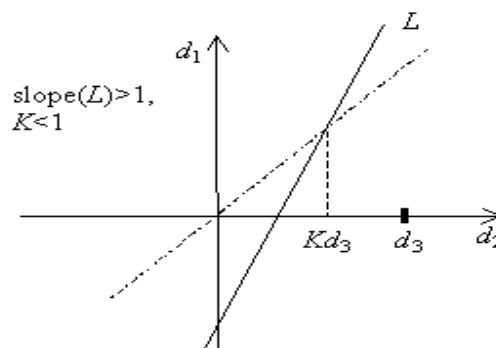


Figure 5: Case 3.

Proof: Let $a > 2$. Then

$$(a - 1)n_3 > n_3 > n_1 - n_2,$$

and it results

$$\frac{an_3 - n_1}{n_3 - n_2} > 1.$$

As $(a - 1)n_2 < (a - 1)n_3$, then

$$\frac{an_2 - n_1}{n_3(a - 1) + n_2 - n_1} < 1.$$

So it suffices to choose $d_2 \in (Kd_3, d_3)$. QED

Example 4.3 Let $\mathbf{n} = (-1, 3, 5)$. Then we can choose $a = 3$, $\mathbf{d} = (-d_3 - 2, -d_3 - 1, d_3)$, so that \mathbf{n} is contained in the 2-flat₃ generated by \mathbf{v}_3 y \mathbf{d} .

Let π_a be the plane generated by \mathbf{d} and \mathbf{v}_a and $\tilde{\pi}_a$ be the plane generated by $\tilde{\mathbf{d}}$ and \mathbf{v}_a . The curves C and \tilde{C} are found as the intersection of the surface with the planes π_a and $\tilde{\pi}_a$, respectively.

According to Section 3 and Remark 4.2 we represent, in the a -parallel coordinate system, each tangent line to the curve C and \tilde{C} , respectively. These sets, named C_1 and C_2 represent the curves C and \tilde{C} , respectively.

Now we can summarize the procedure as follows.

STEP 1. Choose $d_3 > 0$ and fix any $a > 1$.

STEP 2. Choose d_2 according to Theorem 4.2.

STEP 3. Set d_1 given by (1).

STEP 4. Let $\tilde{\mathbf{d}} = (d'_1, d_2, d_3)$ such that $d'_1 > d_3$.

STEP 5. Draw, in the a -parallel coordinate system, the curves C and \tilde{C} .

5 Example of Application

Let us consider the surface of revolution generated by the rotation of the the curve C in \mathbb{R}^3 given by:

$$\begin{cases} x = -5t, \\ y = 5t^2, \\ z = 8t^2 + t, \end{cases} \quad t > 0, \quad (3)$$

around the line of equation $\mathbf{P}(\lambda) = (-1, 3, 5)\lambda, \lambda \geq 0$. The curve \tilde{C} is given by equation:

$$\begin{cases} x = \cos \alpha, \\ y = \sin \alpha, \\ z = \cos \alpha - 2 \sin \alpha + 26. \end{cases} \quad \alpha \in [0, 2\pi), \quad (4)$$

Choosing $a = 3$ we have, according to Theorem 4.2, $\mathbf{d} = (-d_3, \frac{1}{2}d_3, d_3)$, with $d_3 > 0$ and $\tilde{\mathbf{d}} = (2d_3, \frac{1}{2}d_3, d_3)$. The curve C belongs to π_a , the plane generated by \mathbf{d} and $\mathbf{v}_a = (3, 1, 1)$ and the curve \tilde{C} belongs to $\tilde{\pi}_a$, the plane generated by $\tilde{\mathbf{d}}$ and \mathbf{v}_a .

The points that represent the curve C in the 3-parallel coordinate system are given by:

$$\pi_{12}^3 = \left(\frac{d_3(1-12t)}{2(6t+1)}, \frac{-5t^2}{(6t+1)} \right), \quad t > 0. \quad (5)$$

For the curve \tilde{C} these points are given by the following expression:

$$\tilde{\pi}_{12}^3 = \left(\frac{d_3(6\cos\alpha + \frac{1}{2}\sin\alpha)}{3\cos\alpha + \sin\alpha}, \frac{1}{3\cos\alpha + \sin\alpha} \right), \quad \alpha \in [0, 2\pi). \quad (6)$$

In Figure 6 are shown both C_1 and C_2 , the curves obtained when drawing the points given by equations (5) and (6), with $d_3 = 2$.

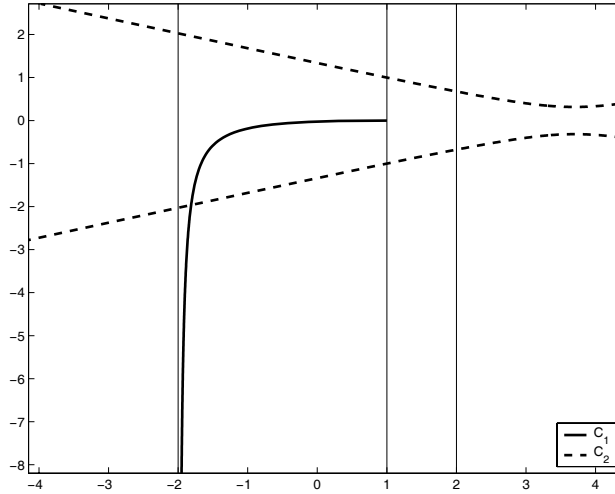


Figure 6: Representation of a surface of revolution in the 3-parallel coordinate system.

6 Conclusions

There are many tools that allow to visualize of 3D functions, but the generalization to functions of greater dimensions is not easy. In this paper we have presented a methodology

that generalizes the work of Hung and Inselberg for representing surfaces of revolution. To do this, it was necessary to define our so called a -parallel coordinate system. This new system allows for the representation of developable, ruled and revolution surfaces in 3D. Although the methodology is restricted to 3D surfaces, we are working on its extension to N -dimensional hypersurfaces of revolution. In this case, all the applications known so far for ruled and developable surfaces could be extended to a wider class of hypersurfaces.

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