Computing Power Indices in Weighted Majority Games

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Abstract

Power indices methodology of weighted majority game is widely used to measure a priori voting power of members of a committee. In this paper we present a computer implementation of the main power indices: Shapley-Shubik, Banzhaf, Johnston, Deegan-Packel and the Holler-Packel power indices. This computer implementation allows comparing the different indices. The system was developed for multiplataform: Linux and Windows. We compute the power indices for two examples and analyze its results.

Keywords: Weighted majority games - Power indices - Computational implementation - Multiplataform

Classification: WASI

1 Introduction

Democratic societies and international organizations use a wide variety of complex rules to reach decisions. Examples where it is not always easy to understand the consequences of the way voting is done include to elect the President or the Parliament of a country, the United Nations Security Council, the governance structure of the World Bank, the European Parliament, etc.

The analysis of power is central in these situations. In general, it is difficult to define the idea of power, but for the special case of voting situations several quantitative measures for evaluating the power of a voter or coalition have been proposed.

Power indices methodology is widely used to measure an a priori voting power of members of a committee. Different power indices have been proposed to assess the a priori distribution of power in voting situations. That is, the distribution of power among the voters for a given decision rule. The two more popular indices are: Shapley-Shubik [10] and Banzhaf [2] . Other power indices are: the Johnston [6], the Deegan-Packel [3] and the Holler-Packel [5] indices.

The results given by these indices may differ quite widely. The computation of these power indices is also complex in practice and requires the use of computational tools.

We will present a computational tool for calculating power indices. This is an extension to the article by Aguirre, Guerrero, Oviedo and Quintas [1]. We compute new indices (Johnston, Deegan-Packel and Holler-Packel). The system also allows comparing the results of different indices and presents statistic reports.

We compute the power indices for two examples and analyze its results.

2 Weighted Majority Games

Cooperative games are used to model interaction among agents or groups of agents when cooperation it possible. Within this theory, there are several solutions measuring the power of each agent.

A cooperative transferible utility (TU) game is given by: $G = (N, v)$, where $N = \{1, 2, \ldots, n\}$ is the set of all the players and v is the characteristic function. Any non empty subset of N is called a *coalition*. An arbitrary coalition is denoted by S, s representing its number of players. This function v is defined on the coalitions $S \subseteq N$ and measures the value or utility $v(S)$ that each coalition S has if it forms. Thus $v(S)$ is the utility that the members of S can obtain by themselves. The meaning of this function is that it measures how much a group could obtain working together.

Definition 1 *A cooperative transferible utility (TU) game is defined by:* $G = (N, v)$*, where* $N = \{1, 2, \ldots, n\}$ *is the set of all players and*
$v : \wp(N) = 2^N \longrightarrow \Re$ *is the characteristic function. This is a real function, defined on the subset of* N*, that fulfill the following properties:*

$$v(\phi) = 0 \tag{1}$$

$$v(\{i\}) \geqslant 0 \qquad \forall i \in N \tag{2}$$

$$v(S \cup T) \geqslant v(S) + v(T) \qquad \forall (S \cap T) = \phi \quad and \quad S, T \subseteq N \tag{3}$$

Condition (1) is only for consistency (the empty coalition has no-power). Condition (2) indicates that the security level of each player is cero.

Condition (3) is known as *Superadditivity Property*, and shows the incentives for the players in conforming bigger coalitions.

In many games (elections for instance) the results is 1 o 0 (winning or losing the elections) thus we have the following definition:

Definition 2 *A simple game is a cooperative game* $G = (N, v)$*, where* $N = \{1, 2, \ldots, n\}$ *is a finte set and* $v : \wp(N) = 2^N \longrightarrow \{0, 1\}$*, such that* $v(\phi) = 0$ *and* $v(S) \leq v(T)$ *whenever* $S \subseteq T \subseteq N$*.*

A coalition is *winning* if $v(S) = 1$, and *losing* if $v(S) = 0$. Let $W(v)$ denote the set of all winning coalitions and $m(v)$ its cardinality. A player i is a *swinger* in a winning coalition S if when removed from the coalition makes it losing, that is to say, if $v(S) = 1$ and $v(S - \{i\}) = 0$. A *minimal winning coalition* is a winning coalition in wich all players are swinger. Let $M(v)$ denote the set of all minimal winning coalitions and $m(v)$ the total number of minimal winning coalitions.

An interesting class of simple games is the class of *Weighted Majority Games* (WMG). It is a system in which voters can have different numbers of votes. The number of votes needed to pass a motion is called the *quota* q.

Definition 3 *A WMG is defined as a TU game* $G = (N, v)$*, where each player has a finite number of votes donoted by* w_i*, with* $w_i > 0$*,* $\forall i \in N$*.*

A WMG is determinated by the structure:

$$[q; w_1, w_2, ..., w_n]. \tag{4}$$

A coalition should obtain at least a quote q of the votes (depending of the type of majority that the decision requires). If we think in terms of the percentage of votes, q could be $\frac{1}{2}$, $\frac{2}{3}$, etc. We will deal with $\frac{1}{2}\sum_{i=1}^{n} w_i < q \leq \sum_{i=1}^{n} w_i$ in order to avoid having two winning coalitions with no empty intersection.

In a WMG we have:

$$v(S) = \begin{cases} 1 & \sum_{i \in S} w_i \geq q \\ 0 & otherwise \end{cases} \tag{5}$$

3 Power Indices

The power of players can be measured by power indices. A power index is a certain vector $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$, where ϕ_i is interpreted as a measure of the influence that player $i \quad (i = 1, 2, \ldots, n)$ can exert on the outcome.

In this study, the following power indices were analized: Shapley-Shubik [9], Banzhaf [2], Johntson [6], Deegan-Packel [3] and Holler-Packel [5].

3.1 Shapley-Shubik Index

Shapley [9] axiomatically characterizes the prospect of having to play a TU game, referred to as the *Shapley Value*. The Shapley-Shubik index (*SSPI*) is the application of this value to simple games [10].

The Shapley Value $\varphi(v) = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ gives an imputation, it means a way to distribute the total amount obtained by the total coalition N, among the players, giving each one at least the amount each player can obtain by himself and taking into account the average marginal contribution by being (or not) member of each coalition. It fullfils:

$$\sum_{i=1}^{n} \varphi_i = v(N) \quad y \quad \varphi_i \geqslant v(\{i\})$$

The Shapley Value is characterized as follows:

Theorem Shapley[9] Let $G = (N, v)$ be an TU game. The Shapley value is a $n - vector$ call $\varphi(v) = (\varphi_1(v), \varphi_2(v), ..., \varphi_n(v))$, such that:

$$\varphi_i(v) = \sum_{S \subseteq N - \{i\}} \frac{s!\,(n - s - 1)!}{n!} \left[v(S \cup \{i\}) - v(S) \right] \tag{6}$$

The Shapley value can be given by the following heuristic explanation. Suppose the players (the elements of N) agree to meet at a specified place and time. Naturally because of random fluctuations, all will arrive at different times; it is assumed, however, that all orders of arrival (permutations of the players) have the same probability : $\frac{1}{n!}$.

$$\left(\underbrace{\cdots\cdots}_{|S|!}, i, \underbrace{\cdots\cdots\cdots}_{(n-|S|-1)!} \right)$$

Suppose that, if a player, i, arrives, and finds the members of the coalition $S - \{i\}$ (and no others) already there, he receives the amount $v(S) - v(S - \{i\})$, i.e., the marginal amount which he contributes to the coalition, as payoff. Then the Shapley Value $\varphi_i(v)$ is the expected payoff to player i under this randomization scheme.

3.2 Banzhaf Index

A second index of power was introduced by Banzhaf [2], which is called the Banzhaf index (*BPI*). This index is based on counting for each player the number of coalitions to which it belongs and it is crucial to win.

The Banzhaf index is given by $\beta(v) = (\beta_1(v), \beta_2(v), \ldots, \beta_n(v))$, where

$$\beta_i(v) = \sum_{\substack{S \subseteq N \\ S \ni i}} \left(\frac{1}{2}\right)^{n-1} [v(S) - v(S - \{i\})] \tag{7}$$

The following normalization:

$$\tilde{\beta}_i(v) = \frac{\beta_i(v)}{\sum_{k \in N} \beta_k(v)} \tag{8}$$

gives the normalized Banzhaf index, that we denote $\tilde{\beta}(v)$.

3.3 Johnston Index

A measure of power should depend on the number of swingers in the coalition. Johnston [6] proposes a modification of the normalized Banzhaf index, which is referred as the Johnston index (*JPI*). In a simple game, let $\zeta(S)$ denote the number of swingers in a winning coalition S. We can then calculate $\gamma(v) = (\gamma_1(v), \gamma_2(v), \ldots, \gamma_n(v))$ where

$$\gamma_i(v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{1}{\zeta(S)} \quad [v(S) - v(S - \{i\})] \tag{9}$$

The summation is only done on the coalitions in which there is at least one swinger. The JPI, that we denote $\tilde{\gamma}(v)$, is obtained by the following normalization:

$$\tilde{\gamma}_i(v) = \frac{\gamma_i(v)}{\sum_{k \in N} \gamma_k(v)} \tag{10}$$

3.4 Deegan-Packel Index

Deegan and Packel [3] proposed an index from the assumption that all minimal winning coalitions have the same probability of forming, and the assumption that the swingers in a minimal winning coalition share equally the spoils.

In a simple game, the Deegan-Packel Power Index (*DPPI*) is given by $\delta(v) = (\delta 1(v), \delta 2(v), \ldots, \delta n(v))$ where

$$\delta_i(v) = \frac{1}{m(v)} \sum_{\substack{S \in M(v) \\ i \in S}} \frac{1}{s} [v(S) - v(S - \{i\})] \tag{11}$$

where $m(v)$ is the total number of minimal winning coalitions.

3.5 Holler-Packel Index

Holler and Packel [5] argued that the preceding indices face the problem of distributing the value of a priori coalitions among their members. There might be no adequate solution to this problem, because the coalition value is a collective good. The private good approach, as implied in the Holler-Packel, might be hence inappropriate if voting in not only a matter of allocating spoils. Holler and Packel[5] proposed a slight modification of the DPPI, which can be derived from the assumption that all minimal winning coalitions have the same probability of forming, and the assumption that all swingers in a minimal winning coalition get all the spoils.

In a simple game, the non-normalized Holler-Packel Power Index is given by $\sigma(v) = (\sigma 1(v), \sigma 2(v), \ldots, \sigma n(v))$ where

$$\sigma_i(v) = \frac{1}{m(v)} \sum_{S \in M(v); i \in S} [v(S) - v(S - i)] \tag{12}$$

The Holler-Packel Power Index (*HPPI*), that we denote $\tilde{\sigma}(v)$, is obtained by the following normalization:

$$\tilde{\sigma}_i(v) = \frac{\sigma_i(v)}{\sum_{k \in N} \sigma_k(v)} \tag{13}$$

In the following section we will introduce the computational implementation.

4 Computing Power Indices

This application uses a client-server paradigm [11], [1]. The server is coded in C and the client in Tcl/Tk [7], [8], [12].

Client software is a Tcl/Tk script that shows a graphical user interface, which can runs over Unix and Windows platforms [13], [14]. It contacts a server, sends a request, and awaits a response. When the response arrives, the client shows the results.

Server software which can run over Unix platform. It create a socket and binds the socket to the port at which it desires to receive requests. It then enters into an infinite loop in which it accepts the next request that arrives from a client, processes the indices and sends the reply back to the client.

When solving a problem, one approach is to divide the problem into smaller specific tasks. Given the nature of the sub-tasks and the sequence in which they be done, we can occasionally split them out to separated processes and, if needed, to make the processes communicate with each other.

UNIX provides system calls to create processes, and it is possible to use these facilities to write parallel programs. We would not get an increased execution speed on a single processor. Actually, the speed would reduce because of the overhead of creating the processes and handling context changes as we swap between processes. A much more efficient mechanism is one in which a concurrent routine is specified that shares the same memory space and global variables. This can be provided by a *thread* mechanism or *lightweight process*.

Each thread has its own execution stack, register set, program counter, thread-specific data, thread-local variables, thread-specific signal mask, and state information. In a multithreaded process each thread executes independently and asynchronously. Problems that consist of multiple individual tasks lend themselves to a threaded solution.

The reason to parallel server programs is to increase the number of computing resources to more efficiently process the indices (i.e. each player is processed in parallel).

The system allows comparing the results of the different implemented indices. It also gives a stadistic report of coalitions, the number of swingers and winning coalitions.

5 Aplication examples

We now present two examples where we use the computer program for calculating and comparing the above described power indices. The first one is a simple example of a Parliament with four parties. The second one is more complex and interesting because it models the actual composition of the European Parliament. It has 732 members gathered in eight multinational parties. The study of the indices shows the share of power each party has in the Parliament.

5.1 Example 1

We consider a Parliament with four parties, having the following voting structure:

1. Party 1 = 15 votes.
2. Party 2 = 13 votes.
3. Party 3 = 3 votes.
4. Party 4 = 1 votes.

The WMG is the following:

$$[17; 15, 13, 3, 1]. \tag{14}$$

Table 1 contains the power indices:

Player	Weight	% Votes	SSPI	BPI	JPI	DPPI	HPPI
Party 1	15	46,86%	0,416666	0,416666	0,5	0,333333	0,285714
Party 2	13	40,62%	0,25	0,25	0,222222	0,277777	0,285714
Party 3	3	9,38%	0,25	0,25	0,222222	0,277777	0,285714
Party 4	1	3,12%	0,083333	0,083333	0,055555	0,111111	0,142857
TOTAL	32	100%	1	1	1	1	1

Table 1:

We observe that some indices can give the same result (in this case SSPI and BPI) while others give a spread of outcomes. They usually gives a different outcome of the percentage of votes each party has. It allows different interpretations of the actual power of the groups.

Now we present a more complex study considering the actual composition of the European Parliament.

5.2 Example 2

The *European Parliament* (EP) is the parliamentary body of the *European Union* (EU), directly elected by EU citizens. Together with the Council of Ministers, it comprises the legislative branch of the institutions of the Union.

The EP [4] has 732 members. Every country has a specified number of seats in the EP.

Member state	**Seats**	**Member state**	**Seats**
Germany	99	Austria	18
France	78	Denmark	14
United Kingdom	78	Finland	14
Italy	78	Slovakia	14
Spain	54	Republic of Ireland	13
Poland	54	Lithuania	13
Netherlands	27	Latvia	9
Belgium	24	Slovenia	7
Czech Republic	24	Cyprus	6
Greece	24	Estonia	6
Hungary	24	Luxembourg	6
Portugal	24	Malta	5
Sweden	19	**Total**	**732**

Table 2: Composition of the European Parliament (June 2004)

However the seats are also allocated as party groups. These parties are supranational and form 8 politics groups. The seats are allocated every five years by elections in the countrys. Table 3 contains the results of most recent elections (June 2004) to the EP:

1. European Peopleś Party - European Democrats (EPP-ED) = 277 votes.
2. Party of European Socialists (PES) = 198 votes.
3. European Liberal Demcrat and Reform Party (ELDR) = 68 votes.
4. European Greens / European Free Alliance (Greens/EFA) = 40 vote.
5. European United Left / Nordic Green Left (EUL/NGL) = 39 votes.
6. Union for a Europe of Nations (UEN) = 27 votes.
7. Europe of Democracies and Diversities (EDD) = 15 votes.
8. Non inscrits (NI)

The WMG is the following:

$$[367; 277, 198, 68, 40, 39, 27, 15, 68]. \tag{15}$$

Table 3 contains the power indices of the EP considering these parties.

Again SSPI and BPI give similar values. JPI gives much power to the largest group . On the other side, DPPI and HPPI distribute the power almost homogenously among all groups.

Player	Weight	% Votes	SSPI	BPI	JPI	DPPI	HPPI
EPP-ED	277	38%	0.426191	0.427273	0.621622	0.222222	0.185185
PES	198	27%	0.178571	0.154545	0.129129	0.106667	0.111111
ELDR	68	9%	0.111905	0.118182	0.077477	0.148889	0.148148
GREENS/EFA	40	5%	0.059524	0.063636	0.033483	0.121111	0.129630
EUL/NGL	39	5%	0.059524	0.063636	0.033483	0.121111	0.129630
UEN	27	4%	0.045238	0.045455	0.023273	0.101111	0.111111
EDD	15	2%	0.007143	0.009091	0.004054	0.030000	0.037037
NIL	68	9%	0.111905	0.118182	0.077477	0.148889	0.148148
TOTAL	732	100%	1	1	1	1	1

Table 3: Power indices of European Parliament

6 Conclusions

The new computational system is a much powerful tool than the system implemented in [1].

The system allows comparing the different indices. It is oriented to WMG making its use very simple. It also allows drawing statistic reports of number of winning coalitions and swings.

The system have been also parallelized by using thread, for an optimal performance. Moreover, the client software is multiplatform. The users can choose the best available platform instead of being constrained only to the availability of a specific platform.

Thus we have obtained a support to game theory researchers, as well as to teachers and students. It has been shown to be particularly useful in complex applications of Cooperative Game Theory.

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