Consequence Operators for Defeasible Argumentation: characterization and logical properties*

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Abstract. Artificial Intelligence (AI) has long dealt with the issue of finding a suitable formalization for commonsense reasoning. Defeasible argumentation has proven to be a successful approach in many respects, proving to be a confluence point for many alternative logical frameworks. Different formalisms have been developed, most of them sharing the common notions of *argument* and *warrant*.
In defeasible argumentation, an *argument* is a tentative (defeasible) proof for reaching a conclusion. An argument is *warranted* when it ultimately prevails over other conflicting arguments. In this context, defeasible consequence relationships for modeling argument and warrant as well as their logical properties have gained particular attention.
This paper discusses two consequence operators for the LDS_{ar} framework for defeasible argumentation. The operators are intended for modeling argument construction and dialectical analysis (warrant), respectively. Their associated logical properties are studied and contrasted with SLD-based Horn logic. We contend that this analysis provides useful comparison criteria that can be extended and applied to other argumentation frameworks.

Key words: defeasible argumentation; knowledge representation; non-monotonic inference; labeled deduction.

1 Introduction and motivations

Artificial Intelligence (AI) has long dealt with the issue of finding a suitable formalization for commonsense reasoning. Defeasible argumentation has proven to be a successful approach in many respects, proving to be a confluence point for many alternative logical frameworks. Different formalisms have been developed, most of them sharing the common notions of *argument* and *warrant*. In defeasible argumentation, an *argument* is a tentative (defeasible) proof for reaching a conclusion. An argument is *warranted* when it ultimately prevails over other conflicting arguments. In this context, defeasible *consequence relationships* for modeling argument and warrant as well as their logical properties have gained particular attention.

The study of logical properties of defeasible argumentation motivated the development of LDS_{ar} [Che01], an argumentation formalism based on the *labeled deduction* methodology [Gab96]. In labeled deduction, the usual notion of formula is replaced by the notion of *labeled formula*, expressed as *Label:f*, where *Label* represents a label associated with the wff *f*. A labeling language $\mathcal{L}_{Label}$ and knowledge-representation language $\mathcal{L}_{kr}$ can be combined to provide an enriched object language, in which labels convey additional information also encoded at object-language level.

This paper introduces two consequence operators C_{arg} and C_{war}, used for argument construction and warrant computation, respectively. These operators were defined within the

* The paper summarizes some of the main results of the first author's PhD Thesis [Che01] written under the direction of Guillermo Simari. An electronic version of the Thesis is available at http:\\cs.uns.edu.ar\ ~cic

LDS_{ar} framework [Che01]. Logical properties that characterize the behavior of these operators are discussed and contrasted with those present in a logic programming setting.

The paper is structured as follows: first, in section 2 we present an overview of the basic notions concerning consequence operators, non-monotonic inference and their properties. Section 3 introduces the definitions of the operators C_{arg} and C_{war}, based on the inference relationships provided by LDS_{ar} to build arguments (defeasible proofs) and compute warrant by performing a dialectical analysis.[1] In sections 4 and 5 different logical properties of these operators are presented and analyzed. Section 6 concludes and summarizes the role of these properties in the context of the proposed formalism, discussing also some related work.

2 Non-monotonic Inference Relationships: fundamentals[2]

In classical logic, inference rules allow us to determine whether a given wff γ follows via "$\vdash$" from a set Γ of wffs. In classical logic the "$\vdash$" relationship is a *consequence relationship* (satisfying idempotence, *cut* and monotonicity). As non-monotonic and defeasible logics evolved into a valid alternative to formalize commonsense reasoning, a similar concept was needed to capture consequence without demanding some of these requirements (*e.g.* monotonicity). This led to the definition of a more generic notion of inference, namely *inference relationships*. New properties were defined and gained interest in this setting. In this section we will introduce the definition of (non-monotonic) inference relationship, as well the definitions some distinguished properties that characterize them.

2.1 Inference relationship. Pure logical properties

Definition 2.1 (Inference relationship $\mid\!\sim$. Inference operator $C(\Gamma)$). *Let Γ be a set of wffs in a language $\mathcal{L}$ and let γ be a wff in Γ. We will write $\Gamma \mid\!\sim \gamma$ if γ is a (non-monotonic) consequence of Γ. We define $C(\Gamma) = \{\gamma \mid \Gamma \mid\!\sim \gamma\}$.*

Given an inference relationship "$\mid\!\sim$ " and a set Γ of sentences, the following are called *basic* (or *pure*) properties associated with *any* inference operator $C(\Gamma)$:

1. **Inclusion**: $\Gamma \subseteq C(\Gamma)$
2. **Idempotence**: $C(\Gamma) \subseteq C(C(\Gamma))$
3. **Cut**: $\Gamma \subseteq \Phi \subseteq C(\Gamma)$ implies $C(\Phi) \subseteq C(\Gamma)$
4. **Cautious monotonicity**: $\Gamma \subseteq \Phi \subseteq C(\Gamma)$ implies $C(\Gamma) \subseteq C(\Phi)$.
5. **Cummulativity**: $\Gamma \subseteq \Phi \subseteq C(\Gamma)$ implies $C(\Gamma) = C(\Phi)$.
6. **Monotonicity**: $\Gamma \subseteq \Phi$ implies $C(\Gamma) \subseteq C(\Phi)$

The intuitive meaning of inclusion and idempotence should be clear without further comments. The *cut* rule states that expanding the information in Γ by adding new propositions from $C(\Gamma)$ does not result in new conclusions being obtained. Cautious monotonicity constitutes somehow the 'inverse' of *cut*: adding new lemmas does not decrease inference power, *i.e.* the set of conclusions that can be obtained from a given theory. Combining *cut* and cautious monotonicity we get *cummulativity*, which states that intermediate proofs (lemmas) can be

[1] In order to make the paper self-contained, an appendix is included with the main definitions that characterize argument construction and warrant in the LDS_{ar} framework.

[2] This section is based on the excellent overview on consequence and inference relationships given in [Ant96].

used as part of other (more complex) proofs without affecting the soundness of their conclusions.

These properties are called *pure*, since they can be applied to any language $\mathcal{L}$, and are abstractly defined for an arbitrary inference relationship "$\mid\!\sim$". Nevertheless, other properties which link classical inference with an arbitrary inference relationship can be stated. These properties will be discussed next. In what follows we will assume that Th stands for an operator that characterizes classical inference, whereas C corresponds to some (non-monotonic) inference relationship "$\mid\!\sim$".

2.2 Horn and non-Horn logical properties

A common name for cataloging non-pure properties is the distinction between *Horn properties* and *non-Horn properties*. Horn properties have the form *"from the presence of some particular inferences, the presence of some other inferences can be assured"*. Non-Horn properties, on the other hand, have the form *"from the absence of some particular inferences, the absence of some other inferences can be assured"*. Next we summarize the most important non-pure properties:[3]

Horn properties

1. **Supraclassicality**: $Th(A) \subseteq C(A)$
2. **Left logical equivalence**: $Th(A) = Th(B)$ implies $C(A) = C(B)$
3. **Right weakening**: If $x \supset y \in Th(A)$ and $x \in C(A)$ then $y \in C(A)$.[4]
4. **Conjunction of conclusions**: If $x \in C(A)$ and $y \in C(A)$ then $x \wedge y \in C(A)$.
5. **Subclassical cummulativity**: If $A \subseteq B \subseteq Th(A)$ then $C(A) = C(B)$.
6. **Left absorption**: $Th(C(\Gamma)) = C(\Gamma)$.
7. **Right absorption**: $C(Th(\Gamma)) = C(\Gamma)$.

Non-Horn properties

1. **Rationality of negation**: if $A \mid\!\sim z$ then either $A \cup \{x\} \mid\!\sim z$ or $A \cup \{\neg x\} \mid\!\sim z$.
2. **Disjunctive rationality**: if $A \cup \{x \vee y\} \mid\!\sim z$ then $A \cup \{x\} \mid\!\sim z$ or $A \cup \{y\} \mid\!\sim z$.
3. **Rational monotonicity**: if $A \mid\!\sim z$ then either $A \cup \{x\} \mid\!\sim z$ or $A \mid\!\sim \neg x$.

3 Capturing argument construction and warrant in LDS_{ar}

LDS_{ar} provides a unified logical framework based on labeled deduction for modeling *defeasible argumentation*. In defeasible argumentation, an *argument* is a tentative (defeasible) proof for reaching a conclusion. An argument is *warranted* when it ultimately prevails over other conflicting arguments. In what follows, we will consider these concepts in the context of the LDS_{ar} framework. The reader is referred to the appendix A, where basic concepts and definitions of LDS_{ar} are summarized in order to make this paper self-contained.[5]

When defining consequence operators in LDS_{ar}, there is an important difference to take into account: LDS_{ar} is based on an *extension* of logic programming, where literals can be

[3] An in-depth discussion of these properties can be found in [Ant96].

[4] It should be noted that "$\supset$" stands for material implication, to be distinguished from the symbol " $\leftarrow$ " used in a logic programming setting.

[5] A complete description can be found in [Che01], Chapter 3.

preceded by strong negation, and some pieces of information can be distinguished (labeled) as 'defeasible'. Clearly the notion of *theorem* in a logic programming setting is more restricted than the one used in classical logic. This leads us to consider a *specialized* consequence operator for Horn-like logics. Formally:

Definition 3.1 (Consequence Operator $\boldsymbol{Th}_{sld}(\Gamma)$). *Given an argumentative theory Γ, we define* $Th_{sld}(\Gamma) = \{[\emptyset, \{\mathsf{n_i}\}]{:}h \mid \Gamma \mathrel{|\!\!\sim}_{Arg} [\emptyset, \{\mathsf{n_i}\}]{:}h\}$

According to definition 3.1, 'classical' consequences from an argumentative theory Γ will be arguments (defeasible proofs) whose support set is empty (*i.e.*, they do not rely on any defeasible information). We want to compare this notion of derivation with respect to inference relationships $\mathrel{|\!\!\sim}_{Arg}$ and $\mathrel{|\!\!\sim}_{\mathcal{T}}$. We will define two specialized consequence operators for those arguments that can be derived from Γ

Definition 3.2 (Consequence operators $\boldsymbol{C}_{arg}(\Gamma)$ and $\boldsymbol{C}_{war}(\Gamma)$). *Given an argumentative theory Γ, we will define two consequence operators $C_{arg}(\Gamma)$ and $C_{war}(\Gamma)$ as follows:*

$$C_{arg}(\Gamma) = \{\mathcal{A}{:}\alpha \mid \Gamma \mathrel{|\!\!\sim}_{Arg} \mathcal{A}{:}\alpha \text{ and } \alpha \text{ is a literal in } \mathcal{L}_{\mathsf{KR}}\}$$
$$C_{war}(\Gamma) = \{[\emptyset, \{\mathsf{n_i}\}]{:}h \mid \Gamma \mathrel{|\!\!\sim}_{\mathcal{T}} h^U\}$$

Operators $C_{arg}(\Gamma)$ and $C_{war}(\Gamma)$ can be associated with the process of argument construction and warrant computation, respectively. In what follows we will analyze the different logical properties discussed in section 2 in the context of LDS_{ar}.

4 Logical properties of $\boldsymbol{C}_{arg}$

Inclusion does not hold in general for C_{arg}, since not every piece of defeasible information can be used as part of an argument. However, it does hold for (non-defeasible) facts. Therefore we refer to it as *restricted* inclusion.

Proposition 4.1 (Restricted inclusion). *The operator $C_{arg}(\Gamma)$ only satisfies inclusion wrt the non-defeasible information present in Γ.*

Proof. Let Γ be an argumentative theory, and let $[\emptyset, \{\mathsf{n_i}\}]{:}\alpha$ be a non-defeasible formula in Γ. Clearly, from rule Intro-NR we can derive $[\emptyset, \{\alpha\}]{:}\mathsf{n_i}$ from Γ, *i.e.* $\Gamma \mathrel{|\!\!\sim}_{Arg} [\emptyset, \{\mathsf{n_i}\}]{:}\alpha$. Then if $[\emptyset, \{\mathsf{n_i}\}]{:}\alpha \in \Gamma$ then $[\emptyset, \{\mathsf{n_i}\}]{:}\alpha \in C_{arg}(\Gamma)$.

The logical properties of idempotence and monotonicity do not hold. Formally:

Proposition 4.2 (Idempotence). *The operator $C_{arg}(\Gamma)$ does not satisfy idempotence, i.e.* $C_{arg}(\Gamma) \not\subseteq C_{arg}(C_{arg}(\Gamma))$

Proof. A counterexample suffices. Consider $\Gamma = \{ [\emptyset, \{\mathsf{n_1}\}]{:}\mathrm{q}, [\{p \leftarrow q\}, \{\mathsf{d_1}\}]{:}p \leftarrow q \}$. Then $C_{arg}(\Gamma) = \{\mathcal{A}_1{:}q, \mathcal{A}_2{:}q\}$, with $\mathcal{A}_1 = \emptyset$ and $\mathcal{A}_2 = \{p \leftarrow q\}$ resp. If we consider $C_{arg}(C_{arg}(\Gamma))$, it follows that $\mathcal{A}_1{:}q \in C_{arg}(C_{arg}(\Gamma))$ but $\mathcal{A}_2{:}p \notin C_{arg}(C_{arg}(\Gamma))$, since $\mathrel{|\!\!\sim}_{Arg}$ does not allow to introduce empty arguments.

Proposition 4.3 (Monotonicity). *The operator $C_{arg}(\Gamma)$ does not satisfy monotonicity.*

Proof. A counterexample suffices. Consider the argumentative theory $\Gamma = \{ [\emptyset, \{\mathsf{n}_1\}]{:}q \ , \ [\{p \leftarrow q\}, \{\mathsf{d}_1\}]{:}p \leftarrow q \ \}$. Clearly $\Gamma \mid\!\sim_{Arg} \mathcal{A}{:}p$, with $\mathcal{A} = \{p \leftarrow q\}$. But $\Gamma' = \Gamma \cup \{[\emptyset, \{\mathsf{n}_1\}]{:}{\sim}p\}$ is such that $\Gamma \not\mid\!\sim_{Arg} \mathcal{A}{:}p$.

Semi-monotonicity is an interesting property suggested by D.Makinson [MS91] for analyzing non-monotonic consequence relationships. It is satisfied if all defeasible consequences from a given theory are preserved when the theory is augmented with new *defeasible* information. Next we show that semi-monotonicity holds for $\mid\!\sim_{Arg}$.

Proposition 4.4 (Semi-monotonicity). *The operator $C_{arg}(\Gamma)$ satisfy semi-monotonicity, i.e. $C_{arg}(\Gamma) \subseteq C_{arg}(\Gamma \cup \Gamma')$, where Γ' is a theory involving only defeasible information.*

Proof. Proof is direct from the structure of the inference rules. Assume $\Gamma \mid\!\sim_{Arg} \mathcal{A}{:}\alpha$, and consider $\Gamma \cup \Gamma'$ as stated above. Clearly, the sequence of steps in the proof $\Gamma \cup \Gamma' \mid\!\sim_{Arg} \mathcal{A}{:}\alpha$ is still valid, since all preconditions in inference rules are defined wrt $\Pi(\Gamma) = \Pi(\Gamma \cup \Gamma')$. Hence $\Gamma \cup \Gamma' \mid\!\sim_{Arg} \mathcal{A}{:}\alpha$.

Cummulativity holds for argument construction. The importance of this property will be discussed later in section 6.

Lemma 4.5 (Cummulativity). [6] *Let Γ be an argumentative theory, and let α_1 and α_2 be wffs in ProgClauses($\mathcal{L}_{KR}$). Then $\Gamma \mid\!\sim_{Arg} \mathcal{A}_1{:}\alpha_1$ implies that*

$$\Gamma \ \cup \ \{\mathcal{A}_1{:}\alpha_1\} \mid\!\sim_{Arg} \mathcal{A}_2{:}\alpha_2 \quad \textit{iff} \quad \Gamma \mid\!\sim_{Arg} \mathcal{A}_2{:}\alpha_2$$

Proposition 4.6 (Horn supraclassicality). *The operator $C_{arg}(\Gamma)$ satisfies Horn supraclassicality wrt Th_{sld}, i.e. $Th_{sld}(\Gamma) \subseteq C_{arg}(\Gamma)$.*

Proof. Proof is direct from the definition 3.1. Every member in $Th_{sld}(\Gamma)$ is an empty argument, and as such it is a member in $C_{arg}(\Gamma)$. Therefore $Th_{sld}(\Gamma) \subseteq C_{arg}(\Gamma)$.

Proposition 4.7 (Left-logical equivalence). *The operator $C_{arg}(\Gamma)$ does not satisfy left-logical equivalence (i.e., given two theories Γ and Γ', if $Th_{sld}(\Gamma) = Th_{sld}(\Gamma')$ this does not imply that $C_{arg}(\Gamma) = C_{arg}(\Gamma')$.*

Proof. A counterexample suffices. Consider the theories $\Gamma_1 = \{ \ [\emptyset, \{\mathsf{n}_1\}]{:}\mathrm{p} \ , \ [\{q \leftarrow p\}, \{\mathsf{d}_1\}]{:}q \leftarrow p \ \}$ and $\Gamma_2 = \{ \ [\emptyset, \{\mathsf{n}_3\}]{:}\mathrm{p} \ \}$. Clearly, $Th_{sld}(\Gamma_1) = Th_{sld}(\Gamma_2)$. However, $C_{arg}(\Gamma_1) \neq C_{arg}(\Gamma_2)$, since there is an argument for q in Γ_1 but not in Γ_2.

Note that the property of right weakening cannot be considered (in a strict sense) in LDS_{ar}, since the deductive system associated with Th_{sld} does not allow the application of the deduction theorem. Therefore, wffs of the form "$x \leftarrow y$" cannot be derived via $\vdash_{\text{SLD}}$. However, an alternative approach can be intended, introducing a new property in which right weakening is restricted to Horn-like clauses:

Proposition 4.8 (Horn Right Weakening). *The operator $C_{arg}(\Gamma)$ satisfies Horn right weakening, i.e. if $\mathcal{A}{:}y \in C_{arg}(\Gamma)$ and $[\emptyset, \{\mathsf{n}_i\}]{:}x \leftarrow y \in Th_{sld}(\Gamma)$, then $\mathcal{A}'{:}x \in C_{arg}(\Gamma)$.*

[6] Proof not included for space reasons.

Proof. Suppose $\mathcal{A}{:}y \in C_{arg}(\Gamma)$. Clearly, $\Pi(\Gamma) \cup \{x \ \leftarrow \ y\} \not\vdash_{\text{SLD}} \bot$ (otherwise it would not have been possible to infer $\mathcal{A}{:}y$). Then applying rule Elim-$\leftarrow$ it holds that $\mathcal{A}{:}x$ can be derived from Γ, or equivalently $\mathcal{A}{:}x \in C_{arg}(\Gamma)$.

Proposition 4.9 (Conjunction of conclusions). [7] *The operator $C_{arg}(\Gamma)$ does not satisfy conjunction of conclusions, i.e. if $x \in C_{arg}(\Gamma)$ and $y \in C_{arg}(\Gamma)$, then it does not hold that $x \wedge y \in C_{arg}(\Gamma)$.*

Proof. A counterexample suffices. Consider the following theory $\Gamma = \{ [\{p \leftarrow q\}, \{\mathsf{d}_1\}]{:}p \leftarrow q \ , \ [\{r \leftarrow z\}, \{\mathsf{d}_2\}]{:}r \leftarrow z \ , \ [\emptyset, \{\mathsf{n}_1\}]{:}\mathrm{q} \ , \ [\emptyset, \{\mathsf{n}_2\}]{:}\mathrm{z} \ , \ [\emptyset, \{\mathsf{n}_3\}]{:}\mathrm{w} \ , \ [\emptyset, \{\mathsf{n}_4\}]{:}{\sim}w \leftarrow p, r \ \}$. Then there exists an argument $\mathcal{A}_1{:}p$ with $\mathcal{A}_1 = \{ p \leftarrow q \}$, and an argument $\mathcal{A}_2{:}r$ with $\mathcal{A}_2 = \{ r \leftarrow z \}$. However, the formula $\mathcal{A}_3{:}p, r$ cannot be derived (nor any other with conclusion p, r) since $\Pi(\Gamma) \cup \{p, r\} \vdash_{\text{SLD}} \bot$.

Proposition 4.10 (Subclassical cummulativity). *The operator $C_{arg}(\Gamma)$ satisfies subclassical cummulativity, i.e. $\Gamma \subseteq \Gamma' \subseteq Th_{sld}(\Gamma)$ implies $C_{arg}(\Gamma) = C_{arg}(\Gamma')$*

Proof. If Γ is an argumentative theory, then $Th_{sld}(\Gamma)$ involves only *facts* (*i.e.* literals that can be derived via SLD from $\Pi(\Gamma)$). By assumption $\Gamma \subseteq Th_{sld}(\Gamma)$. Then Γ involves only facts. But if Γ involves only facts, then every wff in Γ follows trivially via SLD from $\Pi(\Gamma)$. Therefore $\Gamma = Th_{sld}(\Gamma)$, and by assumption it holds that $\Gamma' = \Gamma$. Then it holds that $C_{arg}(\Gamma) = C_{arg}(\Gamma')$, as we wanted to proof.

Proposition 4.11 (Left absorption). *The operator $C_{arg}(\Gamma)$ does not satisfy left absorption, i.e. $Th_{sld}(C_{arg}(\Gamma)) \neq C_{arg}(\Gamma)$.*

Proof. $Th_{sld}(C_{arg}(\Gamma))$ involves only empty arguments that can be derived from $C_{arg}(\Gamma)$. But there can exists non-empty arguments in $C_{arg}(\Gamma)$. Therefore $Th_{sld}(C_{arg}(\Gamma)) \neq C_{arg}(\Gamma)$.

Proposition 4.12 (Right absorption). *The operator $C_{arg}(\Gamma)$ does not satisfy right absorption, i.e. $C_{arg}(Th_{sld}(\Gamma)) \neq C_{arg}(\Gamma)$.*

Proof. As shown in proposition 4.7, when computing $Th_{sld}(\Gamma)$ all defeasible information that might appear in Γ is lost. However, this information could appear in arguments in $C_{arg}(\Gamma)$. Then it holds that $C_{arg}(Th_{sld}(\Gamma)) = C_{arg}(\Gamma)$.

Proposition 4.13 (Rational negation). *The operator $C(\Gamma)$ does not satisfy rational negation.*

Proof. A counterexample suffices. Consider the following theory $\Gamma = \{ \ [\emptyset, \{\mathsf{n}_1\}]{:}{\sim}p \leftarrow x \ , \ [\emptyset, \{\mathsf{n}_2\}]{:}{\sim}p \leftarrow {\sim}x \ , \ [\emptyset, \{\mathsf{n}_3\}]{:}r \ , \ [\{z \leftarrow p\}, \{\mathsf{d}_1\}]{:}z \leftarrow p \ , \ [\{p \leftarrow r\}, \{\mathsf{d}_2\}]{:}p \leftarrow r \ \}$. Then $\Gamma \mid\!\sim_{Arg} \mathcal{A}{:}z$, with $\mathcal{A} = \{z \leftarrow p, \, p \leftarrow r \ \}$. However, $\Gamma \cup \{x\} \not\mid\!\sim_{Arg} \mathcal{A}{:}z$, and $\Gamma \cup \{{\sim}x\} \not\mid\!\sim_{Arg} \mathcal{A}{:}z$ (since in both cases the use of d_2 is not valid because of consistency constraints).

Proposition 4.14 (Rational Monotonicity). *The operator $C(\Gamma)$ does not satisfy rational monotonicity.*

Proof. Consider example 4, where $\Gamma \mid\!\sim_{Arg} \mathcal{A}{:}z$, with $\mathcal{A} = \{z \leftarrow p, \, p \leftarrow r \ \}$. If we consider now $\Gamma \cup \ \{x\}$ it holds that $\Gamma \cup \ \{x\} \not\mid\!\sim_{Arg} \mathcal{A}{:}z$, and $\Gamma \not\mid\!\sim_{Arg} {\sim}x$. Therefore rational monotonicity is not satisfied.

Clearly, the operator C_{arg} does not satisfy disjunctive rationality either, since disjunctions cannot be expressed as formulas in $\mathcal{L}_{\mathsf{KR}}$.

[7] Conjunction of conclusions in an argument is not possible in a strict sense (since conclusions are restricted to literals). However, we consider the general case as it is allowed by the inference rules in $\mid\!\sim_{Arg}$.

5 Logical properties of C_{war}

Next we will analyze some relevant logical properties of C_{war}.

Proposition 5.1 (Restricted inclusion). *The operator $C_{war}(\Gamma)$ only satisfies inclusion wrt (non-defeasible) facts in Γ.*

Proof. Let Γ be an argumentative theory, and let $[\emptyset, \{\mathsf{n_i}\}]{:}p$ be a non-defeasible fact in Γ. Clearly, from rule Intro-NR it follows that $\Gamma \mid\!\sim_{Arg} [\emptyset, \{\mathsf{n_i}\}]{:}p$. The formula $[\emptyset, \{\mathsf{n_i}\}]{:}p$ provides an argument for p that has no defeaters. Therefore p is a warranted literal, or equivalently $p \in C_{war}(\Gamma)$.

Proposition 5.2 (Idempotence). *The operator $C_{war}(\Gamma)$ satisfies idempotence, i.e. $C_{war}(\Gamma) \subseteq C_{war}(C_{war}(\Gamma))$*

Proof. If we consider $C_{war}(\Gamma)$ as a set of non-defeasible facts, it is clear that all they are warranted wrt $C_{war}(\Gamma)$. Therefore $C_{war}(\Gamma)= C_{war}(C_{war}(\Gamma))$, and in particular, $C_{war}(\Gamma) \subseteq C_{war}(C_{war}(\Gamma))$.

Proposition 5.3 (Monotonicity). *The operator $C_{war}(\Gamma)$ does not satisfy monotonicity.*

Proof. A counterexample suffices. Consider the example given in proposition 4.3. In that case, $\Gamma \mid\!\sim_{\mathcal{T}} p^U$ hence p is warranted. However, in $\Gamma \cup \{[\emptyset, \{\mathsf{n_1}\}]{:}q\}$ there is no argument with conclusion p (and consequently p is not warranted).

In contrast with $\mid\!\sim_{Arg}$, semi-monotonicity does not hold for $\mid\!\sim_{\mathcal{T}}$. The reason is the following: adding new defeasible information cannot invalidate existing arguments, but it can enable building new arguments that were not derivable before. Hence, dialectical relationships among arguments are different. Arguments that were warranted may therefore no longer keep that status. Formally:

Proposition 5.4 (Semi-monotonicity). *The operator $C_{war}(\Gamma)$ does not satisfy semi-monotonicity, i.e. $C_{war}(\Gamma) \not\subseteq C_{war}(\Gamma \cup \Gamma')$, where Γ' is an argumentative theory which involves only defeasible information.*

Proof. Consider the following counterexample. Given the theory $\Gamma = \{ [\emptyset, \{\mathsf{n_1}\}]{:}\mathrm{q}, [\{p \leftarrow r\}, \{\mathsf{d_1}\}]{:}p \leftarrow r, [\{r \leftarrow q\}, \{\mathsf{d_1}\}]{:}r \leftarrow q \}$, it is clear that $p \in C_{war}(\Gamma)$, since there exists an argument $\mathcal{A}{:}p$ with $\mathcal{A}=\{p \leftarrow r, r \leftarrow q\}$, such that $\mathcal{A}{:}p$ has no defeaters (and hence p is warranted).

Consider now $\Gamma'=\Gamma \cup \{[\{\sim p \leftarrow q\}, \{\mathsf{d_2}\}]{:}\sim p \leftarrow q\}$. In this case, an argument $\mathcal{B}{:}\sim p$ could be obtained, with $\mathcal{B}=\{\sim p \leftarrow q\}$, such that $\mathcal{B}{:}\sim p$ defeats $\mathcal{A}{:}p$. Therefore $p \notin C_{war}(\Gamma')$.

Proposition 5.5 (Cummulativity). *The relationship $\mid\!\sim_{\mathcal{T}}$ does not satisfy cummulativity.*

Proof. Consider the theory $\Gamma = \{ [\emptyset, \{\mathsf{n_1}\}]{:}\sim s \leftarrow q, [\emptyset, \{\mathsf{n_2}\}]{:}\mathrm{p}, [\{q \leftarrow p\}, \{\mathsf{d_1}\}]{:}q \leftarrow p, [\{s \leftarrow p\}, \{\mathsf{d_2}\}]{:}s \leftarrow p, [\{\sim q \leftarrow s\}, \{\mathsf{d_3}\}]{:}\sim q \leftarrow s \}$. Consider the arguments $\mathcal{A}{:}q$ with $\mathcal{A}=\{ q \leftarrow p \}$, and $\mathcal{B}{:}\sim q$ with $\mathcal{B}=\{ \sim q \leftarrow s, s \leftarrow p \}$, based on Γ. Note that $\Gamma \mid\!\sim_{\mathcal{T}} s^U$, *i.e.* s is a warranted literal in Γ. Consider now $\Gamma' = \Gamma \cup \{[\emptyset, \{\mathsf{n_3}\}]{:}\mathrm{s}\}$. Note that $\Gamma' \mid\!\sim_{\mathcal{T}} \sim q^U$ (*i.e.*, $\sim q$ is warranted). But $\Gamma \not\mid\!\sim_{\mathcal{T}} \sim q^U$ (since $\mathcal{A}{:}q$ defeats $\mathcal{B}{:}\sim q$).

Proposition 5.6 (Horn Supraclassicality). *The operator $C_{war}(\Gamma)$ satisfies Horn supraclassicality wrt Th_{sld}, i.e. $Th_{sld}(\Gamma) \subseteq C_{war}(\Gamma)$.*

Proof. All literals in $Th_{sld}(\Gamma)$ have arguments with no defeaters. Therefore they are warranted, and hence they are members of $C_{war}(\Gamma)$.

Proposition 5.7 (Left-logical equivalence). *The operator $C_{war}(\Gamma)$ does not satisfy left-logical equivalence, i.e. given two theories Γ and Γ', such that $Th_{sld}(\Gamma) = Th_{sld}(\Gamma')$ then it does not follow that $C_{war}(\Gamma) = C_{war}(\Gamma')$.*

Proof. Consider the counterexample given in proposition 5.7.

Note that right weakening cannot be considered wrt $\mid\!\sim_{T}$, since there is no way of warranting formulas of the form "$x \leftarrow y$". However, as in the case of $\mid\!\sim_{Arg}$, an alternative version can be provided.

Proposition 5.8 (Horn Right Weakening). *The operator $C_{war}(\Gamma)$ satisfies* Horn right weakening, *i.e. if $[\emptyset, \{\mathsf{n_i}\}]{:}y \in C_{war}(\Gamma)$ and it holds that $[\emptyset, \{\mathsf{n_j}\}]{:}x \leftarrow y \in Th_{sld}(\Gamma)$, then $[\emptyset, \{\mathsf{n_k}\}]{:}x \in C_{war}(\Gamma)$.*

Proof. Assume that $[\emptyset, \{\mathsf{n_k}\}]{:}x \notin C_{war}(\Gamma)$. Assume there is an argument $\mathcal{A}{:}y$, such that y is warranted. In particular, the empty set is $\mathcal{A}$ is also an argument for x, *i.e.* $\mathcal{A}{:}x$ via $\mathsf{Elim}\text{-}\leftarrow$. Argument $\mathcal{A}{:}x$ and argument $\mathcal{A}{:}y$ have the same set of associated defeaters. Therefore, if $\mathcal{A}{:}y$ is warranted, then so is $\mathcal{A}{:}x$. Hence $[\emptyset, \{\mathsf{n_k}\}]{:}x \in C_{war}(\Gamma)$ (contradiction).

Proposition 5.9 (Conjunction of conclusions). *The operator $C_{war}(\Gamma)$ does not satisfy conjunction of conclusions, i.e. if $x \in C_{war}(\Gamma)$ and $y \in C_{war}(\Gamma)$, then it does not hold $x \wedge y \in C_{war}(\Gamma)$.*

Proof. Consider the counterexample shown in proposition 4.9.

Proposition 5.10 (Subclassical Cummulativity). *The operator $C_{war}(\Gamma)$ satisfies subclassical cummulativity, i.e. $\Gamma \subseteq \Gamma' \subseteq Th_{sld}(\Gamma)$ implies $C_{war}(\Gamma) = C_{war}(\Gamma')$*

Proof. We can reason as in proposition 4.10. Given a theory Γ, clearly $Th_{sld}(\Gamma)$ will involve only *facts* (*i.e.* literals derivable via SLD from $\Pi(\Gamma)$). By assumption, $\Gamma \subseteq Th_{sld}(\Gamma)$. Then Γ is formed only by facts. But if this is the case, then it follows that every formula in Γ is trivially provable via SLD from $\Pi(\Gamma)$. Therefore $\Gamma = Th_{sld}(\Gamma)$, and by assumption it follows that $\Gamma' = \Gamma$. Hence $C_{war}(\Gamma) = C_{war}(\Gamma')$.

The C_{war} operator satisfies left absorption, but not right absorption. This follows from the epistemic status assigned to warranted literals: if they are incorporated as new facts into a given theory, clearly they will be also derivable via SLD. The converse is not true, since not every warranted literal is derivable via SLD.

Proposition 5.11 (Left absorption). *The $C_{war}(\Gamma)$ operator satisfies left absorption, i.e. $Th_{sld}(C_{war}(\Gamma)) = C_{war}(\Gamma)$.*

Proof. ($\Longrightarrow$): Suppose h is warranted, and consequently $[\emptyset, \{\mathsf{n_i}\}]{:}h \in C_{war}(\Gamma)$. Clearly $\Gamma \mid\!\sim_{Arg} [\emptyset, \{\mathsf{n_i}\}]{:}h$, and in particular from def. 3.1, it follows that $[\emptyset, \{\mathsf{n_i}\}]{:}h \in Th_{sld}(C_{war}(\Gamma))$.
($\Longleftarrow$): Suppose $[\emptyset, \{\mathsf{n_i}\}]{:}h \in Th_{sld}(C_{war}(\Gamma))$. Then from def. 3.1 there exists an empty argument $\emptyset{:}h$. Therefore h is warranted, or equivalently $[\emptyset, \{\mathsf{n_i}\}]{:}h \in C_{war}(\Gamma)$.

Property	$\mid\!\sim_{Arg}$	$\mid\!\sim_{T}$	C	Comments
Inclusion	yes	yes	P	Restricted to non-defeasible information. Propositions 4.1 and 5.1.
Idempotence	no	yes	P	Propositions 4.2 and 5.2.
Cummulativity	yes	no	P	Lemma 4.5 and proposition 5.5.
Monotonicity	no	no	P	Propositions 4.3 and 5.3.
Horn Supraclassicality	yes	yes	H	Supraclassicality restricted to Horn-like formulas (Prop. 4.6 and 5.6)
Left-logical equivalence	no	no	H	Prop. 4.7 and 5.7
Horn Right Weakening	yes	yes	H	Weakening restricted to Horn-like formulas (Propositions 4.8 and 5.8)
Conjunction of conclusions	no	no	H	Propositions 4.9 and 5.9.
Subclassical cummulativity	yes	yes	H	Propositions 4.10 and 5.10.
Left absorption	no	yes	H	Propositions 4.11 and 5.11.
Right absorption	no	no	H	Propositions 4.12 and 5.12.
Rational Negation	no	no	NH	Propositions 4.13 and 5.13.
Disjunctive Rationality	no	no	NH	Not considered due to object language constraints.
Rational monotonicity	no	no	NH	Propositions 4.14 and 5.14.

NOTE: Column C denotes the kind of property (P=pure; H=Horn; N=non-Horn)

Fig. 1. Logical properties in LDS_{ar}: summary

Proposition 5.12 (Right absorption). *The $C_{war}(\Gamma)$ operator does not satisfy right absorption, i.e. $C_{war}(Th_{sld}(\Gamma)) \neq C_{war}(\Gamma)$.*

Proof. A counterexample suffices. Consider the theory $\Gamma = \{\ [\{p \leftarrow q\}, \{\mathsf{d}_1\}]{:}p \leftarrow q\ \ ,$ $[\emptyset, \{\mathsf{n}_1\}]{:}\mathrm{q}\ \ \}$. Clearly p and q are warranted literals. In particular $[\emptyset, \{\mathsf{n}_i\}]{:}\mathrm{p} \in C_{war}(\Gamma)$. Note that $Th_{sld}(\Gamma)$=$\{\ [\emptyset, \{\mathsf{n}_1\}]{:}\mathrm{q}\ \ \}$, and consequently $C_{war}(Th_{sld}(\Gamma)) = Th_{sld}(\Gamma)$. But $[\emptyset, \{\mathsf{n}_i\}]{:}\mathrm{p} \notin C_{war}(Th_{sld}(\Gamma))$.

Proposition 5.13 (Rational negation). *The $C_{war}(\Gamma)$ operator does not satisfy rational negation.*

Proof. Consider example 4. In this case there is only one argument for z, namely $\Gamma \mid\!\sim_{Arg} \mathcal{A}{:}z$, and such an argument has no defeaters. Consequently z is warranted wrt Γ.

However, $\Gamma \cup\ \{[\emptyset, \{\mathsf{n}_i\}]{:}x\} \not\mid\!\sim_{Arg} \mathcal{A}{:}z$, and $\Gamma \cup\ \{[\emptyset, \{\mathsf{n}_i\}]{:}{\sim}x\} \not\mid\!\sim_{Arg} \mathcal{A}{:}z$ (since in both cases the use of d_2 is not allowed by consistency constraints). Therefore z is not warranted in either of these cases.

Proposition 5.14 (Rational monotonicity). *The $C_{war}(\Gamma)$ operator does not satisfy rational monotonicity.*

Proof. Consider again example 4, where $\Gamma \mid\!\sim_{Arg} \mathcal{A}{:}z$, with $\mathcal{A} = \{z \leftarrow p,\ p \leftarrow r\ \}$. If we consider now $\Gamma \cup\ \{x\}$ it holds that $\Gamma \cup\ \{x\} \not\mid\!\sim_{Arg} \mathcal{A}{:}z$, and $\Gamma \not\mid\!\sim_{Arg} {\sim}x$. Therefore rational monotonicity does not hold.

Finally, note that the C_{war} operator does not satisfy *disjunctive rationality*. The reasons are the same as those discussed for the C_{arg} operator.

6 Conclusions. Related work

Research in logical properties for defeasible argumentation was started by G.Vreeswijk [Vre93] and H.Prakken [PV99]. In particular, the work of S.Sardiña [SS98] focused on logical properties of the original Simari-Loui framework [SL92] and of *defeasible logic programming* [Gar00]. This research is partly motivated by these results.

As we have shown in this paper, LDS_{ar} provides a useful framework for analyzing different logical properties of defeasible argumentation, providing a better understanding of how argument construction and warrant behave. Figure 1 provides a summary of the logical properties discussed before.

When formalizing argument construction (operator C_{arg}), restricted inclusion ensures that non-defeasible facts can be ontologically understood as empty arguments. Cummulativity allows to keep any argument obtained from a theory Γ as an 'intermediate proof' (lemma) to be used in building more complex arguments. Horn supraclassicality indicates that every conclusion that follows via SLD can be considered as a special form of argument (namely, an empty argument), whereas Horn right weakening tells us that strong rules in LDS_{ar} preserve the intuitive semantics of a Horn rule(the existence of a strong rule $[\emptyset, \{\mathsf{n_i}\}]{:}y \leftarrow x$ makes every argument $\mathcal{A}$ for x be also an argument for y) Finally, subclassical cummulativity indicates that two argumentative theories Γ and Γ' whose information is a subset of those literals that can be derived via SLD from Γ (or Γ') are equivalent when considering the arguments that can be obtained from them.

Computing warrant, on the other hand, can also be better understood in the light of some logical properties of C_{war}. Restricted inclusion ensures that any non-defeasible fact in a theory Γ can be considered as warranted. Idempotence indicates that successive applications of C_{war} on a the set S of warranted literals returns exactly the same set. From Horn supraclassicality it follows that every conclusion obtained via SLD is a particular case of warranted literal, whereas Horn right weakening indicates that non-defeasible rules behave as such in the meta-level (a strong rule $[\emptyset, \{\mathsf{n_i}\}]{:}y \leftarrow x$ ensures that every warrant $\mathcal{A}$ for a literal x is also a warrant for y). From subclassical cummulativity it follows that two theories Γ and Γ', whose information is a subset of the conclusions that can be obtained from Γ (or Γ') are equivalent when considering the set of literals that can be warranted from them. Finally, left absorption in C_{war} wrt C_{arg} indicates that once a set of warranted literals have been obtained, SLD derivation does not add any inferential power.[8]

We contend that a formal analysis of defeasible consequence is mandatory in order to get an in-depth understanding of the behavior of argumentation frameworks. The logical properties discussed in this paper provide a natural tool for characterizing that behavior, as well as useful comparison criteria when developing new argumentation frameworks, or assessing their expressive power.

A Appendix: LDS_{ar} fundamentals

In this section we will introduce a *knowledge representation language* $\mathcal{L}_{\mathsf{KR}}$ for performing defeasible inference, together with a *labeling language* $\mathcal{L}_{\mathsf{Labels}}$. These languages will be used to define the object language $\mathcal{L}_{\mathsf{Arg}}$. Following Gabbay's terminology [Gab96], the basic information units in $\mathcal{L}_{\mathsf{Arg}}$ will be called *declarative units*, having the form *Label:wff*. In our approach we will restrict wffs in labeled formulas to ground literals. A ground literal can be understood as a *conclusion* of an *argument*, which will be defined by the label.

A label in a formula $L{:}\alpha$ will provide three elements which are convenient to take into account when formalizing defeasible argumentation, namely:

1. For every declarative unit $L{:}\alpha$ the label L will distinguish whether that declarative unit corresponds to *defeasible* or *non-defeasible* information.
2. The label L will also provide an *unique name* to identify a wff in the knowledge base Γ.

[8] This is due to the limitations of our formalism, in the sense that only literals can be warranted.

3. When performing the inference of a declarative unit $L{:}\alpha$ from a set Γ of declarative units, the label L will provide a *trace* of the wffs needed to infer $L{:}\alpha$ from Γ.

Wffs in our knowledge representation language $\mathcal{L}_{\mathsf{KR}}$ will be a subset of a classic propositional language $\mathcal{L}$, restricted to *rules* and *facts*. The set of all rules and facts in $\mathcal{L}_{\mathsf{KR}}$ will be denoted $\mathsf{Rules}(\mathcal{L}_{\mathsf{KR}})$ and $\mathsf{Facts}(\mathcal{L}_{\mathsf{KR}})$, resp. We define $\mathsf{ProgClauses}(\mathcal{L}_{\mathsf{KR}}) = \mathsf{Rules}(\mathcal{L}_{\mathsf{KR}}) \cup \mathsf{Facts}(\mathcal{L}_{\mathsf{KR}})$. A modality (label) will be attached to wffs in $\mathcal{L}_{\mathsf{KR}}$, indicating whether they are *defeasible* or *non-defeasible*.

Definition A.1 (Language $\mathcal{L}_{\mathsf{KR}}$. Wffs in $\mathcal{L}_{\mathsf{KR}}$). *The language $\mathcal{L}_{\mathsf{KR}}$ will be composed of*

1. *A countable set of* propositional atoms*, possibly subindicated. We will denote propositional atoms with lowercase letters. Example: a, b, c, d, e, ..., a_1, a_2, a_3 are propositional atoms.*
2. *Logical connectives $\wedge$, $\neg$ and $\leftarrow$.*

Wffs in $\mathcal{L}_{\mathsf{KR}}$ will be defined as follows:

1. *If α is an atom in $\mathcal{L}_{\mathsf{KR}}$, then α and $\sim\alpha$ are wffs called* literals *in $\mathcal{L}_{\mathsf{KR}}$.*
2. *If $\alpha_1, \ldots \alpha_k$, β are literals in $\mathcal{L}_{\mathsf{KR}}$, then $\beta \leftarrow \alpha_1, \ldots \alpha_k$ is a wff in $\mathcal{L}_{\mathsf{KR}}$.*

For the sake of simplicity, when referring to the language $\mathcal{L}_{\mathsf{KR}}$ the following conventions will be used: Greek lowercase letters α, β, γ will refer to any wff in $\mathcal{L}_{\mathsf{KR}}$. Lowercase letters (such as h, q, etc.) will be used for referring to ground literals in $\mathcal{L}_{\mathsf{KR}}$. Greek uppercase letters Υ, Φ, Γ will refer to a *set* of wffs in $\mathcal{L}_{\mathsf{KR}}$. The conjunction $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k$ will be simply written as $\alpha_1, \alpha_2, \ldots, \alpha_k$.

Definition A.2 (Labeling constants). *A set Labels $= \{\mathsf{n}_1, \mathsf{n}_2, \ldots, \mathsf{d}_1, \mathsf{d}_2, \ldots\}$ of* labeling constants *will include constant names with the form $\mathsf{n_i}$ and $\mathsf{d_i}$, standing for* non-defeasible *and* defeasible *information, resp. A* set *of labeling constants will be denoted as L_1, L_2, ..., L_k.*

Definition A.3 (Labeling language $\mathcal{L}_{\mathsf{Labels}}$). *A label L in our labeling language $\mathcal{L}_{\mathsf{Labels}}$ can be either an* argument label *or a* dialectical label.[9] *An* argument label *will be a tuple $\langle \mathsf{L}_i, \Phi\rangle$ where $\mathsf{L}_i \subseteq$ Labels and $\Phi \subseteq \wp(Wffs(\mathcal{L}_{\mathsf{KR}}))$.*

Definition A.4 (Defeasible Labeled Language $\mathcal{L}_{\mathsf{Arg}}$). *If $\mathcal{L}_{\mathsf{Labels}}$ is a labeling language, and $\mathcal{L}_{\mathsf{KR}}$ is a knowledge representation language, then the* defeasible labeled language*, denoted $\mathcal{L}_{\mathsf{Arg}}$, is defined as $\mathcal{L}_{\mathsf{Arg}} = (\mathcal{L}_{\mathsf{Labels}}, \mathcal{L}_{\mathsf{KR}})$*

Definition A.5 (Argumentative formula). *Given a declarative unit f=Label:α, where Label=$[\Phi, \mathsf{L}]$ is an argument label, then f will be called an* argumentative formula*. The set Φ in $[\Phi, \mathsf{L}]{:}\alpha$ will be called* argument support *in α. The set L will provide a trace of the wffs used for deriving α. Alternatively, we will use the notation f=$\mathcal{A}{:}\alpha$, where $\mathcal{A} = \Phi$.*

Since $\mathcal{L}_{\mathsf{KR}}$ is a Horn-like logic language, we will assume an underlying inference mechanism $\vdash_{\text{SLD}}$ equivalent to SLD resolution [Llo87], properly extended to handle a negated literal $\sim p$ as a new constant name *no_p*. Given $P \subseteq \mathsf{ProgClauses}(\mathcal{L}_{\mathsf{KR}})$, we will write $P \vdash_{\text{SLD}} \alpha$ to denote that α follows from P via $\vdash_{\text{SLD}}$.

Definition A.6 (Contradictory set of wffs in $\mathcal{L}_{\mathsf{KR}}$). *Given a set P of wffs in $\mathcal{L}_{\mathsf{KR}}$, P will be called* contradictory *iff literals p and $\overline{p}$ can be derived via $\vdash_{\text{SLD}}$ from P. This situation will be denoted as $P \vdash_{\text{SLD}} \perp$.*

Definition A.7 (Argumentative theory Γ). *A finite set $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ with $\Gamma \subseteq \mathsf{Wffs}(\mathcal{L}_{\mathsf{Arg}})$ will be called an* argumentative theory.

We will assume that the set of non-defeasible information $\Pi(\Gamma)$ in any argumentative theory Γ is non-contradictory.

Some distinguished sets will be considered. $\mathsf{Strict}(\Gamma)$ is the set of all non-defeasible formulas in Γ; $\mathsf{Defeasible}(\Gamma) = \Gamma - \mathsf{Strict}(\Gamma)$; $\Pi(\Gamma)$ is the set of all $\mathcal{L}_{\mathsf{KR}}$ formulas in Γ whose support set is empty; $\Delta(\Gamma)$ is the set of all $\mathcal{L}_{\mathsf{KR}}$ formulas in Γ whose support set is non-empty.

A.1 Formalizing argument construction and warrant

1. **Introducing non-defeasible information** (Intro-NR): Non-defeasible argumentative formulas can be introduced in a proof.

$$\frac{}{[\emptyset, \{\mathsf{n_i}\}]{:}\alpha}$$

for any $[\emptyset, \{\mathsf{n_i}\}]{:}\alpha \in \mathsf{Strict}(\Gamma)$.

[9] A formal treatment of dialectical labels is outside the scope of this appendix. For details see [Che01].

2. **Introducing defeasible information** (Intro-RE): Defeasible argumentative formulas can be introduced in a proof whenever its support set is non-contradictory wrt $\Pi(\Gamma)$.

$$\frac{\Pi(\Gamma) \cup \Phi \not\vdash_{\text{SLD}} \bot}{\Gamma,\ [\Phi, \{\mathsf{d_i}\}]{:}\alpha}$$

for any $[\Phi, \{\mathsf{d_i}\}]{:}\alpha \ \in \mathsf{Defeasible}(\Gamma)$.

3. **Introducing conjunction** (Intro-$\wedge$): If $[\Phi_1, \mathsf{L_1}]{:}\alpha_1$, $[\Phi_2, \mathsf{L_2}]{:}\alpha_2, \ldots, [\Phi_k, \mathsf{L_k}]{:}\alpha_k$, are wffs such that $\Phi_1 \cup \ \Phi_2 \ldots \cup \Phi_k \not\vdash_{\text{SLD}} \bot$, then the conjunction $\alpha_1, \alpha_2, \ldots, \alpha_k$ can be derived.

$$\frac{\Gamma, \quad [\Phi_1, \mathsf{L_1}]{:}\alpha_1 \qquad [\Phi_2, \mathsf{L_2}]{:}\alpha_2 \qquad \ldots\ [\Phi_k, \mathsf{L_k}]{:}\alpha_k \qquad \Pi(\Gamma) \cup \bigcup_{i=1\ldots k} \Phi_i \not\vdash_{\text{SLD}} \bot}{\Gamma, \quad [\bigcup_{i=1\ldots k} \Phi_i, \bigcup_{\mathsf{i=1\ldots k}} \mathsf{L_i}]{:}\alpha_1, \alpha_2, \ldots, \alpha_k}$$

4. **Eliminating implication** (Elim-$\leftarrow$):

$$\frac{\Gamma,\ [\Phi_1, \mathsf{L_1}]{:}\beta{\leftarrow}\alpha_1, \ldots, \alpha_k \qquad [\Phi_2, \mathsf{L_2}]{:}\alpha_1, \ldots, \alpha_k \qquad \Pi(\Gamma) \cup \Phi_1 \cup \ \Phi_2 \not\vdash_{\text{SLD}} \bot}{\Gamma,\ [\Phi_1 \cup \Phi_2, \mathsf{L_1} \cup \mathsf{L_2}]{:}\beta}$$

Definition A.8 (Generalized argument). *Let Γ be an argumentative theory, and let $h \in \mathsf{Lit}(\mathcal{L}_{\mathsf{KR}})$ such that $\Gamma \mid\!\sim_{Arg} \mathcal{A}{:}h$ Then $\mathcal{A}$ will be called a* generalized argument *for h.*

If it is not the case that $\Gamma \mid\!\sim_{Arg} \mathcal{B}{:}h$, with $\mathcal{B} \subset \mathcal{A}$, then $\mathcal{A}{:}h$ is called a minimal argument *or just* argument.

Given an argument $\mathcal{A}{:}h$ derivable from a theory Γ, there may be other conflicting arguments also supported by Γ which *defeat* it according to some preference criterion. A common syntactic preference criterion is *specificity* [SL92], which prefers those arguments which are more informed or more 'direct'. However, any partial order on the set of all possible arguments could be used. Since defeaters are arguments, they may be on its turn defeated, and so on. This leads to a recursive analysis, in which a tree structure rooted in $\mathcal{A}{:}h$ results. If $\mathcal{A}{:}h$ ultimately prevails over other conflicting arguments, then $\mathcal{A}{:}h$ is called a *warrant*. In LDS_{ar}, this situation is formalized in terms of an inference relationship $\mid\!\sim_{\mathcal{T}}$.

For space reasons we just give a brief sketch of the notion of warrant in order to understand some logical properties discussed in section 5. A full discussion can be found in [Che01].

Definition A.9 (Warrant (sketch)). *Let Γ be an argumentative theory, such that $\Gamma \mid\!\sim_{Arg} \mathcal{A}{:}h$, and $\mathcal{A}{:}h$ is an argument such that: a) it has no defeaters; or b) every defeater for $\mathcal{A}{:}h$ is ultimately defeated. Then $\mathcal{A}{:}h$ is a warranted argument. In that case, we will also say that h is warranted, denoting this as $\Gamma \mid\!\sim_{\mathcal{T}} h^U$.*

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