Computation of Indices of Power with Incompatibilities

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Abstract

In this paper we present the Shapley Value and Banzhaf–Coleman Index of Power with incompatibilities. There are restrictions that some groups present in formation a coalition. It is the case in many applications when there are some antagonistic groups.

We developed a computational system which allows the study of applications to voting systems.

The data input can be done in two different ways: by considering all the possible coalitions, or only the basic coalitions (political parties, sectorial groups, etc.). We have improved a previous development by allowing a much easier way of introducing the data and allowing the user to introduce group incompatibilities without having to modify the characteristic function (the program manages it). We present a parallel computational implementation of Indices of Power with incompatibilities. It improves the the performance of the system.

We present an application of the Power Indices in the Argentinean Deputy House under alternatives scenarios.

The resulting system model allows users to work over Windows and Linux platforms.

Keywords: Cooperative Games Theory - Indices of Electoral Power with Incompatibilities - Computational implementation - Client server paradigm - Scripting - Parallel - Multiplatform

1 Introduction

Game Theory is a mathematical theory which models agents interactions in situations of strategic conflict. A Game is a situation in which two or more players interact. It includes the model of the interactions among firms, groups etc. in Economic or Politic Scenarios.

Each player has partial control of the situation, but in general, no player controls it totally. Each player or group of players have certain personal preferences on the set of possible results and tries to obtain the one that is best for him. These preferences can be described by some utility functions, in which each player is characterized by a numerical function.

Games can be divided in two categories: "noncooperative" and "cooperatives". In the first one, only selfishness is assumed. In the second approach, besides this assumption, we consider the possibility of forming coalitions and the groups of players act cooperatively.

In the present article we will work only with cooperative games. Within this theory, there are several solutions measuring the power of each agent. These are Power Indices.

We will present a computational tool for calculating the classical Power Indices (Shapley Value and Banzhaf-Coleman Index) when some agents are incompatible. This is an extension to the article by Aguirre, Oviedo and Quintas [1]. Now we have a more sophisticated tool. The incompatibilities are introduced by the user and the system modifies the characteristic function and makes all the computation. The system have been also parallelized for an optimal performance. We also present an application to the Argentinean Deputy House.

2 Cooperative Games

These games are those where the individuals are free to communicate, to negotiate and to sign contracts to obtain better results. Let us suppose that a game has two or more results and at least two players agree (in preference) in a result, then they (2 or more) will sign an agreement to induce this result as the solution of the game.

In the cooperative games it naturally appears the concept of coalition (players that sign an agreement to induce a result of the game).

A cooperative game is given by: $G = (N, v)$, where $N = \{1, 2, ..., n\}$ is the set of all the players and v is the characteristic function. This function is defined on the coalitions $S \subseteq N$ and measures the value or utility $v(S)$ that each coalition S has if it forms. Thus $v(S)$ is the utility that the members of S can obtain by themselves.

Definition 1 *A cooperative $n-person$ game or simply a cooperative game is defined by: $G = (N, v)$, where $N = \{1, 2, ..., n\}$ is the set of all players and $v : P(N) = 2^N \longrightarrow R$ is the characteristic function. This is a real valuated function, defined on the subset of N, that fulfill the following properties:*

$$v(\phi) = 0 \tag{1}$$

$$v(\{i\}) \geqslant 0 \qquad \forall i \in N \tag{2}$$

$$v(S \cup T) \geqslant v(S) + v(T) \qquad \forall (S \cap T) = \phi \tag{3}$$

Condition (1) is only for consistency (the empty coalition has no-power). Condition (2) indicates that the security level of each player is cero.

Condition (3) is known as Superadditivity Property, and shows the incentives for the players in conforming bigger coalitions.

We will deal with a family of Cooperative Games named Weighted Majority Games [16]:

Definition 2 *A Weighted Majority Games is defined as a Cooperative n-person game $G = (N, v)$, where each player has a finite number of votes: $w_1, w_2, ..., w_n$.*

Thus for a coalition S we have:

$$v(S) = \sum_{i \in S} w_i.$$

These games appear in the Parliaments and other Directive Committees. A coalition should obtain at least a quote q of the votes, depending of the type of majority that the decision requires; q is a parameter depending of the voting system . If we think in terms of the percentage of votes, q could be $\frac{1}{2}$, $\frac{2}{3}$, etc.

S is a winning coalition if $v(S) \geq q$, where q is the type of majority selected (we will deal with $q > \frac{1}{2}v(N)$ in order to avoid having two winning coalitions with no empty intersection)

A Weighted Majority Games is determinate by the structure:

$$[q; w_1, w_2, ..., w_n].$$

3 Power Indices

The cooperative games admit different types the solutions. These are possible forms to distribute the total amount provided by the coalition among the players.

To give a solution or result of the cooperative game is to find a vector ($n - vector$) where each component says how much each player get.

Among the solutions more widely spread for cooperative games we mention: the Shapley Value, the Core, the Banzhaf–Coleman Power Index, the Nucleolus, etc. (see [13], [14]).

The Power Indices are mathematical indicators of the real possibilities that the agents have in the game. There are many applications of them to several sciences including Political Sciences, Economy, Sociology, etc.

In this work we will study the Shapley Value [15], and the Banzhaf–Coleman Power Index [7], [3], [4]. Both make some average of the marginal contribution of each agent to the possible coalitions he/she could enter.

We will implement both solutions.

3.1 The Shapley Value

The Shapley Value $\varphi(v) = (\varphi_1, \varphi_2, ..., \varphi_n)$ gives an imputation of the game $G = (N, v)$. An imputation is a way to distribute the total amount obtained by the total coalition N, among the players, giving each one at least the amount each player can obtain by himself and taking into account the average marginal contribution by being (or not) member of each coalition. In symbols we have:

$$\sum_{i=1}^{n} \varphi_i = v(N) \quad y \quad \varphi_i \geqslant v(\{i\}).$$

The Shapley Value can be computed in the following way:

Theorem 3 *Let $G = (N, v)$ be a cooperative game. The Shapley value is a $n-$ vector $\varphi(v) = (\varphi_1(v), \varphi_2(v), ..., \varphi_n(v))$, such that:*

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!\,(n-s-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right],$$

where s is the cardinality of S.

We can associate the following heuristic explanation to the Shapley Value. Suppose the players (the elements of N) agree to meet at a specified place and time. Naturally because of random fluctuations, all will arrive at different times. It is assumed, however, that all orders of arrival (permutations of the players) have the same probability : $\frac{1}{n!}$.

$$\left(\underbrace{\cdots\cdots}_{|S|!}, i, \underbrace{\cdots\cdots}_{(n-|S|-1)!}\right)$$

Suppose that, if a player, i, arrives, and finds the members of the coalition $S - \{i\}$ (and no others) already there, he receives the amount $v(S) - v(S - \{i\})$, i.e., the marginal amount which he contributes to the coalition, as payoff. Then the Shapley value $\varphi_i[v]$ is the expected payoff to player i under this randomization scheme.

The Shapley Value always exists and it is unique.

3.2 The Banzhaf–Coleman Power Index

A second index of power has been suggested by Banzhaf [3], [4] and Coleman [7]. It is known as the Banzhaf-Coleman Power Index.

This index is defined on simple games and is based on counting for each player the number of coalitions to which it belongs and it is crucial to win.

Lets assume that $G = (N, v)$ is a simple game normalized to $(0, 1)$. It is a game where $v(s) = 0$ or 1. We say a S is a winning coalition when $v(S) = 1$. For each winning coalition S, when $v(S - \{i\}) = 0$, is said to be a *swing* for player i.

For a game (N, v), lets suppose that $\sigma_i(N, v)$ is the number of swings for i. Lets suppose that $\sigma_0(N, v) = \sum_{i \in N} \sigma_i(N, v)$ is the total number of swings of all the players of the game. The normalized Banzhaf -Coleman Index $(b_1(N, v), b_2(N, v), ..., b_n(N, v))$ gives an imputation and it is defined by

$$b_i(N, v) = \frac{\sigma_i(N, v)}{\sigma_0(N, v)}.$$

This index can be generalized to general Cooperative Games [13] by the following formula:

$$b_i(N,v) = \sum_{S \subset N} \left(\frac{1}{2}\right)^{n-1} [v(S) - v(S - \{i\})] .$$

The difference between the Shapley Value and the Banzhaf-Coleman Power Index is that in the first case it is considered the probability of each orders of arrival of the players to form a coalition S, while in the second case we give the same probability to all coalitions.

3.3 Groups Incompatibilities

Within a cooperative environment it is very usual finding some restrictions for coalition formation. These restrictions could be of two types: communications restrictions or incompatibilities among some groups. The model should take into account these facts in order that the results could get close to reality.

The first case it is modeled by a communication graph [10]. It represents the possible communications ways among the players. Thus it could be some pivot players serving as a gate for several players interaction which won´t negotiate in an independent manner.

On the other hand sometimes there exist incompatibilities (of preferences, politics, etc.) among players, which in practice never form a join coalition. It can be modeled by modification the characteristic function in such a way that we only consider the compatible coalition values. It was done for the Shapley Value in Simple Games by Carreras [6]. Later on, it was generalized to general Cooperative Games by Bergantiños, Carreras and Garcia Jurado [5].

In this paper we extend a previous version of Cooperative Games with incompatibilities because of their practical use for analyzing politic interactions and we will compute both the Shapley Value and the Banzhaf-Coleman Index. Thus the characteristic function v can be computed in the following way:

$$v(S) = \max_{\substack{T \subseteq S \\ \nexists J \in I:\ J \subseteq T}} v(T),$$

where I is the set of the incompatibilities players.

The computational implementation will simplify this task to the user by doing it automatically. Thus the system computes the maximum coalition values over all the compatible sets. Otherwise, making this without the computer aid would be difficult and time demanding to the user.

4 The computational implementation

This application uses a client-server paradigm [18], [1]. The server is coded in C and the client in Tcl/Tk [11], [12], [19].

Client software is a Tcl/Tk script that shows a graphical user interface, which can run over Unix and Windows platforms [20], [21]. It contacts a server, sends a request, and awaits a response. When the response arrives, the client shows the results.

Server software run over Unix platform. It creates a socket and binds the socket to the port at which it desires to receive requests. It then enters into an infinite loop in which it accepts the next request that arrives from a client, processes the Banzhaf-Coleman and Shapley Value formula, and sends the reply back to the client.

Figure 1 shows the previous method [1] for computing the Indices of Power:

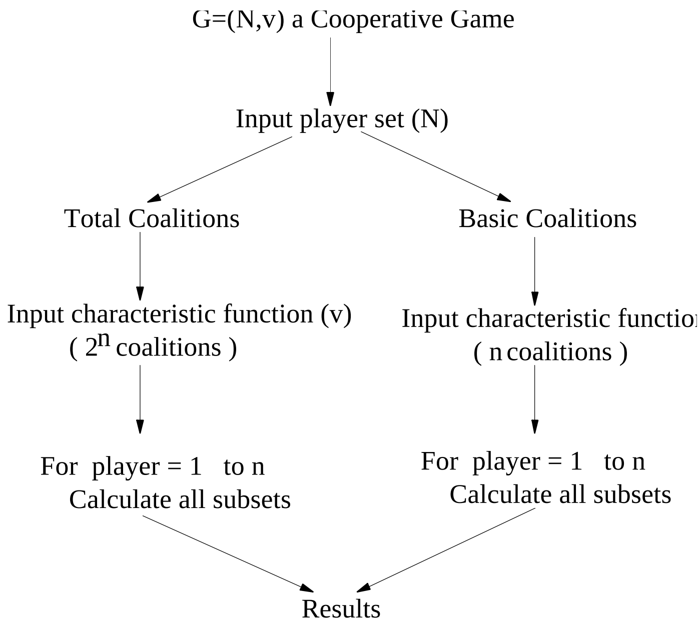

Figure 1

When solving a problem, one approach is to divide the problem into smaller specific tasks. Given the nature of the sub-tasks and the sequence in which they be done, we can occasionally split them out to separated processes and, if needed, to make the processes communicate with each other.

UNIX provides system calls to create processes, and it is possible to use these facilities to write parallel programs. We would not get an increased execution speed on a single processor. Actually, the speed would reduce because of the overhead of creating the processes and handling context changes as we swap between processes. A much more efficient mechanism is one in which a concurrent routine is specified that shares the same memory space and global variables. This can be provided by a *thread* mechanism or *lightweight process*.

Each thread has its own execution stack, register set, program counter, thread-specific data, thread-local variables, thread-specific signal mask, and state information. In a multithreaded process each thread executes independently and asynchronously. Problems that consist of multiple individual tasks lend themselves to a threaded solution.

The reason to parallel server programs is to increase the number of computing resources to more efficiently process the indices (i.e. each player is processed in parallel).

Figure 2 shows the new method for computing the Indices of Power:

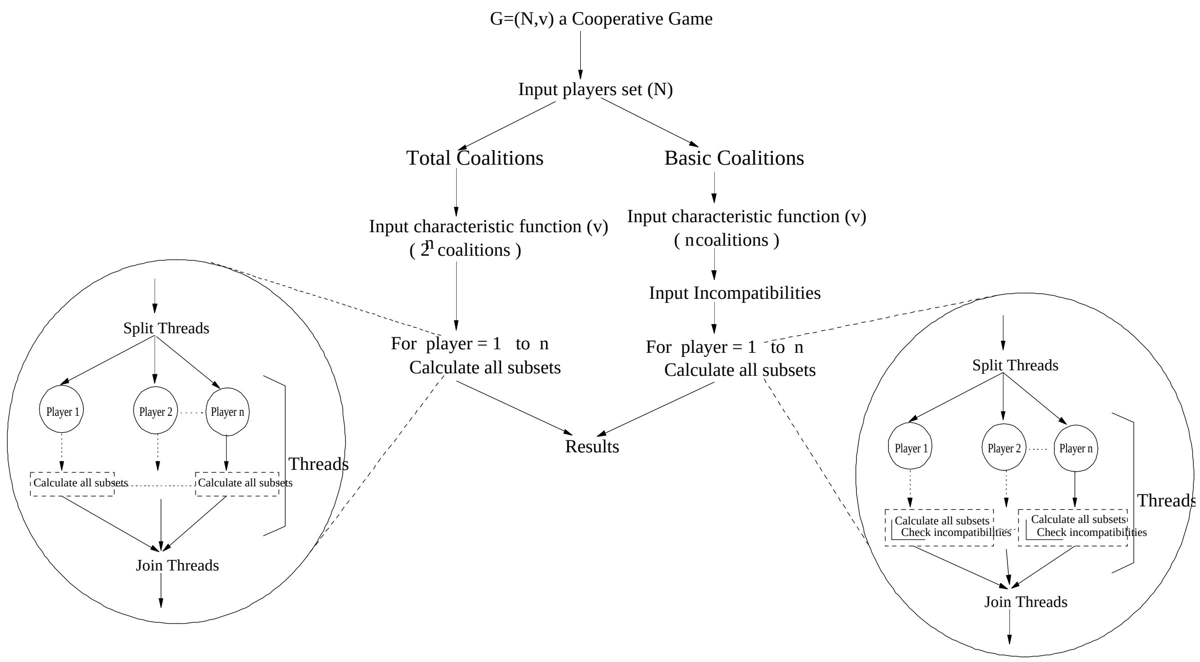

Figure 2

In next section we describe an application of the Indices of Power with Incompatibilities

5 An Application to the Argentinean Deputy House

We will use the present system in order to compute the Power Indices in the Argentinean Parliament. We will focus on the Argentinean Deputy House because here there is no party having a proper majority The actual composition of the Argentinean Deputy House is:

Players	**Politic Party**	**Votes**
A	Alianza	108
PJ	Partido Justicialista	100
IF	InterBloque Federal	14
IAR	Interbloque Acción por la República	11
ARI	Argentinos por una República de Iguales	7
FC	Frente para el Cambio	4
FPN	Frente Partido Nuevo	3
FR	Fuerza Republicana	3
MCS	Movimiento Cívico y Social	2
MPN	Movimiento Popular Nequino	2
FCS	Frente Cívico y Social	2
UB	Unidad Bonaerense	1
	Total:	257

Scenario 1

We will first compute the Power Indices without making any adition assumption about the possible relation about the parties.

Thus we will have the Weighted Majority Games $G^1 = (N^1, v^1)$ defined by:

$N^1 = \{A,\ PJ,\ IF,\ IAR,\ ARI,\ FC,\ FPN,\ FR,\ MCS,\ MPN,\ FCS,\ UB\}$,

$v^1 = [\mathbf{129};\ 108,\ 100,\ 14,\ 11,\ 7,\ 4,\ 3,\ 3,\ 2,\ 2,\ 2,\ 1]$,

where 129 is a simple majority game of the Argentinean Deputy House.

Next table shows the Indices of Power:

Players	**Votes**	**% Votes**	**Shapley**	**Banzhaf-Coleman**
A	108	42,02 %	35,73 %	33,88 %
PJ	100	38,91 %	22,08 %	18,85 %
IF	14	5,45 %	13,68 %	15,04 %
IAR	11	4,28 %	9,21 %	10,30 %
ARI	7	2,72 %	5,97 %	6,18 %
FC	4	1,56 %	3,23 %	3,71 %
FPN	3	1,17 %	2,38 %	2,78 %
FR	3	1,17 %	2,38 %	2,78 %
MCS	2	0,78 %	1,52 %	1,85 %
MPN	2	0,78 %	1,52 %	1,85 %
FCS	2	0,78 %	1,52 %	1,85 %
UB	1	0,38 %	0,78 %	0,93 %
Total:	**257**	**100 %**	**100 %**	**100 %**

This shows a picture which is not very close to reality. The major parties have the greatest indices (Alianza 35% and PJ 20% approximately) but the intuition size that PJ has more power and it is not shown in this analysis. It is so because in practice many of the small parties work together with Alianza and PJ respectively. There are also some parties incompatibilities and this has to be taken into account. Thus we have the following scenario.

Scenario 2

The votes of the following parties are summed up together because of their political affinity

$\mathcal{A} = A + FCS + IAR + MCS = 123$

$\mathcal{PJ} = PJ + FC + IF = 118$

The cooperative game $G^2 = (N^2, v^2)$ is now defined by

$N^2 = \{\mathcal{A},\ \mathcal{PJ},\ ARI,\ FPN,\ FR,\ MPN,\ UB\}$,

$v^2 = [\mathbf{129};\ 123,\ 118,\ 7,\ 3,\ 3,\ 2,\ 1]$,

We also consider the following group incompatibilities:

$I = \{UB - \mathcal{A},\ ARI - \mathcal{A}\}$

It is done in this way because ARI and UB are parties which a clear opposite position to the government. Next table shows the new Indices of Power:

Players	**Votes**	**% Votes**	**Shapley**	**Banzhaf-Coleman**
$\mathcal{A}$	123	47,86 %	29,76 %	29,41 %
$\mathcal{PJ}$	118	45,91 %	34,76 %	33,33 %
ARI	7	2,72 %	9,76 %	9,80 %
FPN	3	1,17 %	11,42 %	11,76 %
FR	3	1,17 %	11,42 %	11,76 %
MPN	2	0,78 %	1,44 %	1,97 %
UB	1	0,39 %	1,44 %	1,97 %
Total:	**257**	**100 %**	**100 %**	**100 %**

This seems to be a more real scenario. Both Alianza and PJ have the major power. The small parties still have a relative important place in the game. Their votes in some cases are important in order to make a winning coalition.

Scenario 3

So far we have seen a relation between PJ and Alianza of confrontation but no incompatibility (the PJ have given quorum and many times votes the Alianza proposal). We finally consider an hypothetic scenario of plain confrontation between Alianza and PJ (incompatibility) The cooperative game $G^3 = (N^3, v^3)$ with incompatibilities $A - PJ$ is defined by:

$N^3 = \{A, PJ, IF, IAR, ARI, FC, FPN, FR, MCS, MPN, FCS, UB\}$,

$v^3 = [\mathbf{129};\ 108,\ 100,\ 14,\ 11,\ 7,\ 4,\ 3,\ 3,\ 2,\ 2,\ 2,\ 1]$,

The next table shows the Indices of Power:

Players	**Votes**	**% Votes**	**Shapley**	**Banzhaf-Coleman**
A	108	42,02 %	24,69 %	23,34 %
PJ	100	38,91 %	11,04 %	8,99 %
IF	14	5,45 %	20,73 %	21,52 %
IAR	11	4,28 %	14,41 %	14,74 %
ARI	7	2,72 %	8,70 %	8,85 %
FC	4	1,56 %	4,99 %	5,31 %
FPN	3	1,17 %	3,62 %	3,98 %
FR	3	1,17 %	3,62 %	3,98 %
MCS	2	0,78 %	2,34 %	2,65 %
MPN	2	0,78 %	2,34 %	2,65 %
FCS	2	0,78 %	2,34 %	2,65 %
UB	1	0,38 %	1,18 %	1,34 %
Total:	**257**	**100 %**	**100 %**	**100 %**

It shows that in this case some small groups will be decisive in the new scenario. Besides some other political features, it also reveal the importance for the major parties Alianza and PJ to work together.

6 Conclusions

The new computational system is a much powerful tool than the previous version [1].

On one side the incompatibilities are now introduced by the user directly throw a window box, the system modifies the characteristic function (it would be time demanding for the user making it by hand) and makes all the computations.

On the other hand the system have been also parallelized by using thread, for an optimal performance. Moreover, the client software is multiplatform and the user can choose the best available platform instead of being constrained only to the availability of a specific platform.

Thus we have obtained a support to game theory researchers, as well as to teachers and students. It has been shown to be particularly useful in complex applications of Cooperative Game Theory

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