

On the Logic of Theory Change:
Wr
Contraction Without Recovery

1 Introduction

The postulate of *Recovery*, among the six postulates for theory contraction, formulated and studied by Alchourrón, Gärdenfors and Makinson [AM82, AGM85, AM85, Mak87, Gar88] has been the one that more controversies provoked [Mak87, HAN91, see].

There are clearly cases in which the recovery postulate seems to be contrary to intuition¹.

There are two important issues:

¹Basically, proposed counter-examples cave in into the following case:

Let a theory K , and let $x, y \in K$. Suppose we wish to eliminate x and y ; so we proceed to contract by their disjunction, namely $x \vee y$. If later on I'm informed that either x or y are actually true, without being told which one is true, I shall expand my knowledge, beliefs by $x \vee y$. After performing the expansion as sanctioned by the AGM model, the resulting set will restore both x and y , contrary to what is expected.

1. It is very hard to find a reasonable contraction function without recovery applicable to theories; existing works has concentrated in developing contraction function over knowledge bases; i.e. the finite case.
2. When dealing with standard AGM functions, the Levi and Harper identities allow for the interdefinability of contraction and revision functions. As a direct consequence, we can take any of the two functions as primitive. However, there is no identity relating contractions functions that lack recovery with revision functions.

Contraction functions without recovery have been dubbed *withdrawal functions* [Mak87]

This paper does not attempt to indulge in the polemic about the recovery postulate, so we deliberately avoid this discussion. The purpose of this note is to define a sensible *withdrawal function* over theories and to establish the connection with classical AGM revision functions. The connection is given through satisfaction of an identity along the lines of the Harper identity.

In section 2 we present the AGM model. In section 3 we develop a contraction function for theories that do not satisfy the recovery postulate. In section 4 we relate this contraction function with the classical AGM revision function.

2 The AGM Model [AGM85, Gar88]

In this model an individual's beliefs of a rational agent are represented by a belief set \mathbf{K} , closed under logical consequence C_n , where C_n satisfies the following

properties: $\mathbf{K} \subseteq Cn(\mathbf{K})$ for any set \mathbf{K} of propositions, $Cn(Cn(\mathbf{K})) \subseteq Cn(\mathbf{K})$ and $Cn(\mathbf{K}) \subseteq Cn(\mathbf{H})$ if $\mathbf{K} \subseteq \mathbf{H}$. We assume that Cn includes classical logical consequences, satisfies the rule of introduction of disjunction into premises and is compact.

A theory is understood to be any set \mathbf{K} of proposition closed under Cn ; $Cn(\mathbf{K}) = \mathbf{K}$.

Formally, we define the expansion function $+$ from $\mathbf{K} \times \mathbf{L}^2$ to \mathbf{K} , such that (\mathbf{K}_x^+) denotes the expansion of \mathbf{K} by x and is defined by $(\mathbf{K}_x^+) = Cn(\mathbf{K} \cup \{x\})$.

The six basic postulates for contraction are:

(\mathbf{K}_1^-) \mathbf{K}_x^- is a theory whenever \mathbf{K} is a theory. (closure)

(\mathbf{K}_2^-) $\mathbf{K}_x^- \subseteq \mathbf{K}$ (inclusion)

(\mathbf{K}_3^-) If $x \notin \mathbf{K}$, then $\mathbf{K}_x^- = \mathbf{K}$ (vacuity)

(\mathbf{K}_4^-) If $\not\vdash x$, then $x \notin \mathbf{K}_x^-$ (success)

(\mathbf{K}_5^-) If $\vdash x \leftrightarrow y$ then $\mathbf{K}_x^- = \mathbf{K}_y^-$ (preservation)

(\mathbf{K}_6^-) $\mathbf{K} \subseteq (\mathbf{K}_x^-)_x^+$ whenever \mathbf{K} is a theory (recovery)

A contraction function satisfies the following property:

Prop. 1: Whenever \mathbf{K} is a theory, if $x \in \mathbf{K}$, then $\mathbf{K} = (\mathbf{K}_x^-)_x^+$.

In [AGM85, Gar88] we see that $\mathbf{K}_x^- = \cap \mathbf{S}(\mathbf{K} \perp x)$, where $\mathbf{K} \perp x$ is the set of all inclusion-maximal subsets A of \mathbf{K} such that x is not a logical consequence of A , \mathbf{S} is a selection function, such that: $\mathbf{S}(\mathbf{K} \perp x)$ is a non-empty subset of $\mathbf{K} \perp x$, unless the latter is empty, in which case $\mathbf{S}(\mathbf{K} \perp x) = \{\mathbf{K}\}$.

²Where \mathbf{L} is the set of all the sentences of the language

The six basic postulates for revision are:

- (\mathbf{K}_1^*) \mathbf{K}_x^* is a theory. (closure)
- (\mathbf{K}_2^*) $x \in \mathbf{K}_x^*$ (success)
- (\mathbf{K}_3^*) $\mathbf{K}_x^* \subseteq \mathbf{K}_x^+$ (inclusion)
- (\mathbf{K}_4^*) If $\neg x \notin \mathbf{K}$ then $\mathbf{K}_x^+ \subseteq \mathbf{K}_x^*$ (vacuity)
- (\mathbf{K}_5^*) $\mathbf{K}_x^* = \mathbf{K}_\perp$ iff $\vdash \neg x$ (consistency)
- (\mathbf{K}_6^*) If $\vdash x \leftrightarrow y$ then $\mathbf{K}_x^* = \mathbf{K}_y^*$ (preservation)

2.1 Relation between Contraction and Revision:

We have seen that contraction and revision are defined by two different sets of postulates. These postulates are independent in the sense that the postulates of revision do not refer to contraction and vice versa. However it's possible define the revision function in terms of contraction function, and vice versa, by means of the formulas of Levi and Harper respectively:

$$\text{Def. Levi}(-) = (\mathbf{K}_{\neg x}^-)_x^+$$

$$\text{Def. Harper}(\ast) = \mathbf{K} \cap \mathbf{K}_{\neg x}^*$$

Theorem 1 : Let \mathbf{K} be a belief set and ‘ $-$ ’ an operator for \mathbf{K} that satisfies the contraction-postulates (\mathbf{K}_1^-) – (\mathbf{K}_5^-). Then $\text{Levi}(-)$ is an operator for \mathbf{K} that satisfies the revision-postulates (\mathbf{K}_1^*) – (\mathbf{K}_6^*).

Theorem 2: Let \mathbf{K} be a belief set and $'*'$ an operator for \mathbf{K} that satisfies the revision-postulates $(\mathbf{K}_1^*) - (\mathbf{K}_6^*)$. Then $\mathbf{Harper}(*)$ is an operator for \mathbf{K} that satisfies the contraction-postulates $(\mathbf{K}_1^-) - (\mathbf{K}_6^-)$.

In [Mak87], Makinson incorporate the following results

Theorem 3: Let \mathbf{K} be a belief set and $'-'$ an operator for \mathbf{K} that satisfies the contraction-postulates $(\mathbf{K}_1^-) - (\mathbf{K}_6^-)$. Then $\mathbf{Harper}(\mathbf{Levi}(-)) = -$.

Theorem 4: Let \mathbf{K} be a belief set and $'*'$ an operator for \mathbf{K} that satisfies the revision-postulates $(\mathbf{K}_1^*) - (\mathbf{K}_6^*)$. Then $\mathbf{Levi}(\mathbf{Harper}(*)) = *$.

3 How to construct a contraction function without recovery

We can ask firstly why a contraction function satisfies recovery. The answer is the following:

Obs. 1: If $x_i \in \mathbf{K}$ and $x_i \notin \mathbf{K}_x^-$, then $x \rightarrow x_i \in \mathbf{K}_x^-$.

Definition 1: Let $\mathbf{V}_x = \{x \rightarrow x_i : x_i \in \mathbf{K} \text{ and } x_i \notin \mathbf{K}_x^-\}$

We see clearly that the contraction-function satisfies recovery because all the members of \mathbf{V}_x are in \mathbf{K}_x^- and, if we want have not this property, it is necessary to eliminate

from \mathbf{K}_x^- some of the member of \mathbf{V}_x .

Definition 2 wr: Let wr be a contraction function such that:

$$\mathbf{K}_x^{\text{wr}} = \mathbf{K}_x^- \cap \mathbf{K}_{\mathbf{w}_x}^-, \text{ where } \mathbf{w}_x = x \rightarrow x_j; x_j = \text{Sel}_x(\mathbf{W})^3,$$

$$\mathbf{W} = \{x_i : x_i \in \mathbf{K} \text{ and } x_i \notin \mathbf{K}_x^-\}$$

Theorem 5: \mathbf{K}_x^{wr} defined as **Definition 2** satisfies $(\mathbf{K}_1^-) - (\mathbf{K}_5^-)$.

Theorem 6: \mathbf{K}_x^{wr} defined as **Definition 2** does not satisfy (\mathbf{K}_6^-) iff $x \not\vdash x_j$.

4 Relation between 'wr' and '*'

As the Levi's and Harper's functions for the AGM model, we will define new functions to relate 'wr' and '*'.³

³ $\text{Sel}_x(\mathbf{W})$ select an element of \mathbf{W} ; and this is equivalent to select someone finite subset of \mathbf{W} , because if two sentences are in \mathbf{W} , their conjunction is in \mathbf{W} too (the demonstration is trivial). If $\mathbf{W} = \emptyset$, then $\text{Sel}_x(\mathbf{W}) = x$.

The selection function must satisfies:

Prop. 2: $\text{Sel}_x(\mathbf{W}) \equiv \text{Sel}_y(\mathbf{W})$ if $\vdash x \leftrightarrow y$

$$\text{Def. Levi}'(\underline{\mathbf{w}}\mathbf{r}) = (\mathbf{K}_{-x}^{\underline{\mathbf{w}}\mathbf{r}})^+$$

$$\text{Def. Harper}'(*) = \mathbf{K} \cap \mathbf{K}_{-x}^* \cap \mathbf{K}_{-\mathbf{w}x}^*$$

Theorem 7: $\text{Levi}'(\underline{\mathbf{w}}\mathbf{r}) = \text{Levi}(\underline{\mathbf{w}}\mathbf{r})$.

Theorem 8: Let \mathbf{K} be a belief set and $'*'$ an operator for \mathbf{K} that satisfies the revision-postulates $(\mathbf{K}_1^+) - (\mathbf{K}_6^+)$. Then $\text{Harper}'(*)$ is an operator for \mathbf{K} that satisfies the contraction-postulates $(\mathbf{K}_1^-) - (\mathbf{K}_5^-)$.

Theorem 9: Let \mathbf{K} be a belief set and $'\underline{\mathbf{w}}\mathbf{r}'$ an operator for \mathbf{K} that satisfies the contraction-postulates $(\mathbf{K}_1^-) - (\mathbf{K}_5^-)$. Then $\text{Harper}'(\text{Levi}'(\underline{\mathbf{w}}\mathbf{r})) = \underline{\mathbf{w}}\mathbf{r}$.

Theorem 10: Let \mathbf{K} be a belief set and $'*'$ an operator for \mathbf{K} that satisfies the revision-postulates $(\mathbf{K}_1^+) - (\mathbf{K}_6^+)$. Then $\text{Levi}(\text{Harper}'(*)) = *$.

5 Conclusions

We have defined $\underline{\mathbf{w}}\mathbf{r}$: a contraction function without recovery applicable to theories. We have obtained this function as a combination of two standard AGM contraction functions, so that we have preserved the full strength and elegance of the classical AGM model. We have presented LEVI' and HARPER', two identities relating our withdrawal function with the classical AGM revision function and finally showing that they become reciprocally dual.

Appendix: Proofs

Obs. 1:

Suppose that $x_i \in \mathbf{K}$ and $x_i \notin \mathbf{K}_x^-$; then $\mathbf{K} \neq \mathbf{K}_x^-$; then (by (\mathbf{K}_3^-)) $x \in \mathbf{K}$; then (by (\mathbf{K}_6^-)) $x_i \in (\mathbf{K}_x^-)^+$; hence $x \rightarrow x_i \in \mathbf{K}_x^-$.

Theorem 5:

(\mathbf{K}_1^-) , (\mathbf{K}_2^-) are trivial.

(\mathbf{K}_3^-) . If $x \notin \mathbf{K}$, then $\mathbf{K}_x^- = \mathbf{K}$, then $\mathbf{W} = \emptyset$, then $\mathbf{K}_{\mathbf{w}_x}^- = \mathbf{K}_{x \rightarrow x}^- = \mathbf{K}$, hence $\mathbf{K}_x^{\mathbf{w}\Gamma} = \mathbf{K}$

(\mathbf{K}_4^-) is trivial, because if $\not\vdash x$, then $x \notin \mathbf{K}_x^-$, hence $x \notin \mathbf{K}_x^{\mathbf{w}\Gamma}$

(\mathbf{K}_5^-) . $\mathbf{w}_x \equiv \mathbf{w}_y$ by **Prop. 2**. The rest is trivial.

Theorem 6:

\Leftarrow) Suppose that $\mathbf{K}_x^{\mathbf{w}\mathbf{r}}$ satisfies (\mathbf{K}_6^-) , then $\mathbf{K} \subseteq (\mathbf{K}_x^{\mathbf{w}\mathbf{r}})_x^+$; $x_j \in \mathbf{K}$ ⁴, then $x_j \in (\mathbf{K}_x^{\mathbf{w}\mathbf{r}})_x^+$; then $x \rightarrow x_j \in \mathbf{K}_x^{\mathbf{w}\mathbf{r}}$, then $x \rightarrow x_j \in \mathbf{K}_{x \rightarrow x_j}^-$, and hence (by (\mathbf{K}_4^-)) $\vdash x \rightarrow x_j$, it is $x \vdash x_j$.

\Rightarrow) Suppose that $x \vdash x_j$, then $\vdash x \rightarrow x_j$, then $\mathbf{K}_{x \rightarrow x_j}^- = \mathbf{K}$, then $\mathbf{K}_x^{\mathbf{w}\mathbf{r}} = \mathbf{K}_x^-$, that satisfies (\mathbf{K}_6^-) .

Theorem 7:

First we show that $\vdash x \equiv x \wedge \mathbf{w}_{\neg x}; \mathbf{w}_{\neg x} = \neg x \rightarrow x_j \equiv x \vee x_j. x \equiv x \wedge (x \vee x_j)$.

Hence $(\mathbf{K}_{\neg x}^{\mathbf{w}\mathbf{r}})_x^+ = (\mathbf{K}_{\neg x}^- \cap \mathbf{K}_{\mathbf{w}_{\neg x}}^-)_x^+ = (\mathbf{K}_{\neg x}^-)_x^+ \cap (\mathbf{K}_{\mathbf{w}_{\neg x}}^-)_x^+$; where $(\mathbf{K}_{\mathbf{w}_{\neg x}}^-)_x^+ = (\mathbf{K}_{\mathbf{w}_{\neg x}}^-)_{x \wedge \mathbf{w}_{\neg x}}^+ = ((\mathbf{K}_{\mathbf{w}_{\neg x}}^-)_{\mathbf{w}_{\neg x}}^+)_x^+ =$ (by **Prop. 1**, because $\mathbf{w}_{\neg x} \in \mathbf{K}$) $= \mathbf{K}_x^+$, so that $(\mathbf{K}_{\neg x}^{\mathbf{w}\mathbf{r}})_x^+ = (\mathbf{K}_{\neg x}^-)_x^+ \cap \mathbf{K}_x^+ = \mathbf{K}_{\neg x}^-)_x^+$

Theorem 8:

Harper'(*) $= \mathbf{K} \cap \mathbf{K}_{\neg x}^* \cap \mathbf{K}_{\neg \mathbf{w}_x}^* = \mathbf{K} \cap \mathbf{K}_{\neg x}^* \cap \mathbf{K} \cap \mathbf{K}_{\neg \mathbf{w}_x}^* =$ (by **Harper(*)**)
 $= \mathbf{K}_x^- \cap \mathbf{K}_{\mathbf{w}_x}^-$.

⁴Is not true if $\mathbf{W} = \emptyset$, but in this case $x_j = x$ and the rest is trivial.

Theorem 9:

$$\text{Harper}'(\text{Levi}'(\underline{\mathbf{w}})) = \mathbf{K} \cap (\mathbf{K}_x^-)_{\neg x}^+ \cap \mathbf{K} \cap (\mathbf{K}_{\mathbf{w}_x}^-)_{\neg \mathbf{w}_x}^+ = (\text{by Theorem 3}) = \mathbf{K}_x^- \cap \mathbf{K}_{\mathbf{w}_x}^- = \mathbf{K}_x^{\underline{\mathbf{w}}}$$

Theorem 10:

$$\text{Levi}(\text{Harper}(*)) = (\mathbf{K} \cap \mathbf{K}_x^* \cap \mathbf{K} \cap \mathbf{K}_{\neg \mathbf{w} \neg x}^*)_{\neg x}^+ = (\mathbf{K})_x^+ \cap (\mathbf{K}_x^*)_x^+ \cap (\mathbf{K}_{\neg \mathbf{w} \neg x}^*)_x^+ =$$

(because $\neg x \in \mathbf{K}_{\neg \mathbf{w} \neg x}^*$) $= \mathbf{K}_x^+ \cap \mathbf{K}_x^* \cap \mathbf{K}_{\perp} = \mathbf{K}_x^*$.

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