A SIMILARITY-BASED LUKASIEWICZ'S MANY-VALUED MODAL LOGIC

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Abstract

In our opinion, approximate reasoning is one of the most fascinating branches of I.A., and it has generated an extensive literature. One of the goals of approximate reasoning models is to cope with inference patterns more flexible than those of classical reasoning. Amongst them, similarity-based reasoning aims at modelling notions of resemblance or proximity between propositions and consequence relations which make sense insuch a setting. Considering both approximate reasoning in a general context and a particular one as similarity-based reasoning, uncertainty and vagueness appear as two main notions. We shall associate the term uncertainty with degree of belief regarding a proposition (with itself is crip and may be true or false); on the other hand, vagueness (fuzziness) is associated degree of truth of a proposition (which may be fuzzy, i.e., admits non-extremal degree of truth). Both truth degree of fuzzy proposition and belief degree of crisp proposition are coded by reals from the unit interval $[0, 1]$ (in most cases; we shall not discuss exceptions) but they are handled differently. Our claims are that truth degree should not be mistaken for degree of belief and vice versa and that it is posible to combine them. We shall consider that logical systems corresponding to fuzziness are many-valued logics, whereas systems corresponding to uncertainty are related to various generalizations of modal logics and that presence of both fuzziness and uncertainty gives rise to many-valued modal logics. We shall try to build a logic of both vagueness and uncertainty. With this propose, we explore a modal approach to similarity-based reasoning that is a modal logic over a Rational Pavelka's logic RPL-like extension of the infinitely Lukasiewicz's logic. We define a many-valued modal system (which is a many-valued counterpart of classical S5 modal system) with many-valued similarity-based Kripke model semantics.

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1 Introduction

In our opinion, approximate reasoning is one of the most fascinating branches of I.A., and it has generated an extensive literature. One of the goals of approximate reasoning models is to cope with inference patterns more flexible than those of classical reasoning. Amongst them, similarity-based reasoning aims at modelling notions of resemblance or proximity between propositions and consequence relations which make sense insuch a setting. Considering both approximate reasoning in a general context and a particular one as similarity-based reasoning, uncertainty and vagueness appear as two main notions. We shall associate the term uncertainty with degree of belief regarding a proposition (with itself is crip and may be true or false); on the other hand, vagueness (fuzziness) is associated degree of truth of a proposition (which may be fuzzy, i.e., admits non-extremal degree of truth). Both truth degree of fuzzy proposition and belief degree of crisp proposition are coded by reals from the unit interval $[0, 1]$ (in most cases; we shall not discuss exceptions) but they are handled differently. Our claims are that truth degree should not be mistaken for degree of belief and vice versa and that it is posible to combine them. We shall consider that logical systems corresponding to fuzziness are many-valued logics, whereas systems corresponding to uncertainty are related to various generalizations of modal logics and that presence of both fuzziness and uncertainty gives rise to many-valued modal logics. We shall try to build a logic of both vagueness and uncertainty. With this propose, we explore a modal approach to similarity-based reasoning that is a modal logic over a Rational Pavelka's logic RPL-like extension of the infinitely Lukasiewicz's logic. We define a many-valued modal system (which is a many-valued counterpart of classical S5 modal system) with many-valued similarity-based Kripke model semantics. A many-valued similarity-based Kripke model is a structure $\langle W, S, e\rangle$, in which W is the set of possible worlds, e represents an evaluation assigning to each atomic formula φ and each interpretation $w \in W$ a true valued $e(\varphi, w)$ in the unit interval $[0, 1]$ and S is a similarity fuzzy relation on W (i.e. $S : W \times W \to [0, 1]$), satisfying the following properties: reflexivy ($S(w, w) = 1$), symmetry ($S(w, w') = S(w', w)$) and t-norm-transitivy ($S(w, w') \otimes S(w', w") \leq S(w, w")$) (a t-norm $\otimes$ is a nondecreasing, associative and commutative function over the unit interval with identity 1 and absorbing element 0). The last function captures a notion of semantical proximity (or indistinguishbility) between possible worlds, with value 1 corresponding to the identity of possible worlds and value 0 indicating that knowledge of fuzzy proposition in one of them does not provide any indication about proposition that are true in the other. With this semantic we try to cover inference patterns such as:

$$if\ A\ is\ \mu\ then\ B\ is\ \nu,\ and\ we\ observe\ A\ is\ \mu',\ then\ it\ is\ plausible, at\ some\ extent,$$
$$to\ conclude\ B\ is\ \nu'\ \ when\ \nu'\ is\ as\ close\ to\ \nu\ as\ \mu\ is\ close\ to\ \mu'$$

where $A\ is\ \mu$, $B\ is\ \nu$, $A\ is\ \mu'$ and $B\ is\ \nu'$ are fuzzy propositions. We take the closeness between two fuzzy proposition by considering the similarity of their models. Note that $B\ is\ \nu'$ is concluded with uncertainty; i.e. indistinguishbility between models produces uncertainty between formulas. We provide soundness and completeness results for this system with respect to some classes of the above structures. The paper is organized as follows. After this introduction we survey in Section 2 the Rational Pavelka's Logic – a generalization of Łukasiewicz's logic discovered by Pavelka and simplified by Hájek. In Sections 3 and 4 we present our logic system $SLMV$ and its model theory. In the sections 5 and 6, we give the proofs of soundness and completeness respectively. Finally, some comments are provided in Section 7.

2 Rational Pavelka's Logic

Łukasiewicz's infinitely-valued logic only allows us to prove 1-tautologies, but in fuzzy logic we are interested in inference from partially true assumptions, admitting that the conclusion will also be partially true. Rational Pavelka's Logic RPL is an extension of Łukasiewicz's infinitely-valued logic admitting truth constants $\overline{r}$ for each *rational* $r \in [0, 1]$. It is described in a simple formalization in [4]. Since the approach described in this paper strongly relies on this logic, here we present the main notions and properties of it.

In Łukasiewicz's logic *formulas* are built from propositional variables $p_1, p_2, \ldots$ and connectives $\rightarrow$ and $\neg$. Other connectives are defined from these ones. In particular

$$\begin{array}{rll} \varphi \,\&\, \psi & \text{stands for} & \neg(\varphi \rightarrow \neg\psi) \\ \varphi \,\underline{\vee}\, \psi & \text{stands for} & \neg\varphi \rightarrow \psi \\ \varphi \vee \psi & \text{stands for} & (\varphi \rightarrow \psi) \rightarrow \psi \\ \varphi \wedge \psi & \text{stands for} & \neg(\neg\varphi \vee \neg\psi) \\ \varphi \leftrightarrow \psi & \text{stands for} & (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \end{array}$$

An *evaluation* of atoms is now a mapping of atomic propositions into $[0, 1]$. Such mappings can be extended uniquely to an evaluation of all formulas. The following are *axioms of Łukasiewicz's logic*:

Ł1: $\varphi \rightarrow (\psi \rightarrow \varphi)$.

Ł2: $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$.

Ł3: $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$.

Ł4: $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$.

The *deduction rule* of Ł is modus ponens.

Łukasiewicz's logic satisfies an standard completeness theorem, i. e. Ł $\vdash \varphi$ iff φ is a tautology over the standard MV-algebra, i. e. the real interval $[0, 1]$ with Lukasiewicz's operations as truth functions. One inconvenience is the fact that the usual strong completeness of theories fails for Łukasiewicz's logic.

Rational Pavelka Logic (RPL) is an extension of Lukasiewicz logic by adding to language a truth constant $\overline{r}$ for each rationals $r \in [0, 1]$ and to axiomatic the following two bookeeping axioms for truth constants:

r1: $\neg(\bar{r}) \equiv \overline{(1-r)}$.

r2: $\bar{r} \to \bar{s} \equiv \overline{r \otimes\!\!\longrightarrow s}$.

An evaluation e of propositional variables by reals from $[0,1]$ extends to an evaluation of all formulas as in Lukasiewicz logic provided that $e(\overline{r}) = r$ for each rational r.

The completeness of Łukasiewicz logic extends to RPL but in this case a theorem of strong completeness can be obtained. Two principal notion will be introduced now.

Definition 1 *Let T be a theory and define the* truth degree *of a formula φ in T as $||\varphi||_T = \inf\{e(\varphi) \mid e$ is a model of $T\}$, and* the provability degree *of φ over T as $|\varphi|_T = \sup\{r \mid T \vdash \overline{r} \to \varphi\}$.*

Then the completeness of RPL says that the provability degree of φ in T is just equal to the truth degree of φ over T, this is, $||\varphi||_T = |\varphi|_T$.

3 The Logic SŁMV

The languaje of SŁMV is this of RPL logic plus the modality $\Box$ ($\Diamond$definable). We denote this language as $\mathcal{L}$. Now, we present our SŁMV modal logic as follows:

Definition 2 *The similarity-based Lukasiewicz's many-valued modal logic $SLMV$ is an extension of Rational Pavelka Logic (RPL) by adding the following axioms:*

K$_r$: $\Box(\bar{r} \to \varphi) \to (\Box\bar{r} \to \Box\varphi)$.

T : $\Box\varphi \to \varphi$.

B : $\varphi \to \Box\Diamond\varphi$.

4 : $\Box\varphi \to \Box\Box\varphi$.

Def: $\neg\Box\neg\varphi \equiv \Diamond\varphi$.

Equ1: $\bar{r} \equiv \Box\bar{r}$.

Equ2: $\Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$.

Deductions rules are modus ponens and the following rule:

RR: *From $\varphi \to \psi$ infer $\Box\varphi \to \Box\psi$.*

Note that the known necessitation rule:

RN: From φ infer $\Box\varphi$.

is a particular case of RR.

Next, we define the notion of a similarity Kripke model and state the truth and validity conditions for modal sentences in a world, in a model and in a class of models.

4 Similarity Kripke Models

According to the kripkean idea, a sentence is possibly true at the actual world if it is true in some world accesible from it. Intuitively, a world is a full interpretation of all atomic formulas $p_1, p_2, \ldots$. Hence, two different worlds determine two different assignment of truth to $p_1, p_2, \ldots$.

Definition 3 *Similarity Kripke models are structures $\mathcal{M} = \langle W, S, e\rangle$ where:*

W : *is a non-empty set of objects that we call worlds.*

$S : W \times W \longrightarrow [0,1]$ *is a similarity function, reflexive, symmetric and $\otimes$-transitive, where $\otimes$ is a continuous t-norm.*

$e : Var \times W \longrightarrow [0,1]$ *is a valuation function assigning to each propositional variable φ in Var and each world w in W a truth value $e(\varphi, w)$. The valuation e is extended to all formulas in $\mathcal{L}$, any world w, a continuous t-norm $\otimes$ and its respective residue $\otimes\!\!\longrightarrow$ as follows:*

1. $e(\varphi \to \psi, w) = e(\varphi, w) \otimes\!\!\longrightarrow e(\psi, w)$.
2. $e(\neg\varphi, w) = e(\varphi, w) \otimes\!\!\longrightarrow 0$.
3. $e(\Box\varphi, w) = \inf_{w'} \{S(w, w') \otimes\!\!\longrightarrow e(\varphi, w')\}$.
4. $e(\Diamond\varphi, w) = \sup_{w'} \{S(w, w') \otimes e(\varphi, w')\}$.

Taking into account the Łukasiewicz's truth functions for $\to$ and $\neg$(i.e. $\neg r = 1 - r$ and $r \to s = \min(1, 1 - r + s)$) , it is easy to check that the truth functions for the above connectives are the following ones:

$$\begin{aligned}
&e(\varphi \to \psi, w) = \min(1, 1 - e(\varphi, w) + e(\psi, w)).\\
&e(\neg\varphi, w) = 1 - e(\varphi, w).\\
&e(\Box\varphi, w) = \inf_{w'} \{\min(1, 1 - S(w, w') + e(\varphi, w')\}.\\
&e(\Diamond\varphi, w) = \sup_{w'} \{\max(0, S(w, w') + e(\varphi, w') - 1\}.
\end{aligned}$$

and for the derived connectives:

$$\begin{aligned}
e(\varphi \,\&\, \psi, w) &= \max(0, e(\varphi, w) + e(\psi, w) - 1).\\
e(\varphi \veebar \psi, w) &= \min(e(\varphi, w) + e(\psi, w), 1).\\
e(\varphi \wedge \psi, w) &= \min(e(\varphi, w), e(\psi, w)).\\
e(\varphi \vee \psi, w) &= \max(e(\varphi, w), e(\psi, w)).\\
e(\varphi \leftrightarrow \psi, w) &= \min(1 - e(\varphi, w) + e(\psi, w), 1 - e(\psi, w) + e(\varphi, w)).
\end{aligned}$$

We shall write "$(\mathcal{M}, w) \models \varphi$" to mean that φ is true at the possible world w in the model $\mathcal{M}$.

The notion of satisfiability in a model is formalized next.

Definition 4 *Let w be a world in a model $\mathcal{M} = \langle W, S, e\rangle$ then:*

$$(\mathcal{M}, w) \models \varphi \;\; \textit{iff} \;\; e(\varphi, w) = 1.$$

The notions of a formula being valid in a model and in a class of models is as usual.

Definition 5 *A formula φ is valid in a model $\mathcal{M}$, written $\mathcal{M} \models \varphi$, iff for every world w in $\mathcal{M}$ it holds that $(\mathcal{M}, w) \models \varphi$. A formula φ is valid in a class of models $\mathcal{C}$, written $\models_{\mathcal{C}} \varphi$, if it is valid in every model $\mathcal{M} \in \mathcal{C}$.*

In this paper we shall focus our attention on some particular classes of similarity models, defined by $\otimes$-similarity relations, where $\otimes$ is the Łukasiewicz's t-norm, with values on $[0, 1]$.

Definition 6 *Given a t-norm binary operation $\otimes$ on $[0, 1]$, we define the class of structures $\mathcal{C}_{\otimes}$ as the set of similarity structures $\mathcal{M} = \langle W, S, e\rangle$ where S is a $\otimes$-similarity on W.*

Now we will prove that the logic $S\text{Ł}MV$ is sound with respect to the class $\mathcal{C}_{\text{Ł}}$ of structures $\mathcal{C}_{\otimes}$ for the Łukasiewicz's t-norm $\otimes_{\text{Ł}}$.

5 Soundness

Proposition 1 *The axiom schemes of Definition 2 are valid in the class $\mathcal{C}_{\text{Ł}}$, for the Lukasiewicz's t-norm on $[0, 1]$. Furthemore, the rules that were presented in that definition preserve validity in $\mathcal{C}_{\text{Ł}}$.*

Proof. We may pay attention only to the modal formulas because the rest of the formulas correspond to the classical axiomatization of Łukasiewisz's calculus (for a more detailed account see [5]).

K$_r$. $e((\Box(\bar{r} \to \varphi) \to (\Box\bar{r} \to \Box\varphi)), w) = 1$ if and only if $e(\Box(\bar{r} \to \varphi), w) \leq e(\Box\bar{r} \to \Box\varphi, w)$. Now we take the left side and we write its definition:

$$\inf_{w1} \{S(w, w1) \otimes\!\!\longrightarrow (e(\bar{r}, w1) \otimes\!\!\longrightarrow e(\varphi, w1))\}$$

and by classical properties in the Łukasiewicz's logic we obtain that the last expression is equal to:

$$\inf_{w1} \{e(\bar{r}, w1) \otimes\!\!\longrightarrow (S(w, w1) \otimes\!\!\longrightarrow e(\varphi, w1))\}$$

and as the true value $e(\bar{r}, w1)$ is r for every worlds, it results:

$$\inf_{w1} \{r \otimes\!\!\longrightarrow (S(w, w1) \otimes\!\!\longrightarrow e(\varphi, w1))\} =$$
$$r \otimes\!\!\longrightarrow \inf_{w1} \{(S(w, w1) \otimes\!\!\longrightarrow e(\varphi, w1))) = r \otimes\!\!\longrightarrow e(\Box\varphi, w))\}$$

and by considering $\bar{r} \equiv \Box\bar{r}$ by Equ1, we have the proof that we wanted.

T. $e(\Box\varphi \to \varphi, w) = 1$ if and only if $e(\Box\varphi, w) \le e(\varphi, w)$. By Definition 3 we have:

$$e(\Box\varphi, w) = \inf_{w'} \{S(w,w') \otimes\!\!\longrightarrow e(\varphi, w')\} \le S(w,w) \otimes\!\!\longrightarrow e(\varphi, w) = e(\varphi, w)$$

since $S(w, w) = 1$.

B. $e(\varphi \to \Box\Diamond\varphi, w) = 1$ if and only if $e(\varphi, w) \le e(\Box\Diamond\varphi, w)$. Now by using Definition 3 over the right hand side we obtain:

$$\inf_{w1} \{S(w, w1) \otimes\!\!\longrightarrow \sup_{w2} (S(w1, w2) \otimes e(\varphi, w2))\}$$

and in particular that is greater than or equal to:

$$\inf_{w1} \{S(w, w1) \otimes\!\!\longrightarrow (S(w1, w) \otimes e(\varphi, w))\}$$

and by proposition 1.17 from [1] and continuity of $\otimes_{Ł}$, the last expression is greater or equal than:

$$\inf_{w1} \{S(w, w1) \otimes\!\!\longrightarrow S(w1, w)\} \otimes e(\varphi, w)$$

and that is equal to $e(\varphi, w)$ since for all $w1 : S(w, w1) \otimes\!\!\longrightarrow S(w1, w) = 1$ due to the symmetry of S.

4. $e(\Box\varphi \to \Box\Box\varphi, w) = 1$ if and only if $e(\Box\varphi, w) \le e(\Box\Box\varphi, w)$. As S is transitive for all $w, w1, w2$ result $S(w, w1) \ge S(w, w2) \otimes S(w2, w1)$ and by general properties of $\otimes\!\!\longrightarrow$ we get:

$$S(w, w1) \otimes\!\!\longrightarrow e(\varphi, w1) \le (S(w, w2) \otimes S(w2, w1)) \otimes\!\!\longrightarrow e(\varphi, w1) = S(w, w2) \otimes\!\!\longrightarrow (S(w2, w1) \otimes\!\!\longrightarrow e(\varphi, w1))$$

and as we are considering $\otimes = \otimes_{Ł}$ and in this case $\otimes_{Ł}\!\!\longrightarrow$ is continuous,

$$\inf_{w1} \{S(w, w1) \otimes_{Ł}\!\!\longrightarrow e(\varphi, w1)\} \le S(w, w2) \otimes_{Ł}\!\!\longrightarrow \inf_{w1} \{S(w2, w1) \otimes_{Ł}\!\!\longrightarrow e(\varphi, w1)\}$$

since the last inequation holds true for any $w2$ one has

$$\inf_{w1} \{S(w, w1) \otimes_{Ł}\!\!\longrightarrow e(\varphi, w1)\} \le \inf_{w2} \{S(w, w2) \otimes_{Ł}\!\!\longrightarrow \inf_{w1} \{S(w2, w1) \otimes_{Ł}\!\!\longrightarrow e(\varphi, w1)\}\}.$$

Def. $e(((\neg\Box\neg\varphi) \equiv \Diamond\varphi), w) = 1$ if and only if $e(\neg\Box\neg\varphi, w) = e(\Diamond\varphi, w)$. Using Definition 3 we obtain:

$$e(\neg\Box\neg\varphi, w) = (\inf_{w'} \{S(w,w') \otimes_{\text{Ł}}\!\!\longrightarrow (e(\varphi, w') \otimes_{\text{Ł}}\!\!\longrightarrow 0))\} \otimes_{\text{Ł}}\!\!\longrightarrow 0 =$$
$$(\inf_{w'}\{(S(w,w') \otimes_{\text{Ł}} e(\varphi, w')) \otimes_{\text{Ł}}\!\!\longrightarrow 0)\} \otimes_{\text{Ł}}\!\!\longrightarrow 0 =$$
$$(\sup_{w'}\{S(w,w') \otimes_{\text{Ł}} e(\varphi, w')\} \otimes_{\text{Ł}}\!\!\longrightarrow 0) \otimes_{\text{Ł}}\!\!\longrightarrow 0 = e(\Diamond\varphi, w)$$

Equ1. $e(\bar{r} \equiv \Box\bar{r}, w)$ if and only if $e(\bar{r}, w) = e(\Box\bar{r}, w)$ which is trivial by using Definition 3.

Equ2. $e((\Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi), w) = 1$ if and only if $e(\Box(\varphi \wedge \psi), w) = e((\Box\varphi \wedge \Box\psi), w) = \min(e(\Box\varphi, w), e(\Box\psi, w))$. But that is trivial if we recall the equivalence: $a \otimes_{\text{Ł}}\!\!\longrightarrow (b \wedge c) = \min(a \otimes_{\text{Ł}}\!\!\longrightarrow b, a \otimes_{\text{Ł}}\!\!\longrightarrow c)$.

RR. From $\varphi \rightarrow \psi$ infer $\Box\varphi \rightarrow \Box\psi$. If suppose that for all w $e(\varphi \rightarrow \psi, w) = 1$, results that $e(\varphi, w) \leq e(\psi, w)$. As Definition 3 $e(\Box\varphi, w) = \inf_{w1} \{S(w, w1) \otimes\!\!\longrightarrow e(\varphi, w1)\}$ and since all t-norm satisfies if $v \leq u$ then $w \otimes\!\!\longrightarrow v \leq w \otimes\!\!\longrightarrow u$, we have $e(\Box\varphi, w) \leq e(\Box\psi, w)$.

So we have proved the soundness of our $S\text{Ł}MV$ logic. Now we prove a stronger result.

Definition 7 *The* provability degree *of a formula φ in a system Σ is* $|\varphi|_\Sigma = \sup\{r \mid \vdash_\Sigma \bar{r} \rightarrow \varphi\}$. *The* truth degree *of φ in a class of model $\mathcal{C}$ is* $\|\varphi\|_C = \inf\{e(\varphi) \mid e \in \mathcal{C}\}$.

Lemma 2 *(strong soundness)* $|\varphi|_{SLMV} \leq \|\varphi\|_{\mathcal{C}_L}$, *i.e. whenever* $\vdash_{SLMV} \bar{r} \rightarrow \varphi$ *then* $\|\varphi\|_{\mathcal{C}_L} \geq r$.

Our next job is to prove completeness.

6 Completeness

As usual, to prove completeness it suffices to show that if a formula φ is $\mathcal{C}_{\text{Ł}}$-valid then there is a proof of φ in $S\text{Ł}MV$. The technique that we use here consists in showing that for any formula φ which is not 1-provable in $S\text{Ł}MV$ (i.e. $|\varphi|_{SLMV} < 1$) we can make a model $\mathcal{M} = \langle W, S, e\rangle$ of the class $\mathcal{C}_{\text{Ł}}$ which does not valid φ (i.e. $\|\varphi\|_{\mathcal{M}} < 1$). For such a construction, we have to note that if the formula φ is not 1-provable in $S\text{Ł}MV$ then there exists a rational $r < 1$ such that $\nvdash_{SLMV} \bar{r} \rightarrow \varphi$ and then we can propose a theory T_0 consistent with $\{\varphi \rightarrow \bar{r}\}$. The above mensioned model $\mathcal{M} = \langle W, S, e\rangle$ is made by extending of all possible manner (consistent and complete) the theory T_0. Before presenting the construction of this models, we need to introduce the concepts of consistent and complete sets of formulas in $S\text{Ł}MV$. The following definitions and lemmas are particular case of ones present by Novak and Hajek:

Definition 8 *A theory in $S\text{Ł}MV$ is a set T of formulas containing the $S\text{Ł}MV$-logic and closed under modus ponens.*

Definition 9 *([6]) A theory is contradictory in $S\L MV$ (or inconsistent-$S\L MV$) if for some rational $r > 0$ there is a proof in $S\L MV$ from T whose last element is $\bar{r} \to \bot$ (we notate it as $T \vdash_{SLMV} \bar{r} \to \bot$); otherwise it is consistent.*

Definition 10 *([2]) A theory T is complete in $S\L MV$ (complete-$S\L MV$) if for each rational r and each formula φ, there is a proof in $S\L MV$ from T whose last element is $\bar{r} \to \varphi$ or $\varphi \to \bar{r}$ (we notate it as $T \vdash_{SLMV} \bar{r} \to \varphi$ or $T \vdash_{SLMV} \varphi \to \bar{r}$).*

Lemma 3 *([2]) For each consistent-$S\L MV$ set of formulas T there is a stronger theory $\hat{T}$ which is consistent-$S\L MV$ and complete-$S\L MV$.*

Definition 11 *([2]) If T is consistent- $S\L MV$ and complete-$S\L MV$, for each φ, let $e_T(\varphi) = sup\{r|T \vdash_{SLMV} \bar{r} \to \varphi\} = inf\{r|T \vdash_{SLMV} \varphi \to \bar{r}\}$*

Lemma 4 *([2]) If T is consistent and complete and e_T is as above then e_T is a valuation, i.e. $e_T(\neg\varphi) = 1 - e_T(\varphi)$, $e_T(\varphi \to \psi) = (e_T(\varphi) \otimes\!\!\longrightarrow e_T(\psi))$.*

Note that the last lemma nothing is saying about the modal sentences. Each valuation e_T considers a modal sentence as an atomic formula and in this sense, the last lemma is truth in $S\L MV$. Now let us consider the following definitions and lemmas.

Definition 12 *A boxed formula is a formula begining by $\Box$.*

Lemma 5 *If $\nvdash_{SLMV} \bar{s} \to \psi$ then there is a consistent theory T_0 extending $\{\psi \to \bar{s}\}$ such that for each boxed formula φ and each truth rational constant $\bar{r}$, $(\bar{r} \to \varphi) \in T_0$ or $(\varphi \to \bar{r}) \in T_0$ (T_0 is complete for boxed formulas).*

Proof. If $\nvdash_{SLMV} \bar{s} \to \psi$ then $\bar{s} \to \psi \notin \Theta = \{\gamma \mid \vdash \gamma\}$ (e.i. the set of formulas provable on our logic) and Θ is a theory. It follows that the closure Θ' of $\Theta \cup \{\psi \to \bar{s}\}$ is a consistent theory. As we may arrange all boxed formula into a sequence $b_0, b_1, b_2, ...$ and apply the usual technique of completion to get the wanted theory T_0. ■

Lemma 6 *For T_0 as above, if $|\Box\psi|_{T_0} = r1$ and $|\Diamond\psi|_{T_0} = r2$ then there is a consistent and complete $T \supseteq T_0$ such that for any rational $r1 \leq r \leq r2$, $\bar{r} \to \psi \in T$.*

Proof. It is trivial from definitions and is left to the reader. ■

Given a formula φ, we consider the following model $\mathcal{M}_\varphi = \langle W_\varphi, S_\varphi, e_\varphi \rangle$ defined as:

- W_φ is the set of index for each consistent complete extension of T_0 (e.i. $W_\varphi = \{w \mid T_0 \subseteq T_w$ and T_w is complete and consistent$\}$).
- $S_\varphi(w, w') = \inf_\psi (|\Box\psi|_{T_w} \otimes\!\!\longrightarrow |\psi|_{T_{w'}})$.
- $e_\varphi(p_i, w) = |p_i|_{T_w}$ for each propsitional variable p_i.

Remark 1 *Note that since T_0 is complete with respect to boxed formulas the provability degree of these is independent of w.*

Theorem 7 **(truth lemma)** *Given a formula φ, the model $\mathcal{M}_\varphi$ satisfies :*

$$\mathcal{M}_\varphi \models \psi \textit{ iff for every: } w \in W_\varphi,\ |\psi|_{T_w} = 1$$

Proof. Only is neccessary to prove that for each world w in W_φ and all formulas ψ, $e_\varphi(\psi, w) = |\psi|_{T_w}$. The proof is by induction in the complexity of ψ. If ψ is not a boxed formula then the proof is similar to one due to Hajek ([2]). Hence, here only one case is considered, that of necesitation.

Given a boxed formula $\Box\psi$, assume that for ψ and for all worlds $w \in W_\varphi$: $e_\varphi(\psi, w) = |\psi|_{T_w}$. We show $e_\varphi(\Box\psi, w) = |\Box\psi|_{T_w}$. First note that by definition, $S_\varphi(w, w') \leq |\Box\psi|_{T_w} \otimes\!\!\longrightarrow |\psi|_{T_{w'}}$. It follows by standard properties of $\otimes\!\!\longrightarrow$ that $|\Box\psi|_{T_w} \leq S_\varphi(w, w') \otimes\!\!\longrightarrow |\psi|_{T_{w'}}$. Thus:

$$e_\varphi(\Box\psi, w) =_{def} \inf_{w'} (S_\varphi(w, w') \otimes\!\!\longrightarrow e(\psi, w')) =_{H.I.} \inf_{w'} (S_\varphi(w, w') \otimes\!\!\longrightarrow |\psi|_{T_{w'}}) \geq |\Box\psi|_{T_w}$$

For the another direction of the inequation, note that by definition $S_\varphi(w, w') = \inf_\psi (|\Box\psi|_{T_w} \otimes\!\!\longrightarrow |\psi|_{T_{w'}})$, hence $S_\varphi(w, w') = 1$ for every w and w'. That is so because T_0 is complete with respect to boxed formulas and contains the axiom T.

Hence $e_\varphi(\Box\psi, w) = \inf_{w'} (S_\varphi(w, w') \otimes\!\!\longrightarrow e(\psi, w')) = \inf_{w'} (e(\psi, w'))$. Now suppose that $|\Box\psi|_{T_w} < e_\varphi(\Box\psi, w)$ (note that by construction and the last observation both sides of last expresion are independent of w). Then there exists a rational z such that $|\Box\psi|_{T_w} < z < e_\varphi(\Box\psi, w)$. By the above observation for every world w : $z < e_\varphi(\Box\psi, w) \leq e_\varphi(\psi, w)$. Thus $\bar{z} \to \psi$ is in T_w for every world w, and therefore it is also in T_0. For Lemma 6 $\bar{z} \to \psi$ must be either logically provable or ψ is either a boxed formula or the negation of a boxed formula. Now, we prove that in the three case we can arrive to the absurdity that $z \leq |\Box\psi|_{T_w}$. In the first case, by applying necessitation we know $\Box(\bar{z} \to \psi)$ is in T_0 and by a modus ponens proof with this formula and K_r axiom we conclude that $\bar{z} \to \Box\psi$ is in T_0. If $\psi = \Box\phi$ then by applying modus ponens rule

from **5:**$(\Box\phi \to \Box\Box\phi)$, $(\bar{z} \to \Box\phi)$ and **Ł2:** $(\Box\phi \to \Box\Box\phi) \to ((\bar{z} \to \Box\phi) \to (\bar{z} \to \Box\Box\phi))$

we infer that $\bar{z} \to \Box\Box\phi = \bar{z} \to \Box\psi$ is in T_0. Finally if $\psi = \Diamond\phi$ (i.e. the negation of a boxed formula) then by noting that $\Diamond\phi \to \Box\Diamond\phi$ is in $S{\L}MV$ (for that consider both a particular instance $\Diamond\phi \to \Box\Diamond\Diamond\phi$ of axiom **B**, and the converse of axiom **5** as $\Diamond\Diamond\phi \to \Diamond\phi$) and applying modus ponens

from $(\Diamond\phi \to \Box\Diamond\phi)$, $(\bar{z} \to \Diamond\phi)$, and **Ł2:** $(\Diamond\phi \to \Box\Diamond\phi) \to ((\bar{z} \to \Diamond\phi) \to (\bar{z} \to \Box\Diamond\phi))$

we get $\bar{z} \to \Box\Diamond\phi = \bar{z} \to \Box\psi \in T_0$, which completes the wanted contradiction . ■

Theorem 8 *(Completeness)* $\|\varphi\|_{\mathcal{C}_L} \leq |\varphi|_{SLMV}$ *, i.e. whenever* $\|\varphi\|_{\mathcal{C}_L} \geq r$ *then* $\vdash_{SLMV} \bar{r} \to \varphi$.

In particular, φ is a 1-tautology if and only if, for each rational $r < 1$, $\vdash_{SLMV} \bar{r} \to \varphi$.

Proof. We suppose that there exists $r < 1$ such that $\nvdash_{SLMV} \bar{r} \to \varphi$ and contruct a model $\mathcal{M}_\varphi$ as above. Then $\|\varphi\|_{\mathcal{M}_\varphi} \leq r$. ■

7 Conclusions

The study here developed is related to investigation of fuzzy logic in the narrow sence, e.i. fuzzy logic as a logical calculus, which is a many-valued generalization of classical logic suitable to deal wich impreceseness (vagueness). The based difference between logics of vagueness as many-valued systems and logics of uncertainty as modal systems has been stressed by many authors. We try to build logics of both vagueness and uncertainty. Here we present a modal logic over a Pavelka-like extension of the infinitely valued Łukasiewicz's logic, which is a many-valued counterpart of classical S5 modal system. This logic is more general than one due to Hájek ([3]).

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