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Maximum entropy, Lie algebras and quantum thermodynamics

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An exactly solvable quantum many-fermion system with an arbitrarily strong two-body interaction is studied and some exact thermodynamic functions (in the thermodynamic limit) are derived within the framework of the statistical inference scheme based on information theory. The solution for the associated su(3) Clebsch–Gordan series (for any number of particles) is given. A very important relation between the (many-body) system's entropy per particle (in the thermodynamic limit) and the multiple Kronecker product multiplicities (for any member of an infinite class of Lie-algebraic exactly solvable models) is demonstrated. A general procedure for the treatment of the full class of solvable models is outlined.

1. Introduction

A good deal of our present understanding of the physics of quantum many-body systems has been gained by recourse to the study of exactly solvable models that are able to successfully mimic some of the more salient features of these fascinating systems [1].

Foremost among these solvable models one finds, of course, that one proposed many years ago by Meshkov, Glick and Lipkin [2], that has become the customary testing ground for most of the novel ideas [3] that, in the intervening years, have purported to illuminate new aspects of the intricate quantum many-body problem. However, this su(2) model, its enormous importance notwithstanding, is too simple to accommodate some facets of this problem, and it is our aim here to present the essentials of a more general theory of exactly solvable (MGL-like) many-fermion models obtained as a direct extrapolation of the su(2) MGL model. We will center our attention in those aspects related to the statistical descriptions of these systems, with emphasis upon the Lie-algebraic techniques necessary to obtain an exact solution. As an example, we also discuss some interesting new traits observed in the thermodynamic behavior of an su(3) MGL model.

The proper generalization of the MGL model from two to three (or more) energy levels must be tackled *by following a path that deviates from the Lipkin one on which it was based* (basically, on the properties of the well-known su(2) angular momentum algebra, the simplest classical Lie algebra). The historical path allowed the original authors to circumvent the detailed exposition of major mathematical (group theoretical) results while centering the attention in the more "physical" aspects of the concomitant problem.

In the present paper, which is inspired by the illuminating work of Gilmore and coworkers [4], we will deal mainly with a three (energy) level system with Ω identical fermions. We will construct an exactly (MGL-like) solvable model by the incorporation of the permutation symmetry in the Hamiltonian of the system, so that additional integrals of motion will arise [2]. We will also summarize a few general mathematical results that will enable the reader to reproduce most steps of the complete procedure needed in order to deal with any other n-single particle quantum numbers, MGL-like, Ω fermion model, either in the restricted Hilbert space, of dimension n^{Ω}, or in the complete Fock space, of dimension $2^{n\Omega}$.

The Hamiltonian of a many-particle quantum system interacting via two-body forces is a sum of linear and quadratic terms in the bilinear products of creation and annihilation operators for the considered (quasi)particles, and can thus be expressed as a function of the basis operators of a semisimple classical Lie algebra.

It is well known that the set of bilinear products of the creation and annihilation operators of n-class fermions realize an so($2n$) Lie algebra [5,6]. Moreover, any classical Lie algebra can be realized by linear and bilinear products of operators that create and destroy particles (either fermions or bosons) [7]. The fermion case, which encompasses many interesting systems of many-body strongly interacting identical particles, arises in a natural fashion within the context of the nuclear shell model theory [8].

Among these algebras, one can single out su(3) as the simplest one that is endowed with both the intrinsically vectorial nature of its structure and the multiple character of its weights. A model constructed on the basis of the su(3) algebra acquires thus a relevant paradigmatic value.

In the three-level many-body problem to be discussed here we will constrain the complete Hilbert space of the system, of dimension $\binom{3\Omega}{\Omega}$, to a "restricted Hilbert space", a Hilbert subspace of dimension 3^{Ω}, as in the su(2) original MGL model, and thereby consider an su(3) Lie algebra in order to simplify the three-level model. Otherwise, we would be led to the more complex so(6) classical semisimple Lie algebra. (An n-level, MGL-like model of many fermions in a restricted Hilbert space gives rise to an su(n) classical Lie algebra.)

Despite this restriction, the su(3) model can be regarded as a valuable paradigm of minimal complexity, which may be useful in dealing with much more involved and perhaps more realistic MGL models which, on the basis of the underlying representation theory, will exhibit just the following two generic properties, that appear in the su(3) case and are absent in the simpler su(2) or su(2) $\oplus$ su(2) [9] models:
(i) Quantities of intrinsically vectorial nature.
(ii) A multiple number of weights in most of the irreducible representations (IRs) of the associated algebra $\mathscr{L}$.

It has to be noted that the treatment of the two-level Lipkin model is usually further restricted to a subspace of dimension $\Omega + 1$ that corresponds to the su(2) IR which contains the ground state of many-fermion system [2].

The precise fashion in which the space of states for the (composite) system is constructed on the basis of the state space of its constituents becomes the essential trait in the concomitant formulation, and the Kronecker product representation the main mathematical device. This construction coincides, of course, with that used, albeit only implicitly, in the su(2) MGL original model [2].

This work also aims at paving a direct road between the realms of finite-level MGL-like exactly solvable quantum many-body models and their concomitant statistical behavior. In particular, some interesting thermodynamical facets will be discussed for the simplest nontrivial su(3) MGL-like Hamiltonian, that are entirely absent in the analogous two-level situation.

2. Remarks on the mathematical environment

Although the forthcoming remarks are of a very general nature, it is perhaps easier to follow the presentation by first introducing the "quasispin-like" operators for the su(3) case.

Let a_{pi} be the annihilation operator for a fermion in a single-particle state characterized by two quantum numbers, $p = 1, \ldots, \Omega$ and $i = 1, 2, 3$. The total "quasispin-like" operators are defined by the relations

$$\hat{h}_1 = \frac{1}{2} \sum_p (a^\dagger_{p1} a_{p1} - a^\dagger_{p2} a_{p2}) , \qquad \hat{h}_2 = \frac{1}{3} \sum_p (a^\dagger_{p1} a_{p1} + a^\dagger_{p2} a_{p2} - 2 a^\dagger_{p3} a_{p3}) , \tag{1.a}$$

$$\hat{e}_{\alpha 1} = \sum_p a^\dagger_{p1} a_{p2} = \hat{e}^\dagger_{-\alpha 1} , \qquad \hat{e}_{\alpha 2} = \sum_p a^\dagger_{p2} a_{p3} = \hat{e}^\dagger_{-\alpha 2} ,$$
$$\hat{e}_{\alpha 3} = \sum_p a^\dagger_{p1} a_{p3} = \hat{e}^\dagger_{-\alpha 3} , \tag{1.b}$$

where the notation employed (not the usual one within the present context [1]) for these operators, that can be traced back to the old Sakata model [5], emphasizes the mathematical background one is referring to here.

Due to the fermionic character of the creation and annihilation operators, the operators (1) satisfy "su(3)" commutation rules. More exactly, these operators realize, when one considers a real numerical field, one of the real forms of the "complexification" [10] of the classical compact simple real Lie algebra su(3). (According to Cartan's classification, the operators (1) belong to the simple complex Lie algebra A_2.) We continue, for the sake of brevity, with the usual (less precise) terminology [10].

The complete list of nonvanishing commutators for these operators is [5]

$$
\begin{aligned}
&[\hat{h}_1, \hat{e}_{\pm\alpha 1}] = \pm\hat{e}_{\pm\alpha 1}\,, && [\hat{e}_{\pm\alpha 1}, \hat{e}_{\pm\alpha 2}] = \pm\hat{e}_{\pm\alpha 3}\,,\\
&[\hat{h}_1, \hat{e}_{\pm\alpha 2}] = \mp\tfrac{1}{2}\hat{e}_{\pm\alpha 2}\,, && [\hat{e}_{\pm\alpha 1}, \hat{e}_{\mp\alpha 3}] = \mp\hat{e}_{\mp\alpha 2}\,,\\
&[\hat{h}_1, \hat{e}_{\pm\alpha 3}] = \pm\tfrac{1}{2}\hat{e}_{\pm\alpha 3}\,, && [\hat{e}_{\pm\alpha 3}, \hat{e}_{\mp\alpha 2}] = \pm\hat{e}_{\pm\alpha 1}\,,\\
&[\hat{h}_2, \hat{e}_{\pm\alpha 2}] = \pm\hat{e}_{\pm\alpha 2}\,, && [\hat{e}_{+\alpha 1}, \hat{e}_{-\alpha 1}] = 2\hat{h}_1\,,\\
&[\hat{h}_2, \hat{e}_{\pm\alpha 3}] = \pm\hat{e}_{\pm\alpha 3}\,, && [\hat{e}_{+\alpha 2}, \hat{e}_{-\alpha 2}] = -\hat{h}_1 + \tfrac{3}{2}\hat{h}_2\,,\\
& && [\hat{e}_{+\alpha 3}, \hat{e}_{-\alpha 3}] = +\hat{h}_1 + \tfrac{3}{2}\hat{h}_2\,.
\end{aligned}
\tag{2}
$$

The Cartan (maximal Abelian) subalgebra basis [10,11] can be chosen to consist of the two commuting operators $\hat{h}_1$ and $\hat{h}_2$, whose eigenvalues, $h_i = \lambda(\hat{h}_i)$, $i = 1, 2$, in any representation will yield all the weights belonging to that representation. Each possible set of eigenvalues h_i can be considered as determining the orthogonal components of a vector, usually called the weight vector. All possible weight vectors belong to the so-called weight space. The complete set of weights for a given irreducible representation (IR) can be derived from the highest weight vector, Λ, which labels that IR [11].

Any IR of a semisimple Lie algebra $\mathscr{L}$ of rank l may be uniquely characterized by l nonnegative integers, $n_1, \ldots, n_l$ [11]. This way of labelling appears naturally in Cartan's construction. (For the su$(l+1)$ algebras this labelling is related to the signature $(f_1, \ldots, f_l)$, $f_1 \geqslant \cdots \geqslant f_l$, which also specifies an su$(l+1)$ IR [12] by $n_j = f_j - f_{j+1}$, $j = 1, \ldots, l$.)

If $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ form the simple root system of the semisimple complex Lie algebra $\mathscr{L}$, then the l fundamental weights $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_l\}$, which also span the weight space, are given by

$$
\Lambda_j = \sum_{k=1}^{l} \alpha_k (A^{-1})_{kj}\,, \qquad A_{jk} = 2\frac{\alpha_j \cdot \alpha_k}{\alpha_k \cdot \alpha_k}\,,
\tag{3}
$$

where A is the Cartan matrix of $\mathscr{L}$ [11,10] and the "dot" product is defined by recourse to the Killing form on the Cartan subalgebra in the Weyl canonical basis [10]. This product takes the form of a common scalar product of vectors when an orthonormal algebra basis is used (see, for example, [13]) (this is not the case for the algebra basis used here which is chosen so that all structure constants are half-integer numbers).

The highest weight of any IR is expressed as

$$\Lambda = n_1\Lambda_1 + n_2\Lambda_2 + \cdots + n_l\Lambda_l \,, \tag{4}$$

where $\{n_i\}$ are nonnegative integers. This IR will be denoted by $[n_1, n_2, \ldots, n_l]$.

For the su(3) case the two fundamental weights are, by (3),

$$\Lambda_1 = \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2 \,, \qquad \Lambda_2 = \tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2 \,. \tag{5}$$

The dimension $d(\Lambda)$ of any IR of a semisimple Lie algebra is given by the Weyl dimensionality formula [14]

$$d(\Lambda) = \prod_{\alpha>0} \frac{\alpha\cdot(\Lambda+\delta)}{\alpha\cdot\delta} \,, \qquad \delta = \frac{1}{2}\sum_{\alpha>0}\alpha \,, \tag{6}$$

where the product is taken over all the positive roots α of the corresponding Lie algebra.

In terms of the weight vector components of the su(3) IR label, the dimension of the IR $\Lambda = [n_1, n_2]$ is, by (6),

$$d(\Lambda) = \tfrac{1}{2}(1+n_1)(1+n_2)(2+n_1+n_2) \,. \tag{7}$$

For any semisimple complex Lie algebra the second-order Casimir operator is defined by [15] (sum on repeated indices)

$$\hat{\mathbb{C}} = g_{ij}\hat{X}^i\hat{X}^j \,, \tag{8}$$

where $g_{ij} = c_{il}{}^k c_{jk}{}^l$ is a contravariant symmetric tensor, $c_{ij}{}^k$ are the structure constants of the algebra in the algebra basis $\{\hat{X}_i\}$ (cf. eq. (2)) and $\hat{X}_i = g_{ij}\hat{X}^j$.

In the standard (nonunique) su(3) algebra basis (1), the second-order Casimir operator is (eqs. (8) and (2))

$$\hat{\mathbb{C}} = \tfrac{1}{3}\hat{h}_1^2 + \tfrac{1}{4}\hat{h}_2^2 + \tfrac{1}{6}\sum_{i=1}^{3}(\hat{e}_{+\alpha_i}\hat{e}_{-\alpha_i} + \hat{e}_{-\alpha_i}\hat{e}_{+\alpha_i}) \,. \tag{9}$$

A general expression for the eigenvalue $\mathbb{C}(\Lambda)$ of the second-order Casimir operator in any IR Λ of a semisimple complex Lie algebra was derived by Racah [16]. This formula expresses $\mathbb{C}(\Lambda)$ explicitly in terms of the highest-weight vector Λ labelling the IR as a second-degree multinomial in the components of Λ as

$$\mathbb{C}(\Lambda) = \Lambda \cdot (\Lambda + 2\delta) , \qquad \delta = \tfrac{1}{2} \sum_{\alpha > 0} \alpha . \tag{10}$$

For the su(3) algebra the above expression reads

$$\mathbb{C}(\Lambda) = \tfrac{1}{9} [n_1^2 + n_2^2 + n_1 n_2 + 3(n_1 + n_2)] , \tag{11}$$

where $[n_1, n_2] = \Lambda$.

3. Kronecker product representations

The consideration of an n-level fermionic many-body problem in the restricted Hilbert space leads one to consider a specific su(n) (reducible) Kronecker product representation of dimension n^{Ω} built up from Ω defining IRs of the su(n) algebra.

The defining representation of su(n) is labelled by the components of the fundamental weight vector relative to the fundamental weight basis (3) as $[1, 0, \ldots, 0] = \Lambda_1$, and the algebra Kronecker product representation, which we will abbreviate as Λ_1^{Ω}, is given by (Ω terms, with Ω factors each)

$$\Lambda_1^{\Omega} \equiv \Omega \otimes \mathsf{I} \otimes \cdots \otimes \mathsf{I} + \mathsf{I} \otimes \Lambda_1 \otimes \cdots \otimes \mathsf{I} + \cdots + \mathsf{I} \otimes \mathsf{I} \otimes \cdots \otimes \Lambda_1 . \tag{12}$$

where I is the $n \times n$ unit matrix. Within the present context, this expression is not an easy one to find (as far as we know) for multiple products. In [10], however, a two-fold algebra Kronecker product expression is given. This is due to the privileged status this representation has (in particle physics), mainly in the calculation of transitions probabilities of two-particle scattering processes and related issues.

A full generalization of the representation structure (12) (and of the Lie-algebraic MGL-like exactly solvable models) which is induced by the system's space of states, restricted only by the requirement that it should lead to a physically viable model, can be given if one permits the defining representation of $\mathscr{L}$, Λ_1, to be any one of the fundamental IRs of the considered algebra (cf. eq. (3)). In the su(3) case, for example, the replacement of Λ_1 by Λ_2 in (12) would lead to a system of Ω fermion "holes" distributed among three single-particle energy levels. This is closely related with the description of antiparticles in particle physics (see, for example [10], and references therein).

Any representation of a semisimple Lie algebra $\mathscr{L}$ is fully reducible by Weyl's theorem, so that one can decompose that Ω-fold multiple Kronecker product Λ_i^{Ω}, with $i=1$, or $2,\cdots$, or l, as a direct sum of IRs of the algebra $\mathscr{L}$. This is expressed through the well-known product relation, which with this notation takes the form

$$\Lambda_i^{\Omega} \approx \sum_{\Lambda} \oplus \mathfrak{M}(\Omega;\Lambda)\,\Lambda\,, \tag{13}$$

where the summation runs over all nonequivalent IRs $\Lambda=[n_1, n_2, \ldots, n_l]$ of the su(n) algebra, and $\mathfrak{M}(\Omega;\Lambda)$ is the multiplicity of the representation Λ contained in the reduction (or Clebsch – Gordan series) (13) of the multiple Kronecker product. These multiplicities are called "Kronecker product multiplicities" or "external multiplicities", so as to distinguish them from the (internal) multiplicities of a weight within a given IR. They will become very important to us because of the relevant physical meaning of a related quantity of fundamental importance in the description and explanation of most natural phenomena, namely, the system's entropy. *This fundamental link between these two previously unrelated concepts will play a central role in our subsequent considerations.*

The important point in the possibility of an exact calculation of any statistical expectation value of a strong interacting many-particle system, within the framework of either the orthodox (classical) statistical mechanics or the conceptually broader information theory approach, as developed by Jaynes [22], is that, due to the Clebsch – Gordan decomposition (13), any trace (evaluated in the system's space of states) of any operator capable to be expressed as an analytical function of the basis operators of the semisimple Lie algebra $\mathscr{L}$ can be decomposed as a sum of partial traces on IRs of $\mathscr{L}$. For example, the canonical partition function can be cast as

$$Z_{\Omega}(\beta) = \sum_{\Lambda} \mathfrak{M}(\Omega;\Lambda)\,\mathrm{tr}[\exp(-\beta \mathsf{H}_{\Lambda})]\,, \tag{14}$$

where tr is the trace over each IR of the associated algebra $\mathscr{L}$ (a subspace of the restricted Hilbert space) and H_{Λ} is the matrix representative, of dimension $d(\Lambda)$ given by (6), of the system's MGL Hamiltonian $\hat{H}$, on the carrier space of the IR with highest weight Λ.

For the su(3) case, the operators describing a particle "jumping" from one single-particle state to another can be realized by linear combinations of the eight Hermitian traceless matrices λ_j, $j=1,\ldots,8$, introduced by Gell-Mann in his SU(3) symmetry scheme [17]. These matrices are the analogs of the Pauli spin matrices for the su(2) algebra.

Denoting by $\Gamma(\hat{X})$ the defining IR (i.e. the IR $\Lambda_1 = [1,0]$) of an su(3) basis element $\hat{X}$ defined in (1), the above correspondence may be established (by an appropriate similarity transformation) by the following relations:

$$\Gamma(\hat{h}_1) = \tfrac{1}{2}\,\lambda_3\,, \qquad \Gamma(\hat{h}_2) = \frac{1}{\sqrt{3}}\,\lambda_8\,, \tag{15.a}$$

$$\Gamma(\hat{e}_{\pm\alpha_1}) = \tfrac{1}{2}(\lambda_1 \pm \mathrm{i}\lambda_2)\,, \qquad \Gamma(\hat{e}_{\pm\alpha_2}) = \tfrac{1}{2}(\lambda_6 \pm \mathrm{i}\lambda_7)\,,$$
$$\Gamma(\hat{e}_{\pm\alpha_3}) = \tfrac{1}{2}(\lambda_4 \pm \mathrm{i}\lambda_5)\,. \tag{15.b}$$

This explicit realization for the defining su(3) IR is naturally related to the Fock representation space of the operators of creation and annihilation of particles, $a^\dagger_{pi}$, a_{pi}, and can be obtained using the Condon and Shortly phase convention [18] for the $\hat{e}_{\pm\alpha_1}$ and the $\hat{e}_{\pm\alpha_2}$ generated su(2) subalgebras. The relationship between these matrices resembles the relation between the sets of operators $\{J_x, J_y, J_z\}$ and $\{J_+, J_-, J_z\}$ of the well-known su(2) angular momentum theory.

The two Gell-Mann diagonal matrices (15.a) can be directly used for low values of Ω), in (12), to find any given Kronecker product representation by counting all its weights and degeneracies in the (two) resulting (diagonal) matrices. However, there exists a more direct and powerful procedure, whose rationale depends on the identification of the Ω-fold Kronecker product representation with the representation of the diagonal subalgebra [10] of the direct sum algebra $\mathrm{su}(3)\oplus\mathrm{su}(3)\oplus\cdots\oplus\mathrm{su}(3)$ (Ω times). This subalgebra is just the su(3) algebra realized by the operators (1).

Because of the fact that any representation is determined up to an equivalence, the eigenvalues $h_i = \lambda(\hat{h}_i)$, $i = 1, 2$, of the su(3) Cartan subalgebra basis (1.a) can simply be generated by assigning all the possible nonnegative integer values to the eigenvalues, N_i, of the number operators that count the number of particles in each Ω-fold degenerate single-particle state labelled with the index i, i.e.

$$\hat{N}_i = \sum_p a^\dagger_{pi} a_{pi}\,, \qquad i = 1, 2, 3\,, \tag{16}$$

with the constraint $N_1 + N_2 + N_3 = \Omega$, in the relations (cf. eq. (1.a))

$$h_1 = \tfrac{1}{2}(N_1 - N_2)\,, \qquad h_2 = \tfrac{1}{2}(N_1 + N_2 - 2N_3)\,. \tag{17}$$

These weights belong to a discrete (hexagonal lattice-like) domain contained in a triangular-shaped region of the weight space (whose size grows with Ω).

This procedure generates all the distinct weights of the multiple Kronecker product representation $[1,0]^{\Omega}$. Moreover, each weight appears in this reducible (if $\Omega > 1$) representation with a degeneracy given by

$$\mathfrak{N}(\Omega;\lambda) = \frac{\Omega!}{N_1(\lambda)!N_2(\lambda)!N_3(\lambda)!}, \tag{18}$$

which, owing to the multinomial formula, verifies

$$\sum_{\lambda} \mathfrak{N}(\Omega;\lambda) = 3^{\Omega}, \tag{19}$$

where the summation is taken over all the distinct weights of the Ω-fold Kronecker product representation. Thus, this counting exhaust all the quantum states of the three-level Ω-fermion system in the restricted Hilbert space.

For a given dominant λ, the $\mathfrak{N}(\Omega;\lambda)$ weights belonging to the (reducible for $\Omega > 1$) Ω-fold Kronecker product are distributed among the IRs (of any Lie-algebraic solvable model) in the following way:

$$\mathfrak{N}(\Omega;\lambda) = \sum_{\Lambda} m^{\Lambda}(\lambda)\, \mathfrak{M}(\Omega;\Lambda), \qquad \lambda \geq 0, \tag{20}$$

where $m^{\Lambda}(\lambda)$ is the (internal) multiplicity of the weight λ, which is zero if the weight λ does not belong to the IR labelled by its highest weight Λ.

In the present effort, a major and necessary result towards the complete statistical solution of the large class of su(3) exactly solvable models of the Ω-fermion system is here to be advanced, namely, the su(3) Ω-fold Kronecker product multiplicities. These are explicitly given in terms of the components' highest-weight vector, $\Lambda = [n_1, n_2]$, labelling each su(3) IR by [19]

$$\mathfrak{M}(\Omega;\Lambda) = \frac{(n_1+1)(n_2+1)(n_1+n_2+2)\Omega!}{[\frac{1}{3}(\Omega+2n_1+n_2)+2]![\frac{1}{3}(\Omega-n_1+n_2)+1]![\frac{1}{3}(\Omega-n_1-2n_2)]!}; \tag{21}$$

an original result as far as we know.

Any IR with highest weight Ω, such that any argument of the factorials becomes noninteger or negative, does not belong to the decomposition (i.e. $\mathfrak{M}(\Omega;\Lambda) = 0$).

For instance, the Clebsch–Gordan series (13) for $\Omega = 10$ is, by (21),

$$\begin{aligned}[1,0]^{10} \approx\ & [10,0] \oplus 9[8,1] \oplus 35[6,2] \oplus 75[4,3] \oplus 90[2,4] \oplus 42[0,5] \\ & \oplus 36[7,0] \oplus 160[5,1] \oplus 315[3,2] \oplus 288[1,3] \oplus 225[4,0] \\ & \oplus 450[2,1] \oplus 252[0,2] \oplus 210[1,0]\,.\end{aligned}$$

For finite Ω, any decomposition can also be calculated by recourse to less powerful iterative methods. However, in order to obtain the system's behavior in the thermodynamic limit, the explicit solution (21) or an asymptotic form of it (cf. eq. (28)) is needed.

The solution (21) for the su(3) Clebsch–Gordan series verifies the sum rule

$$\sum_{\Lambda} \mathfrak{M}(\Omega; \Lambda)\, d(\Lambda) = 3^{\Omega}\,, \tag{22}$$

with $d(\Lambda)$ given by (7).

The solution (21) is the specialization to su(3) of a general formula valid for any semisimple Lie algebra [19] which gives the solution for the Clebsch–Gordan series, in explicit, close form, as required to attain exact analytical as well as efficient numerical solutions to the statistical problem posed by this kind of n-level (or, more generally, n-single-particle state) Ω-fermion, MGL-like models, although it can be used, of course, in any context where a Clebsch–Gordan series for the tensor product of many IRs of any Lie algebra is required. With that formula, only the knowledge of the Ω-fold Kronecker product through its weights and degeneracies is required to this end. It is not necessary to find the weight's internal multiplicities $m^{\Lambda}(\lambda)$ for the IRs of $\mathcal{L}$ (a problem usually solved by recourse to the Freudenthal recursion formula [10,11]). This fact is a nice consequence of the underlying representation theory that allows for the saving of a great amount of tedious numerical (and bookkeeping) work that would otherwise be needed to obtain any nontrivial exact statistical prediction for a given MGL model with Ω finite (and not too large).

The expression (21) is the su(3) analog to the su(2) Wigner spin coupling result [20] that is used along with the MGL original model (see, for example, [3]).

4. The entropy [21]

The most general expression for the statistical operator $\hat{\rho}$ is given in terms of a set of (relevant) operators, $\{\hat{A}_k\}$, $k = 1, \ldots, \mathcal{N}$, whose expectation values are assumed to be (a priori) known [22,23]. It reads

$$\hat{\rho} = \exp\Big(-\lambda_0 - \sum_{k=1}^{\mathcal{N}} \lambda_k \hat{A}_k\Big) , \tag{23}$$

where λ_0 is a Lagrange multiplier so chosen as to guarantee normalization ($\mathrm{Tr}\, \hat{\rho} = 1$). The form (23) for the statistical operator is a consequence of the maximum entropy (MAXENT) variational principle [23,3,24] and is considered to yield the least biased probabilistic prescription, based on the sole knowledge of the relevant information supplied through a set of expectation values (and nothing else). It is asserted that, essentially, all the known results of statistical mechanics (equilibrium and off-equilibrium), are derivable consequences of this principle [24].

The multiplier λ_0 is a function of the remaining λ_k and is related to the generalized partition function by

$$\mathcal{Z}(\lambda_1, \lambda_2, \ldots, \lambda_{\mathcal{N}}) = \exp(\lambda_0) = \mathrm{Tr}\Big[\exp\Big(-\sum_k \lambda_k \hat{A}_k\Big)\Big] . \tag{24}$$

The λ_k are Lagrange multipliers that guarantee fulfillment of the a priori knowledge, namely,

$$\mathrm{Tr}(\hat{\rho}\hat{A}_k) = \langle \hat{A}_k \rangle , \qquad k = 1, \ldots, \mathcal{N} , \tag{25}$$

and are found by solving the coupled set of $\mathcal{N}$ simultaneous equations [22–24,3]

$$\frac{\partial \lambda_0(\lambda_1, \lambda_2, \ldots, \lambda_{\mathcal{N}})}{\partial \lambda_k} + \langle \hat{A}_k \rangle = 0 , \qquad k = 1, \ldots, \mathcal{N} . \tag{26}$$

As in the particular case of the canonical partition function (14), due to the Clebsch – Gordan decompositions (13) (whose existence is guaranteed for any Lie-algebraic solvable model by Weyl's theorem), the generalized partition function (24) can also be written down as

$$\mathcal{Z}(\lambda_1, \lambda_2, \ldots, \lambda_{\mathcal{N}}) = \sum_{\Lambda} \mathcal{M}(\Omega; \Lambda)\, \mathrm{tr}\Big[\exp\Big(-\sum_k \lambda_k \mathsf{A}_k(\Lambda)\Big)\Big] , \tag{27}$$

where we denote as $\mathsf{A}_k(\Lambda)$ the matrix representative of the operator $\hat{A}_k$ in the IR Λ of the algebra $\mathcal{L}$.

The von Neumann entropy [25] is, of course, given by ($k \equiv$ Boltzmann constant)

$$S = -k\, \mathrm{Tr}(\hat{\rho} \log \hat{\rho}) , \tag{28}$$

and the intensive related quantity is the intensive entropy, $s = S/\Omega$. We will redefine (28), as usual, by setting $k = 1$, which corresponds to measuring the temperature in energy units (instead of in Kelvin degrees). This makes the entropy dimensionless and adequate for its interpretation as a measure of the lack of information concerning the system's microscopic state. In the example given in the next section, we will further redefine the temperature scale by taking the unit of energy to be equal to the single-particle energy gap ε of a simple su(3) MGL-like Hamiltonian.

A central result of the present study is now to be advanced. After a rather lengthy but straightforward procedure, it is shown in the appendix that the intensive entropy in the thermodynamic limit, for any n-level MGL-like quantum system and for any set of relevant operators, is simply related to the exterior multiplicities by the Boltzmannian relation

$$s(\nu_0) = \lim_{\Omega\to\infty} \frac{1}{\Omega} \log \mathfrak{M}(\Omega; \Omega\nu_0) , \tag{29}$$

where ν_0 is the particular "intensive weight" selected among these that arise after performing the scaling

$$\Lambda = \Omega\nu , \tag{30}$$

which defines the intensive weights $\nu = [\nu_1, \nu_2, \ldots, \nu_l]$, where l is the rank of the considered algebra $\mathscr{L}$. The intensive "leader" weight ν_0 can be roughly characterized by saying that it is the weight that points towards the IR which produces an overwhelming contribution to the partition function for finite (but large) values of Ω and depends, of course, on the system's macroscopic state as specified by the expectation values of the set of relevant operators (25).

In the thermodynamic limit this so-called "leader IR", $\Lambda_0 = \Omega\nu_0$, will give the unique contribution to the generalized partition function and, consequently, to all the system's thermodynamic functions. The contributions to the partition function of all the other IRs are to be evaluated in order to perform that selection through a maximization procedure (It should be stressed that this procedure has nothing to do with the variational maximum entropy principle, which is already embodied in writing down the generalized partition function).

It is worthwhile to point out that no Lagrange or saddle-point method needs to be invoked neither in order to arrive at the result (29) or in order to evaluate the generalized partition function in the thermodynamic limit (see appendix A).

The existence of a limit of the form (29) for any physical system has been conjectured by Grandy [24], who explicitly stated it as a tentative theorem, and used it as a basis for explaining the "extraordinary effectiveness" or statistical mechanics and as a starting point for a discussion relative to the fundamentals of the second law of thermodynamics for quantum systems. It is perhaps appropriate to refer to it as "Grandy's theorem". The classical version of this theorem (although not stated as such) was given earlier by Jaynes [26] (see also theorems 3 and 4 of [27]).

Here, we are able to *extend* the quantum mechanical Grandy version of this theorem for the infinite class of MGL-like models by identifying the precise mathematical object on the right hand side of (29), namely, the external multiplicities of a proper Ω-fold Kronecker product. (Grandy's theorem deals with an ad-hoc notion of "high-probability manifolds" in the Hilbert space defined through a classification procedure of degenerate subspaces, and asserts, along with a relation like (29) between entropy and a high-probability manifold, that any physical system leads to a "decomposable" Hilbert space in which a high-probability manifold, in the thermodynamic limit, can be singled out.)

Our proof (see appendix A) embraces the possibility of dealing with a noncommuting set of relevant operators, thereby including all the off-equilibrium thermodynamics [24], and is achieved within, although it is not logically dependent on, the general context of statistical inference based on the information theory [27,23].

This theorem, except for the cases in which there exists a continuum portion in the single-particle spectrum, applies to any conceivable quantum MGL-like system, provided that the standard formulation of the general many-body problem (in terms of second quantized operators defined with relation to single-particle noninteracting states as the building blocks of the space of states of any strongly interacting system) be a valid one. The infinite quasicontinuum (denumerable) single-particle spectra should be considered embraced by the theorem as a limiting case, because, as the number of single-particle states is allowed to arbitrarily grow, the theorem, in the present form, still holds.

A particular version of the relation (29) for a two-level MGL model constructed around the $\mathrm{su}(2) \oplus \mathrm{su}(2)$ algebra has previously been found [28], besides the few other cases (not MGL nontrivial models) mentioned in [24].

Using Stirling's asymptotic formula in (29) for the su(3) case (i.e. with $\mathfrak{M}(\Omega; \lambda)$ given by (21)), one finds the following (exact) Shannon - Wiener form for the entropy per particle in the thermodynamic limit valid for *any* su(3) MGL many-body problem:

$$s(\nu) = -\sum_{i=1}^{3} p_i(\nu) \log p_i(\nu) , \qquad \nu = [\nu_1, \nu_2] , \tag{31.a}$$

where

$$p_1(\nu) = \tfrac{1}{3}(1 + 2\nu_1 + \nu_2)\,, \qquad p_2(\nu) = \tfrac{1}{3}(1 - \nu_1 + \nu_2)\,,$$
$$p_3(\nu) = \tfrac{1}{3}(1 - \nu_1 - 2\nu_2)\,, \tag{31.b}$$

with

$$p_i \geqslant 0\,, \qquad \sum_i p_i = 1\,. \tag{31.c}$$

This relation, as well as the result (29), *does not depend on the operators of the relevant set and, in particular, on the form* (*or even the existence*) *of the interaction term in the system's Hamiltonian*, provided one can write these operators in terms of the su(3) basis operators (1), or, in the general case, the basis operators of the considered algebra $\mathscr{L}$. It can be seen that this situation turns out to be always the case in the full class of MGL models.

5. The simplest su(3) nontrivial MGL-like Hamiltonian

We shall illustrate the preceding considerations by recourse to a very simple but nontrivial three-level MGL-like model. A two-body interaction term, which induces transitions between pairs of particles, will be chosen so as to lead to a diagonal Hamiltonian in all the su(3) IRs' carrier spaces.

The model describes Ω fermions distributed in three Ω-fold degenerate energy levels, which are separated by an energy gap ε. The single-particle states, $|pi\rangle$, are labelled by two indices. One of them, say i, takes the values 1, 2 and 3 for the levels with energies $-\varepsilon$, 0 and $+\varepsilon$, respectively, while the other, p, runs from 1 to Ω. For each value of p there are three single-particle states of different energy.

The pertinent Hamiltonian reads

$$\begin{aligned}\hat{H} = {} & \varepsilon \sum_p (a^\dagger_{p3} a_{p3} - a^\dagger_{p1} a_{p1}) \\ & + \tfrac{1}{2} W \sum_{pq} (a^\dagger_{p1} a^\dagger_{q2} a_{q1} a_{p2} + a^\dagger_{p2} a^\dagger_{q3} a_{q2} a_{p3} + a^\dagger_{p1} a^\dagger_{q3} a_{q1} a_{p3} + \text{h.c.})\,,\end{aligned} \tag{32}$$

where W is the coupling constant specifying the strength of that particular interaction which scatters one particle "upstairs" while another is scattered "downstairs" (between corresponding levels), conserving the energy in each one of these two-particle processes. The summations on p and q run between 1 and Ω. This Hamiltonian can be recast, by (1), (2), (9) and the anticommutation relations of the creation and annihilation operators, as

$$\hat{H} = -\varepsilon(\hat{h}_1 + \tfrac{3}{2}\hat{h}_2) - W(3\hat{\mathbb{C}} - \hat{h}_1^2 - \tfrac{3}{4}\hat{h}_2^2 - \tfrac{3}{2}\Omega)\,, \tag{33}$$

and is analogous to the su(2) Lipkin original Hamiltonian with $V = 0$ (see [1]).

As is well known [24,3,23,22], if $\hat{H}$ is the only relevant operator in building up $\hat{\rho}$ (cf. eq. (23)) we shall be describing a situation of thermodynamic equilibrium, a situation that will not vary if additional operators that commute with $\hat{H}$ are also included in the relevant set. However, the number of independent relevant operators that one can choose is limited by the rank of the considered algebra. In the present example we are almost forced to limit our considerations to the first instance so that all the information-theoretic paraphernalia is reduced to the calculation of the standard canonical partition function (14). In doing so, we find a *new* thermodynamic phase which is *absent* in the analogous su(2) model.

The system's energy per particle in the thermodynamic limit is

$$\mathscr{E}(\nu, \eta) = \lim_{\Omega\to\infty} \frac{1}{\Omega} E(\Lambda, \lambda)\,, \qquad \eta \in \nu = [\nu_1, \nu_2]\,, \tag{34}$$

where $\eta = (\eta_1, \eta_2)$, $\eta_i = h_i/\Omega$, is a m intensive weight belonging to the IR labeled by its highest intensive weight ν and $E(\Lambda, \lambda)$ is the energy eigenvalue,

$$\hat{H}|\Lambda\lambda r\rangle = E(\Lambda, \lambda)|\Lambda\lambda r\rangle\,, \qquad r = 1, \ldots, m^{\Lambda}(\lambda)\,. \tag{35}$$

By (33), the system's energy per particle is

$$\mathscr{E}(\nu, \eta) = -\varepsilon[3w\mathscr{C}(\nu) + \mathscr{g}(\eta)]\,, \tag{36}$$

where $w = \Omega W/\varepsilon$ is the finite dimensionless coupling constant for the considered two-body interaction.

$\mathscr{C}(\nu)$ is the quantity defined by

$$\mathscr{C}(\nu) = \lim_{\Omega\to\infty} \frac{1}{\Omega^2} \mathbb{C}(\Lambda) = \tfrac{1}{9}(\nu_1^2 + \nu_2^2 + \nu_1\nu_2)\,, \tag{37}$$

with $\mathbb{C}(\nu)$ given by (11), and the function $\mathscr{g}(\nu)$ is

$$\mathscr{g}(\eta) = \eta_1 + \tfrac{3}{2}\eta_2 - w(\eta_1^2 + \tfrac{3}{4}\eta_2^2)\,. \tag{38}$$

Fig. 1 shows some contour lines for the function $\mathscr{g}(\eta)$ as well as the path traced by the intensive leader weight ν_0 in the intensive weight space as the temperature of the system, measured in the system's natural energy units ε, varies between zero and infinity, for the special case when the strength of the

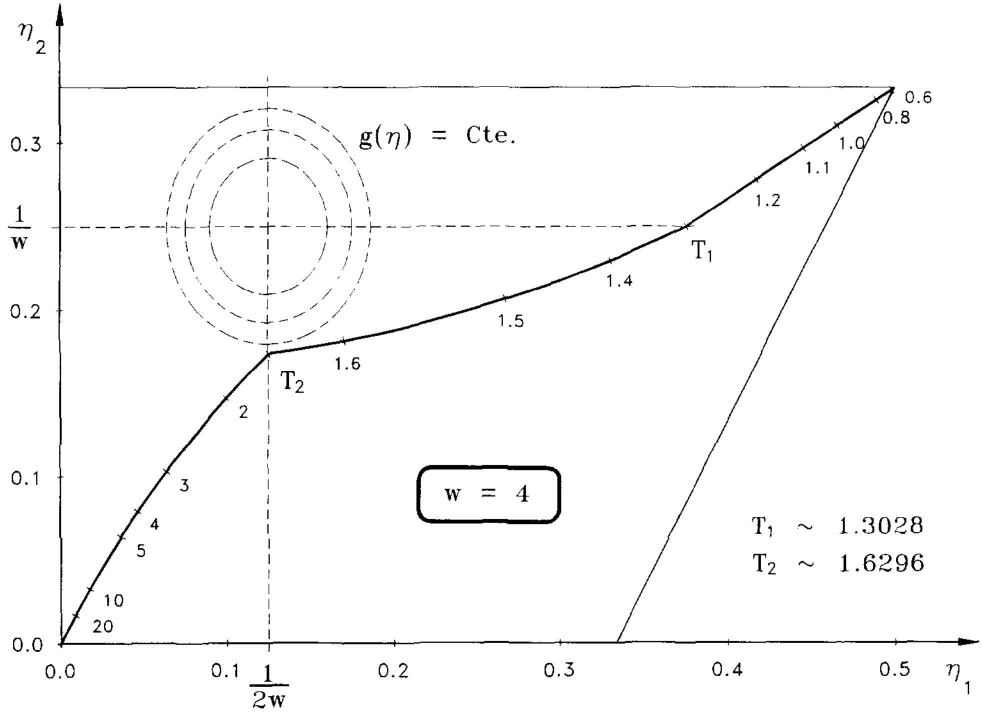

Fig. 1. Path traversed by the intensive leader weight ν_0 in the intensive weight space as the temperature of the system varies between zero and infinity. The strength of the interaction is $w = 4$.

two-body interaction is $w = 4$. Only the upper right part of the intensive weight diagram is showed.

For sufficiently high values of the coupling constant w, the system undergoes two phase transitions at the critical temperatures T_1 and T_2 given by (see fig. 2)

$$T_1 = 3\left(\log\frac{w+6}{w-3}\right)^{-1}, \tag{39.a}$$

$$T_2 = 2\left(\log\frac{w[1+\sigma(w)]+2}{w[1-2\sigma(w)]-1}\right)^{-1}, \quad \sigma(w) = \sqrt{1-\frac{3}{4}\left(1-\frac{1}{w}\right)^2} - \frac{w+1}{2w}. \tag{39.b}$$

If $w < 1$, there does not exist any ordered phase at any temperature. In this case the system behaves as an ideal gas ($w = 0$), even for *finite* interaction strengths. The existence of this finite threshold seems to be a general

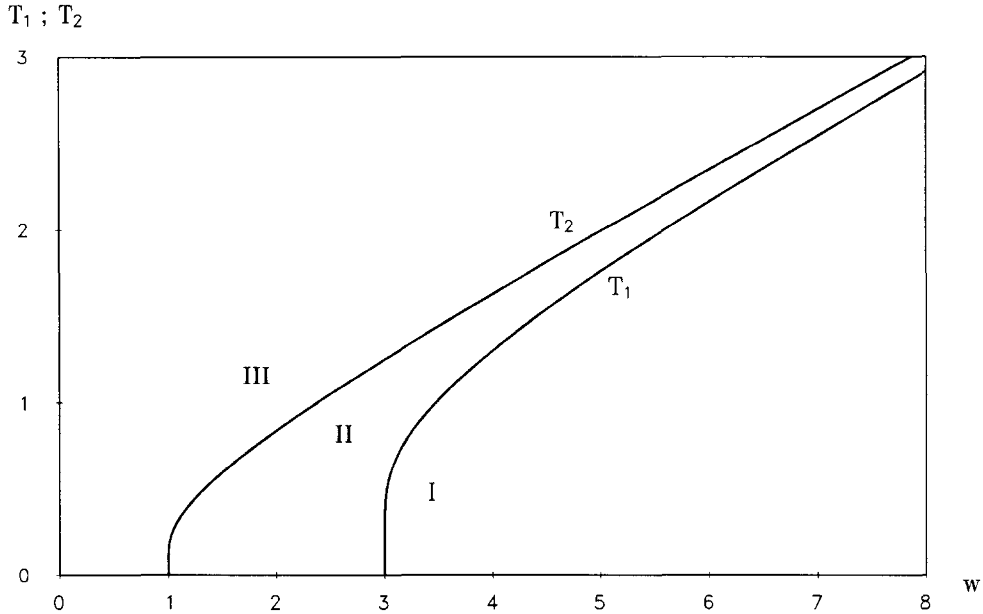

Fig. 2. Critical temperatures T_1 and T_2 as functions of the adimensional intensive coupling constant parameter w.

characteristic, at least in the class of MGL models. We find it in all known cases. The implication of this fact would be, if general, that all the possible interactions (the forces) the (quasi)fermions feel remain hidden to the observer provided that they be sufficiently small. The impossibility of knowing about the existence of statistically concealed forces would remain true even for a finite number of particles, because the threshold existence for fermionic systems is determined by the finiteness of the region of accessible intensive weight space, a circumstance that does not depend on Ω.

Completing the previous results, a straightforward but somewhat lengthy calculation yields [19] the following:

Phase I, $3/w \leq \xi < 1$ $(0 < T \leq T_1)$

$$\sigma = 0\,,$$
$$\mathscr{E} = -\tfrac{1}{3}\varepsilon\left(w\xi^2 + \frac{1}{w}\right),$$
$$T = w\xi\left(\log\frac{1+2\xi}{1-\xi}\right)^{-1}. \tag{40.a}$$

Phase II, $1/w < \xi < 3/w$ $(T_1 < T < T_2)$

$$\sigma \text{ such that } (1-\xi+\sigma)^{w\xi+3}(1+2\xi+\sigma)^{w\xi-3}=(1-\xi-2\sigma)^{2w\xi}\,,$$
$$\mathscr{E}=-\frac{1}{4}\left((2+w\xi)\xi+4\sigma+\frac{1}{2w}\right),$$
$$T=\tfrac{1}{2}(w\xi+3)\left(\log\frac{1+2\xi+\sigma}{1-\xi-2\sigma}\right)^{-1}. \tag{40.b}$$

Phase III, $0<\xi\leqslant 1/w$ ($T_2\leqslant T<\infty$)

$$\sigma=\sqrt{1-\tfrac{3}{4}(1-\xi)^2}-\tfrac{1}{2}(1+\xi)\,,$$
$$\mathscr{E}=-\varepsilon(\xi+\sigma)\,,$$
$$T=2\left(\log\frac{1+2\xi+\sigma}{1-\xi-2\sigma}\right)^{-1}. \tag{40.c}$$

Here $\nu_0=[\xi,\sigma]$ is the intensive weight which labels the leader su(3) IR (which depends on the system's temperature T).

The entropy per particle $s(\nu_0)=s(\xi,\sigma)$ is given by the same expression (31) in the three thermodynamic phases and is depicted in fig. 3.

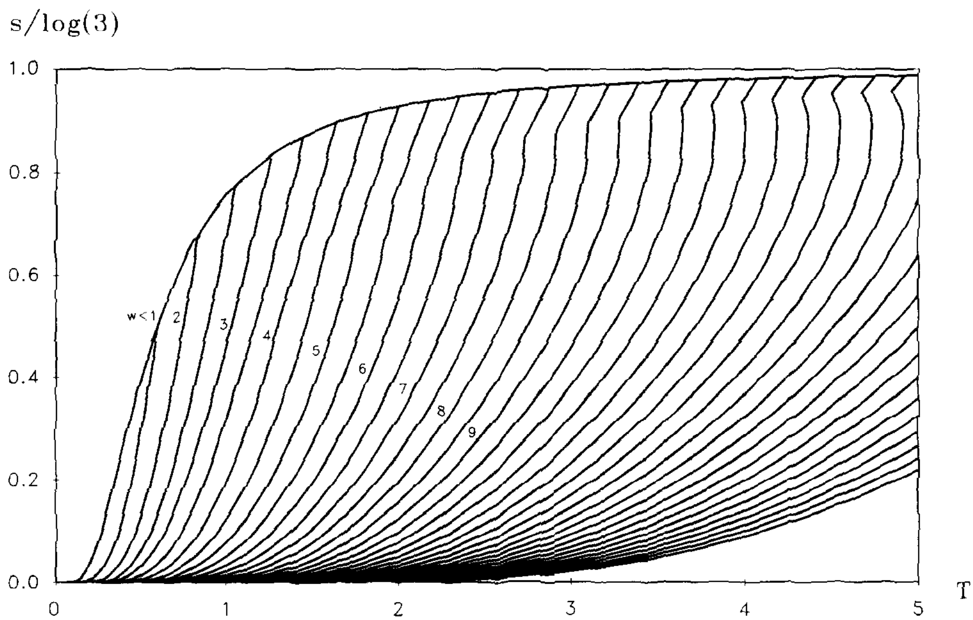

Fig. 3. Intensive entropy versus temperature for several interaction strengths.

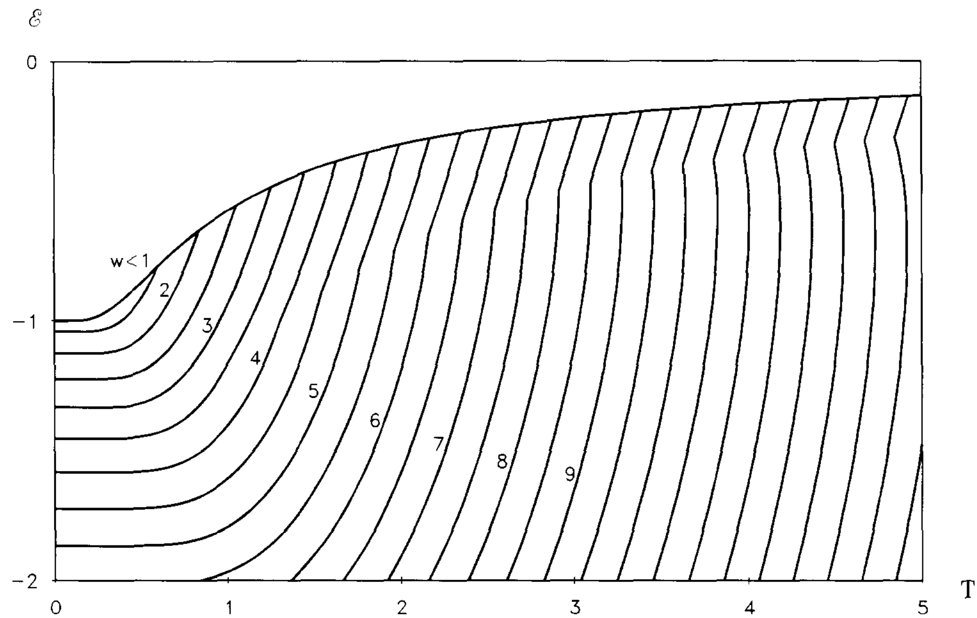

Fig. 4. Energy per particle versus temperature for several interaction strengths.

The energy per particle is plotted in fig. 4. We only give, for the sake of brevity, just a plot for the specific heat, fig. 5, the analytic expression of which is easily derived from (40).

The most salient feature of the simplest su(3) MGL model is the appearance, as we anticipated, of a new thermodynamic phase (if $w > 1$) that does not exist in the analogous two-level su(2) situation.

The specific heat has two finite jumps at each critical temperature, provided that w be greater than three and less than a certain "critical" value, w_∞, for which there exist an infinite jump in the specific heat at the temperature T_2. For greater values of w there exists a temperature interval in which all the considered thermodynamic functions are multi-valued and the specific heat adopts negative values. This critical value for the coupling constant is $w_\infty = 7.95332718$.

The explanation for this anomaly may be found in a sort of inverse symmetry-breaking effect that this strong interaction produces (relative to the energy ordering) on the nonperturbed single-particle states, i.e., the interaction term treats the three possible kinds of transitions that in this three-level system may occur on an equal footing (each giving an equal contribution to the

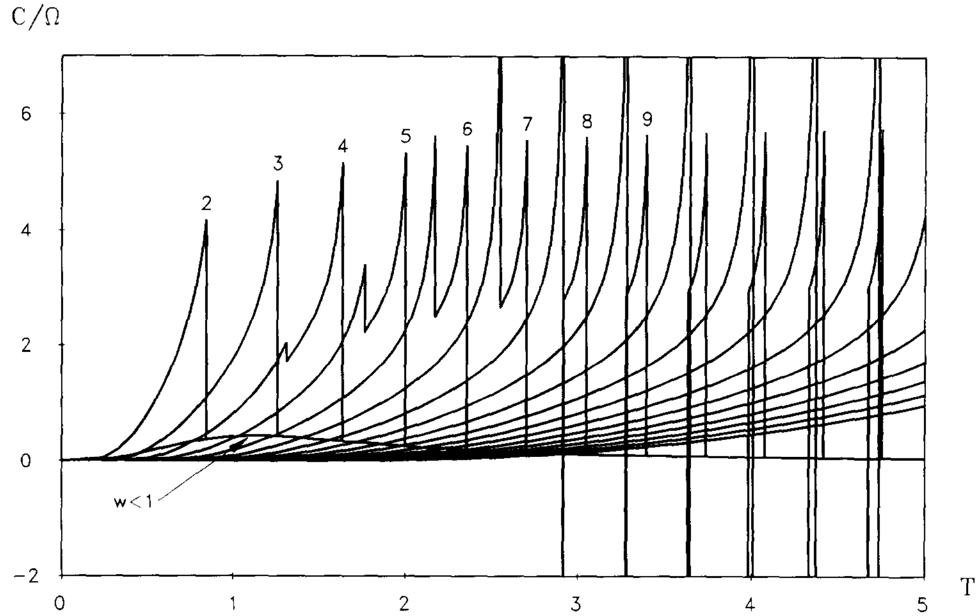

Fig. 5. Specific heat versus temperature for several interaction strengths.

system's energy), while the transitions between the three "nonperturbed" single-particle states are obviously energetically distinct.

This symmetry-breaking effect can be better appreciated by inverting the role of the terms in the systems' Hamiltonian: assume that just the interaction term, proportional to W, specifies the system. The term proportional to the energy gap ε would arise by the application of an external field that interacts with each particle, in close analogy with the Ising model of ferromagnetism [29]. This would be the direct symmetry-breaking effect, although in such a case the calculated thermodynamic functions would lose their original meanings. However, the entropy would remain the same, as it should, because of its distinct role of being a state function and not an observable.

It is worthwhile to point out that this example highlights the secondary role played by the Lagrange multipliers in the sense that assigning a set of values to them does not necessarily completely specify the unique macrostate of the system, while, on the other hand, the specification of the expectation values uniquely determines this macrostate. From the purely formal information-theoretic point of view, this is the first example in which, to our knowledge, these multi-valued functions arise. It has to be noted that the information theory formalism does not break down, even if the physical interpretation may lose its meaning.

6. Conclusions

The present study of solvable MGL models allow us to make significant long-range advances of general validity in the statement, systematization and partial resolution (mainly in that part addressed necessarily by the systems's statistical description) of a broad class, infinite indeed, of exactly solvable quantum models, that may be useful in better understanding the complexities of the quantum many-body problem with arbitrarily strong many-body interaction, the su(3) model being the simpler and paradigmatic first example in this direction. The advances made will enable the statistical (equilibrium or off-equilibrium, numerical or analytical) exact resolution of this kind of quantum many-body systems of strongly interacting particles both in the thermodynamic limit as well as for a finite number of fermions.

Appendix A

We undertake in this appendix the proof of eq. (29), i.e. Grandy's theorem. Let the operator $\hat{F}$ be defined by

$$\hat{F}(\lambda_k) = -\sum_{k=1}^{\mathcal{N}} \lambda_k \hat{A}_k , \tag{A.1}$$

where λ_k are the Lagrange multipliers associated to the operators $\hat{A}_k$ belonging to the relevant set selected for the probability assignment, according to the information theory approach and the MAXENT variational principle [23,3,24,30].

Assume that these operators are analytical functions of the basis operators of a semisimple Lie algebra $\mathscr{L}$, so that the full class of Lie-algebraic exactly solvable models is thereby encompassed.

Then, by the reducibility of the Ω-fold Kronecker product representation, the operator $\hat{F}$ is realized by a matrix that can be brought, through a similitude transformation, to (block) diagonal form. Each submatrix of dimension $d(\Lambda)$, given by the Weyl dimensionality formula (6), is labelled by the components, relative to the fundamental weight basis, of the highest-weight vector Λ of the IRs of the considered Lie algebra $\mathscr{L}$ that appears in the reduction (Clebsch – Gordan series) of the Ω-fold Kronecker product representation. (As the trace is invariant against these similitude transformations, the knowledge of the (Clebsch – Gordan) matrix that performs such a transformation is irrelevant in order to calculate expectation values).

Thus the generalized partition function adopts the appearance

$$\mathfrak{Z}_\Omega(\lambda_k) = \sum_\Lambda \mathfrak{M}(\Omega;\Lambda)\,\mathrm{tr}[\exp \mathsf{F}_\Lambda(\lambda_k)]\,, \tag{A.2}$$

where F_Λ is a $d(\Lambda)$-dimensional matrix representative of the operator $\hat{F}$ on the carrier space of the IR labelled by its highest weight Λ, spanned by the kets $\{|\Lambda\lambda\mu\rangle\}$, $\mu = 1, \ldots, m^\Lambda(\lambda)$, with $m^\Lambda(\lambda)$ being the multiplicity of the weight λ in the IR Λ (i.e. the number of times the weight λ appears in the IR Λ), and $\mathrm{tr}\cdots$ is the trace on each IR given by

$$\mathrm{tr}[\exp \mathsf{F}_\Lambda(\lambda_k)] = \sum_{\lambda\in\Lambda} \sum_{\mu=1}^{m^\Lambda(\lambda)} \langle\Lambda\lambda\mu|\exp \mathsf{F}_\Lambda(\lambda_k)|\Lambda\lambda\mu\rangle\,, \tag{A.3}$$

where the summation on λ is extended over all the *distinct* weights of the IR Λ.

Let $f_i(\Lambda;\lambda_k)$, $i = 1, \ldots, d(\Lambda)$, be the $d(\Lambda)$(not necessarily all different) real eigenvalues of the Hermitian operator $\hat{F}$ on each IR Λ,

$$\mathsf{F}_\Lambda(\lambda_k)|\Lambda_i\rangle = f_i(\Lambda;\lambda_k)|\Lambda i\rangle\,, \qquad i = 1, \ldots, d(\Lambda)\,, \tag{A.4}$$

where $\{|\Lambda i\rangle\}$ are the corresponding orthonormal eigenvectors (after a orthonormalization procedure if necessary). Then the trace (A.3) is

$$\mathrm{tr}(\exp \mathsf{F}_\Lambda) = \sum_{i=1}^{d(\Lambda)} \exp[f_i(\Lambda)]\,. \tag{A.5}$$

Let $\Lambda_0$be highest weight of the leader IR, defined in such a way as to produce an absolute maximum of the function

$$G(\Lambda) = \mathfrak{M}(\Omega;\Lambda)\,\mathrm{tr}(\exp \mathsf{F}_\Lambda) \tag{A.6}$$

for a given set of values of the $\mathcal{N}$ Lagrange multipliers $\{\lambda_k\}$. (Thus the leader IR Λ_0 depends on the macroscopic state of the system specified through the set of Lagrange multipliers.)

Then, as $G(\Lambda)$ is definite positive, the following inequality holds:

$$G(\Lambda_0) \leqslant \mathfrak{Z} \leqslant n(\Omega)\,G(\Lambda_0)\,, \tag{A.7}$$

where $n(\Omega)$ is the number of terms in the summation (A.2), i.e. it is the number of distinct (that is, without counting its multiplicity) inequivalent IRs that appear in the Ω-fold Kronecker product reduction.

The number $n(\Omega)$ equals the number of nonequivalent distinct weights that belong to the IR of the Clebsch – Gordan series which has the maximum highest weight. That is the weight $\Omega\Lambda_j$ for the Kronecker product representation Λ_j^Ω. (Two weights are said to be equivalent if one can be transformed into the other by some sequence of Weyl's reflections [11] of the considered algebra $\mathscr{L}$.)

Thus, $n(\Omega)$ is asymptotically proportional to the l-dimensional volume of the region of the weight space which contains all the weights of the IR $\Omega\Lambda_j$, where $l = \operatorname{rank}(\mathscr{L})$. This region has a linear magnitude of the order of Ω, measured in units of the weight spacing. Thus

$$n(\Omega) \sim a\Omega^l\,, \qquad a < \infty\,. \tag{A.8}$$

For example, for $\mathscr{L} = \mathrm{su}(3)$ and either Kronecker product representation Λ_1^Ω or Λ_2^Ω, $n(\Omega)$ is exactly given by

$$n(\Omega) = 1 + [\tfrac{1}{2}\Omega + \tfrac{1}{12}\Omega^2] \qquad (\sim \tfrac{1}{12}\Omega^2)\,, \tag{A.9}$$

where $[x]$ is the integer part of x.

From (A.7) and (A.8) one deduces the inequality

$$\frac{1}{\Omega}\log G(\Lambda_0) \le \frac{1}{\Omega}\log \mathscr{Z} \le \frac{1}{\Omega}\log G(\Lambda_0) + \frac{l}{\Omega}\log(a\Omega)\,. \tag{A.10}$$

In the thermodynamic limit ($\Omega \to \infty$), the last term in (A.10) vanishes and one has by (A.6)

$$\lambda_0 \equiv \lim_{\Omega\to\infty} \frac{1}{\Omega}\log \mathscr{Z} = m + \phi\,, \tag{A.11}$$

where

$$m = \lim_{\Omega\to\infty} \frac{1}{\Omega}\log \mathfrak{M}(\Omega; \Lambda_0)\,, \tag{A.12}$$

and

$$\phi = \lim_{\Omega\to\infty} \frac{1}{\Omega}\log \operatorname{tr}(\exp \mathsf{F}_{\Lambda_0})\,. \tag{A.13}$$

In (A.11) we have redefined λ_0 (which is related to the generalized partition function by an exponentiation operation) as the intensive Lagrange multiplier associated to the identity operator $\lim_{\Omega\to\infty}\lambda_0/\Omega \to \lambda_0$.

Let $f_0(\Lambda_0; \lambda_k)$ be (one of) the largest eigenvalue $f_i(\Lambda_0; \lambda_k)$, $i = 1, \ldots, d(\Lambda_0)$, (cf. eq. (A.4)) of the leader IR Λ_0. Then, by (A.5) (with $\Lambda \to \Lambda_0$) and the positiveness of the terms in that sum, the following inequality holds:

$$\exp f_0(\Lambda_0) \le \operatorname{tr}(\exp \mathsf{F}_{\Lambda_0}) \le d(\Lambda_0) \exp f_0(\Lambda_0)\,, \tag{A.14}$$

or

$$\frac{1}{\Omega} f_0(\Lambda_0) \le \frac{1}{\Omega} \log \operatorname{tr}(\exp \mathsf{F}_{\Lambda_0}) \le \frac{1}{\Omega} f_0(\Lambda_0) + \frac{1}{\Omega} \log d(\Lambda_0) . \tag{A.15}$$

The dimension $d(\Lambda_0)$ is, by the Weyl dimensionality formula (6), a multinomial expression in the components of the weight Λ_0, $\{n_{01}, \ldots, n_{0l}\}$, with positive (or null) coefficients. As the weights' components can only take non-negative integer values from 0 to Ω, the dimension $d(\Lambda_0)$ has an upper bound given by the dimension of the representation $[\Omega, \Omega, \ldots, \Omega]$ (this representation does not belong to the considered Clebsch – Gordan series). Next we calculate the dimension of this representation.

The Weyl formula can be written as

$$d(\Lambda) = \prod_{\alpha > 0} \left(1 + \frac{\alpha \cdot \Lambda}{\alpha \cdot \delta}\right), \qquad \delta = \frac{1}{2} \sum_{\alpha > 0} \alpha . \tag{A.16}$$

As α is a positive root, it can be spanned as

$$\alpha = \sum_{i=1}^{l} k_i(\alpha)\, \alpha_i , \tag{A.17}$$

where $k_i(\alpha)$ are l positive integers and α_i are the l simple roots of the algebra $\mathscr{L}$.

The highest weight of the bounding IR $\Lambda = [\Omega, \Omega, \ldots, \Omega]$ can be expressed as (cf. eq. (4))

$$\Lambda = \Omega \sum_{j=1}^{l} \Lambda_j . \tag{A.18}$$

Using the identity

$$2 \frac{\Lambda_j \cdot \alpha_i}{a_i \cdot \alpha_i} = \delta_{ij} , \tag{A.19}$$

we find

$$\alpha \cdot \Lambda = \tfrac{1}{2} \Omega \sum_{j=1}^{l} k_j(\alpha)\, \alpha_j \cdot \alpha_j . \tag{A.20}$$

The product $\alpha \cdot \delta$ can be calculated by means of the identity [9, p. 533]

$$\alpha_j \cdot \alpha = \tfrac{1}{2} \alpha_j \cdot \alpha_j , \tag{A.21}$$

giving

$$\alpha \cdot \delta = \frac{1}{2} \sum_{j=1}^{l} k_j(\alpha)\, \alpha_j \cdot \alpha_j \,. \tag{A.22}$$

Let $d(\Omega)$ be the dimension of the bounding representation $[\Omega, \Omega, \ldots, \Omega]$. Then

$$d(\Omega) = \prod_{\alpha>0} (1 + \Omega) \,. \tag{A.23}$$

The number of positive roots depends on the considered algebra $\mathscr{L}$. For su$(l+1)$ that number is $\frac{1}{2}l(l+1)$. For so$(2l+1)$ and sp(l) it is l^2. For so$(2l)$ it is $l(l+1)$. In any of these cases we have

$$d(\Omega) \leq (\Omega + 1)^{l(l+1)} \,. \tag{A.24}$$

Consequently

$$\frac{1}{\Omega} \log d(\Lambda_0) < \frac{1}{\Omega} \log d(\Omega) \leq \frac{l(l+1)}{\Omega} \log(\Omega + 1) \xrightarrow[\Omega\to\infty]{} 0 \,, \tag{A.25}$$

and, in the thermodynamic limit, the last term in (A.15) vanishes and we obtain (exactly)

$$\phi = \lim_{\Omega\to\infty} \frac{1}{\Omega} f_0(\Lambda_0; \lambda_k) \,. \tag{A.26}$$

$f_0(\Lambda_0)$ is the leader eigenvalue of the leader IR, which is the only one contributing to the generalized partition function in the thermodynamic limit. The result (A.26) remains valid also in the case of having more than one (i.e. a degenerate) leader eigenvalue.

The intensive entropy, S/Ω, in the limit $\Omega \to \infty$ (cf. eqs. (28) and (23)), is

$$s(\lambda_k) = \lambda_0 + \sum_{k=1}^{\mathcal{N}} a_k \lambda_k \,, \qquad a_k = \lim_{\Omega\to\infty} \frac{1}{\Omega} \langle \hat{A}_k \rangle \,, \tag{A.27}$$

where a_k are the intensive expectation values of the operators belonging to the relevant set in the thermodynamic limit.

By making the scale transformation in the weight space (30) which defines the intensive weights ν, we are able to write the l equations which satisfy the intensive leader weight ν_0 as

$$\frac{\partial}{\partial \nu} \lambda_0(\lambda_k; \nu)\bigg|_{\nu=\nu_0} = 0 , \tag{A.28}$$

where $\lambda_0(\lambda_k) = \lambda_0(\lambda_k; \nu_0)$.

The leader IR yields an (absolute) maximum of the generalized partition function, and, consequently, of the intensive Lagrange multiplier λ_0.

Deriving (A.11) with respect to λ_j, we obtain (because of (A.28))

$$\frac{\partial \lambda_0}{\partial \lambda_j} = \frac{\partial \phi}{\partial \lambda_j} . \tag{A.29}$$

This derivative is equal to the (intensive expectation value associated to λ_j with opposite sign $(-a_j)$ (cf. eq. (26)).

From (A.13) we have

$$\frac{\partial \phi}{\partial \lambda_j} = \lim_{\Omega\to\infty} \frac{1}{\Omega\, \mathrm{tr}(\exp \mathsf{F}_{\Lambda_0})} \mathrm{tr}\left(\frac{\partial}{\partial \lambda_j} \exp \mathsf{F}_{\Lambda_0}\right) . \tag{A.30}$$

Using the identity [31]

$$\delta \exp \mathsf{F} = \exp \mathsf{F} \int_0^1 \mathrm{d}u \exp(-u\mathsf{F})\, \delta \mathsf{F} \exp(u\mathsf{F}) , \tag{A.31}$$

we obtain

$$\begin{aligned} -\sum_{j=1}^{\mathcal{N}} \lambda_j a_j &= \sum_{j=1}^{\mathcal{N}} \lambda_j \frac{\partial \phi}{\partial \lambda_j} = \lim_{\Omega\to\infty} \frac{1}{\Omega\, \mathrm{tr}(\exp \mathsf{F}_{\Lambda_0})} \mathrm{tr}(\mathsf{F}_{\Omega_0} \exp \mathsf{F}_{\Lambda_0}) \\ &= \lim_{\Omega\to\infty} \frac{\Sigma_i f_i \exp f_i}{\Omega\, \Sigma_i \exp f_i} = \frac{\partial}{\partial \gamma} \left(\lim_{\Omega\to\infty} \frac{1}{\Omega} \log \sum_i \exp(\gamma f_i)\right)_{\gamma=1} \\ &= \frac{\partial}{\partial \gamma} \left(\lim_{\Omega\to\infty} \frac{\gamma}{\Omega} f_0\right)_{\gamma=1} = \phi , \end{aligned} \tag{A.32}$$

where we have used the relations (A.1), (A.4), (A.5) and (A.26).

Substituting (A.11) and (A.32) in (A.27) we finally obtain $s = m$, i.e., by (A.12)

$$s = \lim_{\Omega\to\infty} \frac{1}{\Omega} \log \mathfrak{M}(\Omega; \Lambda_0) , \tag{A.33}$$

with Λ_0 determined from (A.28) and the additional requirement that it produces an absolute maximum of the function $\lambda_0(\lambda_k; \nu)$. This completes our proof.

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