Epistemological obstacles in the learning process of Numeral Systems

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Abstract. This study has two main research objectives. Its main one is to analyze the origin of a systematic error, made by first year students of the Computer Sciences Degree at the National University of Salta. The systematic error arises when the students are dealing with Numeral Systems. From there, a second objective is to develop teaching procedures leading to recreate the error in a conceptual reconstruction process, within a playful environment. The theoretical framework used is the Brousseaunian concept of epistemological obstacles in the learning process. We use a sample of 152 students. In this article, we propose a strategy that allows real processes of error reconstruction and construction, in order to enable real epistemological progress. Resources within a virtual environment are presented as play and interactive alternatives with the aim of reinforcing the concepts studied in the classroom: videos, educational software and a computer game specifically developed for Numeral Systems.

Key words: Epistemological obstacles, Numeral Systems, Error Construction and Reconstruction, Teaching procedures, Liaison.

1 Introduction

This study has two main research objectives. The first one is focused on the origin of a systematic error made by first year students of the Computer Sciences Degree at the National University of Salta, when dealing with mathematical concepts that have been taught in different theoretical and conceptual domains. The second one is to develop an innovative teaching procedure, seeking the reconstructing of the systematic error through a conceptual revision process, within a playful environment.

Activities were carried out within the framework of the CIUNSa Research Project N° 1865/3 called *Virtual Environments for the between Secondary School and Exact Sciences University Degrees* which has been accredited by the Research Council of the National University of Salta. The timeframe for this project is from 2010 to 2013. Students in their last year of Secondary School, who are interested in studying either an university degree in Math, Chemistry or Computer Sciences, take part in this project. In a second stage, a follow-up system is used for students who actually go on to University. It is therefore of interest to detect conceptual problematic foci on each discipline within the University environment, in order to guide the Liaison activities based on these results.

The innovative teaching procedure is applied then to first year students at the University and to Secondary School students taking part in the Liaison project.

The inter-disciplinary team involved in the Project upholds that University teaching and research are complementary and mutually reinforcing activities [1]. Some authors of this study, in their role as teachers, use the teaching procedures developed from their research, in their classrooms. This kind of research improves the teaching process, involving the teachers in a double role, as teachers and researchers, who simultaneously perform at three different levels, Empiric, Theoretical and Meta-theoretical.

2 Application context

The IT subject courses in the first year of the Degree in Systems Analysis (LAS) and University Technical Degree in Programming (TUP), given by the Faculty of Exact Sciences of the National University of Salta, have really high matriculation numbers. In general, the students have recently finished Secondary School. It is not always the case that the students have the required cognitive ability to carry out a certain level of abstraction that enables them to keep the pace in the first year. Neither they have developed study habits or strategies that allow them to settle themselves in a Higher Education System. The subject course called Elements of Programming is part of both the LAS and TUP's syllabi. It is taught in the first semester of the first year, and therefore it is the first IT subject course the students take. The subject's content can be divided in three:

- Initial Programming Concepts, with a focus in algorithms design.
- Basic computing elements that have base on Applied Mathematics: Numeral Systems and Boolean Algebra.
- Complementary and introductory content leading to computing literacy.

3 Objectives

The main aim is to analyze the origin of a systematic error made by the LAS and TUP first year students, when dealing with Numeral Systems. From there we aim to design an innovative teaching procedure, leading to reconstruct the aforementioned error.

The theoretical framework answers to the Brousseaunian concept of epistemological obstacles in the learning process. Due to the nature of the subject contents, which is usually known by the students in other contextual domains -such as counting and solving basic operations within the Decimal System-, we resort to a playful environment for the conceptualization and practice on this and other numeral systems.

3.1 Obstacles of epistemological and didactic origin

Mathematics provides several examples of obstacles that result in different kind of errors. The knowledge of the Decimal System as an obstacle to the learning of other Numeral Systems, which are especially useful for Computer Sciences, such as Binary, Octal and Hexadecimal Systems.

In the Decimal System, the numbers have a specific name that identifies them. Let say, for example, thirteen, which in mathematical notation is $(13)_{10}$ or just 13, when the sub-index that indicates the system's base is omitted. In a different numeral system, such as the Octal System, the number 13 is represented by $(15)_8$. The correct denomination of this octal number is one-five, reading the number's consecutive characters, and not fifteen, as a Decimal System reading would mistakenly suggest.

It can be noticed the need for abstraction that is required to keep stable, not only the quantity, but also the oral and written representation of numbers when they are expressed in different numeral systems. The abstraction reference is essential for this analysis. Guillermo Simari in [2], says:

"Our graduates must have the ability to think in different levels of abstraction and this ability is difficult to acquire, it requires time and practice." And adds: *"The depth and complexity in the change of perspective make it necessary to approach the teaching of these abilities in an early stage, in order to allow enough time for the indispensable comprehension and cognitive maturity process."*

In addition to representational difficulties, there are others whose origin lay in real conflicts that originate in previous knowledge. According to [3]:

> *"... to describe knowledge, understand its usage; explain which advantages it provides compared to previous uses, to which social practices is related, with which techniques and, if it is possible, with which mathematical conditions; to indicate these conceptions in relation to other possible conceptions, especially those that succeeded it, in order to understand its limitations, its difficulties and finally the causes for this conceptions' failure, but at the same time the reasons for a balance that seems to have lasted a long enough time; to search possible reappearances, unexpected environments, if not under its initial appearance, at least under similar forms, and the reasons for that."*

Regarding the balance that lasts over the time, few pieces of knowledge are more general, with such a practical use, and so socially related, as that which states that "any number ending in an even digit, is an even number itself". When venturing into new numeral systems, the validity of this statement is challenged. For example, we can see this in a numeral system whose set of characters has an odd number of elements. The set of characters is C= {0, 1, 2, 3, 4} . The base is $\beta = (10)_5 \equiv (5)_{10}$.

The following chart shows the succession of the first decimal system's positive integers, the numeral system in base 5 and a third one equivalent to the latter, and whose characters are abstract. It's not difficult to notice that the decimal system's even numbers have their corresponding even equivalents in the numeral system in base 5. Some of them, such as 11 (YY) and 13 (YW) do not follow the statement above, despite them definitely being even numbers.

Chart 1. Decimal counting and systems with an odd base.

Base	Succession of positive integers													
In base 10:	0	1	2	3	4	5	6	7	8	9	10	11	12	…
In base 5:	0	1	2	3	4	10	11	12	13	14	20	21	22	…
In base YZ: C = {Z,Y,X,W,V}	Z	Y	X	W	V	YZ	YY	YX	YW	YV	XZ	XY	XW	…

4 Method

The error which is being under study pertains to the subject Numeral Systems, given in Unit No. 6 of the course called Elements of Programming.

It is worth mentioning that in the first part of the course, basic programming concepts are taught. It comprises the Computing problems resolution, Basic algorithms and one-dimensional and bi-dimensional indexed variables. This is important because the algorithmic approach is cross-linked to the rest of the subjects that are taught, trying to encourage abstraction abilities in different problem situations, which are typical of Computer Sciences. Therefore, before studying the Numeral Systems, the student learns to design algorithms in which they frequently uses different algorithms such as the separation of a number's digits, composition of a number from the weighted sum of its digits, parity recognition using the MOD function, among others.

After the units 6 and 7 are taught, students are evaluated through a written test on Numeral Systems and Boolean Algebra. It comprises multiple-choice exercises: 5 for the former subject and 10 for the latter. Each exercise has four answers to choose from. And only one is correct. The exercise on which this study is focused is:

Let C = {Z, Y, X, W, V} be a set of characters of a numeral system in base β = YZ. Please indicate the smallest even natural number that has two significant digits.

The four choices given are: a) YZ, b) ZY, c) YY (correct answer) and d) None of the above. The two first wrong choices answer to the following criteria:

A) This is a choice that fits the expected error. If we translate the presented Numeral System to a system in base 5, the number YZ is equivalent to $(10)_5$. Indeed, if we only analyze the written representation $(10)_5$ or its oral representation *one-zero*, we can conclude that it is an even number because it ends in an even figure. It is clear that $(10)_5$ is an odd number due to the following: a) in the number line, the number $(10)_5$ is between two even numbers, b) the decimal number equivalent to $(10)_5$ is 5, which is an odd number, c) $(10)_5$ cannot be written as *2*n*.

B) This is an incorrect answer that does not follow any of the conditions asked in the exercise. The number ZY is equivalent to $(01)_5$, or $(1)_5$ because zero is not a significant digit in that position, and therefore, it is not a number with two digits. Additionally, 1 is an odd number.

5 Materials and Methods used for the Teaching Procedure

For a few years now, the department has been working with a *B-Learning* method, using a virtual course called *Elements of Programming*, designed on a Moodle platform. This course's goal is to open an alternative meeting space through which the students could establish other communication channels with their peers and teachers, as well as find different activities and material that may help to reinforce their learning process.

The virtual classroom's subjects are organized in blocks of topics, one for each unit of the syllabus. The blocks corresponding to the different units of the syllabus have the theoretical and practical material required to attend classes on campus, a virtual self-evaluation questionnaire and additional material related to the unit's specific subjects. In the block corresponding to Numeral Systems, as seen in Figure No 1, the students can access a series of videos, games and software that provide alternative play and interactive activities to strengthen the knowledge acquired in the on-campus classroom. The students are given the possibility of designing and implementing in a collaborative way the algorithms for conversions between numeral systems, by means of a wiki, and using a pseudocode.

Although all the resources put forward help to prop up the Numeral Systems' concepts, it is of special interest for this study to focus on those resources aimed at re-conceptualizing the error being studied. Most of the resources that are made available have an interactive and play-orientated approach. The use of this kind of resources aims at making available to the students alternatives that are different, motivating and entertaining, whilst educational.

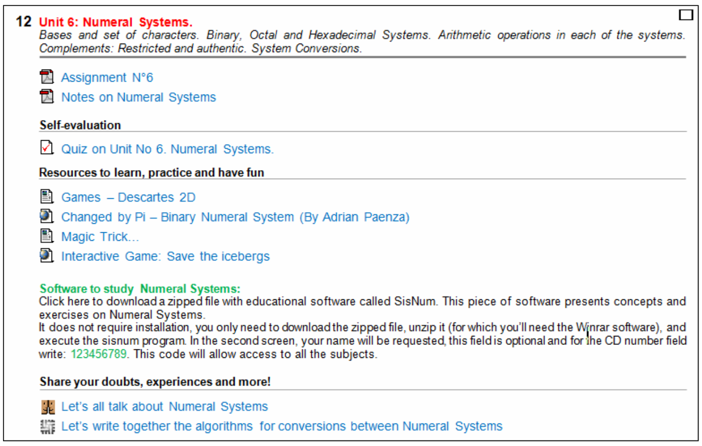

Fig. 1. Numeral Systems Block of the "Elements of Programming" virtual course.

One of these resources is the game "Save the icebergs". This game was developed by the department using Alice 3.1. [http://www.alice.org/]. The objective of this game is that through the solving of riddles presented to the player, he/she can help to save the icebergs from total destruction. The game takes place in a planet where its inhabitants have three fingers and therefore it is natural to count in a numeral system in base 3.

Some of the questions are:

- If the first even number is 2, which is the following one? Answer: 11. Once that it is clear that the following even number is 11, the following questions are asked:
- Given 11+2, is the result an even number? Press 1 for YES or 0 for NO. Answer: YES
- If $11)_3$ equals $4)_{10}$ then, does every even number in this system end with a 1? Press 1 for YES or 0 for NO. Answer: NO

The game is structured in two steps for each riddle, first a challenge is presented to the player, if he/she can solve it, then he/she can move forward towards the solving of other riddles. The second step is when a player fails to answer correctly and, as a consequence of his/her error, an iceberg disappears. It is then when the main character of the game presents to the player the possibility to recover the lost iceberg, if together they try to find the right answer. To this end, the character uses different resources, such as his fingers and some ice rocks, to count. The possibility of reasoning "together", the player and the game's character, aims at showing the correct logical reasoning to achieve the right answer. Finally, the player has a new opportunity to answer the riddle.

In order to show an example, the actions carried out in the game as a result of a "wrong" answer in the first question (it has inverted commas because the student believes there is no error) are described. In this case, the character counts with his fingers, starting with 1 and up to 11 (one-one). Simultaneously, below the character, the ice stones are grouped together until there are 4 in total. At this moment, the player is asked again. If the players gives the correct answer, the lost iceberg is recovered and the games continues, otherwise the player loses the game. Figure 2 show a screenshot of the game.

Fig. 2. Screenshot 1 - "Save the icebergs" game.

6 Results

The test was carried out on a sample of 152 students who took the course Elements of Programming in the first semester of 2013. For the purposes of this study, only the results for the block corresponding to Numeral Systems are considered. Table 2 shows the distribution of answers given by students, divided between those who passed the block and those who failed. In addition, it distinguishes –out of the wrong alternatives– which are those selected by the students.

Table 2. Distribution of answers to the exercise set out.

Answers	Passed			Failed			Total/Percentage
Correct (Alternative C)	40			4			44
Percentage	36%			10%			29%
Incorrect							
Alternative	A	B	D	A	B	D	
	49	4	17	22	6	10	
Total Incorrect	70			38			108
Percentage	70%	6%	24%	58%	16%	26%	71%
Total Answers	110			42			152
Percentage	72%			28%			100%

From the results we can infer a first approximation to the identification of the error as an epistemological obstacle.

- 72% of the students passed the Numeral Systems block, with a minimum of 3 out of 5 correct exercises. The four exercises pose operations or conversions in numeral systems, whose solutions are closely linked to the application of algorithms of arithmetical calculus, which demand an abstraction level that is lower than the problem under study.
- Of the 44 students who passed, it is not surprising that 36% responded correctly to the problem, which is much higher than the 10% that the answer correct in spite of not having passed the block.
- The highest percentage observed when answering incorrectly, corresponds in both cases to the alternative A. 70% among those who passed and 58% among those who failed . This result is coherent with the conjecture of the error that applies the parity rule within the Decimal System's domain.
- The less frequent error belongs to alternative B, which is a distracting alternative. It is natural that there is a lower percentage of choice, 6%, from students that passed, compared to the 16% obtained from students that failed.
- The alternative D has a medium level of frequency. This fact can be explained on the basis that the student who is in doubt tends to choose the answer that denies the other three alternatives.

The results, as previously mentioned, allow for a first conjecture. In the following, we analyzed if it is an epistemological obstacle that generates this error. Brousseau defines it as that which corresponds to a body of knowledge that in a previous time had a progressive direction and allowed access to a certain level of knowledge, but on attempting to widen the domain of such knowledge, difficulties arise. Therefore, the obstacle appears as a hindrance for a deeper and broader comprehension.

In order to specify in detail the idea of obstacle, the conditions that must fulfilled, according to Susana Quaranta and Maria Emilia Wolman, 1995 [4], are stated and applied to this study in table 3, below:

Table 3. Conditions to characterize an epistemological obstacle.

Condition	Characterization of the case under study
The errors correspond to conceptions that support them, which maybe either general or particular. The errors are supported by implicit theories that are coherent and consistent for the student.	There is a double difficulty: on the one hand, the error originates from a more specific conception (the specific fact of having always worked with a Decimal System), and at the same time there is an awareness that this knowledge is of a general nature. In any case, it is clear that the problem lies in the existence of "incomplete" or "relative" knowledge and not in the fact that there is a lack of knowledge, that is, always from the teacher's point of view.
An obstacle has a domain of validity and effectiveness. It exists in an area where that knowledge is correct. Said knowledge is effective within certain boundaries.	It is clear that the boundaries of the domain of knowledge are the problem to be looked into, when solving situations outside that domain. The arising question is the following: to what extent could we talk of a correct piece of knowledge if the same does not behave as such outside the domain, which is a part of a larger whole? In other words, assuming that the Decimal system is adopted, but that it coexists with as many as we would like to define, which is to say that there is an awareness of operating within a part of a whole, wouldn't it be reasonable to adjust to rules whose validity can be guaranteed for the whole and not limited to a part of the domain, which, inevitably, will become wrong for domains that are close, but different? Naturally, these questions are aimed to the teachers.
Brousseau states, "if out of that context, it generates false answers". There are problems where the piece of knowledge held by the person appears to be relevant, but turns out to be false, ineffective and a source of errors. The student is not aware that his/her knowledge is of a local nature, does not know the boundaries beyond which it loses its validity, coherence and validity.	The student is not aware of the boundaries of the domain of his/her knowledge. This is the case, because said knowledge was presented with a kind of universal validity. In this case, the statement "any number ending in an even digit, is an even number itself", gives the word any a strong semantic meaning, which is very difficult to change when assessing the validity of a proposition.
"… this piece of knowledge resists the contradictions it faces and the rooting of a better piece of knowledge. It is not enough to have a better piece of knowledge for the previous one to disappear, this is what differentiates the overcoming of obstacles from Piagetian	The results obtained are an eloquent proof of the resistance exerted by the previous piece of knowledge, when faced with a jeopardizing situation presented by the new piece of knowledge. 71% of the students that had to reconsider their previous concepts, not so much as incorrect but as unfinished before the new piece of knowledge, when tested on their reflections about the behavior of rules formerly

accommodation. It is essential to identify it and to contrast it against the new piece of knowledge. In order to establish its obstacle nature, it is essential to prove and explain the resistance to reject one piece of knowledge and to the rooting of a more accurate one.	known, gave way to the local knowledge mentioned above.
Obstacles do not disappear suddenly. They continue to appear, and keep on arising long after having been conscientiously rejected.	There are no data on the persistence, or not, of the error under this study.

Resuming the subject of obstacles that originate at school, in a conference in the UQAM, Canada, on January 21, 1988, Brousseau stated the following on the link between obstacles and the teaching-learning contract [5]:

"The student's place in the teaching-learning has been reclaimed by different disciplines -psychoanalysis, psychology, pedagogy, etc.- as the place of "reality"". Genetic epistemology has offered the most serious arguments and those closer to knowledge, but other studies are necessary to use its contributions. It is frequent that the student's mistakes are read by the teacher as an inability to reason in general or, at least as a logical error: within a broader teaching-learning contract, the teacher takes charge of the representations, of the direction of knowledge. But, within narrower conditions, he/she is only led to point where the student's answer contradicts previous knowledge, carefully avoiding any diagnosis on the cause of the error. .... This teaching-learning contract provided the teacher the safest defense: He/she only takes responsibility of the knowledge already known within his/her own domain. It is enough for the teacher to set an axiomatic order and then demand axioms as answers."

If the teacher does not enquire about the student's representations, because of his/her own disciplinary and/or pedagogical shortages, there is a risk of upholding teaching procedures that are not the correct ones to teach Numeral Systems. According to Delia Lerner and Patricia Sadovsky [6]:

"The hypothesis according to which the numeral writing arises from its correlation to oral numeracy, leads children to produce non-conventional notations. Why? Because, unlike written numeracy, oral numeracy is not positional".

An example of the statement above, is when the child writes four thousand seven hundred and five, just as he says it: 4 1000 700 5.

In the problem treated in this study, the number one-one, which is the right answer, is presented in a type of representation in which it is difficult to identify an even number. Only through the counting of integers and the acknowledging of the alternation between even and odd numbers, on the number line, or through the conversion of $(11)_5$ to the Decimal System, by applying a weighted sum of powers of the base, or in other words, by resorting to the positional nature of the numeral systems, it is possible to recognize one-one as the equivalent to 6 in the Decimal System, which is an even number in this system and therefore even number in any other system. There is research suggesting that the child apprehends the numeral system, not from the intrinsic characteristics of the system, that is, from its positional condition, but rather from conceptualizations that he/she has on quantities, knowledge that comes from the oral world, before the written representation.

7 Some conclusions

The conclusion is therefore, that when revisiting this body of knowledge in an introductory university course, or within a Liaison experience with future University students, special attention must be put on written representations and representational discourse of numbers. Both types of representations coexist in the numeral system in base 10, which the student uses all throughout school. Incorporating the Binary, Octal, Hexadecimal systems or any other, including systems such as the one presented here with generic characters, requires a higher level of abstraction in order to maintain stable those previous pieces of knowledge and revisiting the validity of the rules in this new domain.

The present landscape, in which scientific-technological University Degrees show a strong negative tendency with regards of student retention and persistence rates, challenges the teacher to revisit his/her practices, searching for teaching procedures that allow real processes for construction and deconstruction of errors, as a real source of epistemological progress. Likewise, the University, continuing with the educational policy of universal admission, should review its support and guidance strategies available for students, who experience complex and heterogeneous learning paths. To assume the challenge of rising numbers of admitted students means to assume the problem of the diversity and inequality of previous knowledge with which they come with when entering University, which is the result of a fragmented education system. We are convinced that Liaison activities can be a very useful tool. Additionally, the University should promote learning environments in which that previous body of knowledge is challenged, directed towards a process of conceptual change. Another fundamental aspects to have in mind are the students' interests, who are just starting to solidify their vocation towards Computer Sciences. In both cases it is essential to approach the syllabus' contents by means of activities that are enjoyable, creative and that require team work.

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