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Research Article

A Note on Holography and Phase Transitions

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Focusing on the connection between the Landau theory of second-order phase transitions and the holographic approach to critical phenomena, we study diverse field theories in an anti de Sitter black hole background. Through simple analytical approximations, solutions to the equations of motion can be obtained in closed form which give rather good approximations of the results obtained using more involved numerical methods. The agreement we find stems from rather elementary considerations on perturbation of Schrödinger equations.

1. Introduction

Much activity has been centered in the last few years on the application of the gauge/gravity duality, originally emerged from string theory [1–3], to analyze strongly interacting field theories by mapping them to classical gravity. More recently such duality was successfully applied to describe, holographically, systems undergoing phase transitions like superconductors, superfluids, and other strongly interacting systems [4–17] (for a wider list of references see e.g., [18]).

In the holographic approach to the study of phase transitions one starts, on the gravity side, with a field theory in asymptotically anti de Sitter (AdS) space-time with temperature arising from a black hole metric, either introduced as a background or resulting from back reaction of matter on the geometry. Then, using the AdS/CFT correspondence, one can study the behavior of the dual field theory defined on the boundary, identify order parameters, and analyze the phase structure of the dual system.

Several models with scalar and gauge fields in a bulk which corresponds asymptotically to an AdS black hole metric have been studied [4–17]. The rather involved systems of nonlinear coupled differential equations that have to be solved require general application of numerical methods. In this way one often finds nontrivial hairy solutions that cease to exist for $T > T_c$, where T is the Hawking temperature associated with the black hole and T_c a critical temperature depending on the space-time dimensions and the parameters of the model on the bulk.

The main point in this AdS/CFT based calculation is that the asymptotics of the solution in the bulk encodes the behavior of the QFT at finite temperature defined on the border. One finds in general a typical scenario of second-order phase transition. The critical exponents can be computed with a rather good precision and coincide with those obtained within the mean field approximation in a great variety of models.

It is the purpose of this paper to get an analytical insight complementing the numerical results, with a focus on the connection between the holographic approach and mean field theory for the calculation of critical exponents. To this end, it will be important to first stress some connections, already signaled in [6], between the Landau phenomenological theory of second-order phase transitions and the gauge/gravity approach. We will expose the importance of analyticity to explain the similitude in the results for relevant physical quantities (critical exponents, dependence of the critical temperature on the charge density, and magnetic field, etc.) in very diverse models. This will be done first by stressing the relationship between the equations of motion and a Schrödinger problem, so that usual perturbative techniques allow to prove the critical behavior. We then show that simple matching conditions lead to results that broadly agree with elaborate numerical calculations.

It should be noted that our calculations assume that the backreaction of dynamical fields is negligible (probe approximation) which is valid when the gauge coupling constant is large. This approximation is useful to study the behavior near the phase transition which is precisely the domain we will analyze, comparing our analytical results with those obtained numerically. In fact, the holographic results we have previously obtained solving numerically the Einstein-Yang-Mills-Higgs equations of motion both considering the probe approximation [7] and the case in which matter backreacts on the geometry [5] show that when parameters are such that condensation takes place the solutions are very similar and the nature of the phase transition is identical.

With this in mind and using an analytic approach proposed in [8], we study the equations of motion for different models, showing how one can determine in a very simple way the critical behavior of the systems defined on the border with a good agreement with numerical results.

2. Holography and Mean Field Results

In the Landau approach to second-order phase transitions one considers an order parameter $\mathcal{O}$ and assumes analyticity of the free energy $F = F(T, \mathcal{O})$, which can be expanded in even powers of $\mathcal{O}$

$$F(T, \mathcal{O}) = F = F_0 + F_2[T]\mathcal{O}^2 + F_4[T]\mathcal{O}^4 + \cdots. \tag{2.1}$$

The dependence of the order parameter on T is obtained by minimizing F as a function of $\mathcal{O}$. The next step is to expand the coefficients F_2 and F_4, in powers of $T - T_c$. To ensure stability and a change of behavior at T_c, one takes $F_2 = a(T - T_c)$ and $F_4 = b/4$ with a, b positive constants. In this way, one finds that the minimum of F is at $\mathcal{O} = 0$ for $T > T_c$ (disordered phase) while for $T < T_c$ one has

$$\mathcal{O} \sim (T_c - T)^{1/2}. \tag{2.2}$$

Let us give a brief description of gauge/gravity approach to phase transitions to connect it to the Landau theory. On the gravity side one considers a classical field theory in a Schwarzschild-AdS black hole background (or one in which backreaction of fields on an asymptotically AdS space leads to a black hole solution). The choice of such a geometry is dictated by the fact that the warped AdS geometry prevents massive charged particles to be repelled to the boundary by a charged horizon and as a result a condensate floating over the horizon can be formed. To find such condensate one should look for nontrivial static solutions for the fields outside the black hole by imposing appropriate boundary conditions. The behavior of fields at infinity then allows to determine the dependence of the order parameter on temperature, identified with the Hawking temperature of the black hole.

Basically, the free energy F is identified with the minimum of the action in the gravitational theory with prescribed boundary condition for the fields. Regularity of the fields at the horizon and smoothness of the background geometry, basic assumptions within the gauge/gravity duality, imply analyticity of this action. This is a first point of contact with Landau theory of second-order phase transitions leading to a mean field behavior. We will now see that if continuity and smoothness of fields are imposed in the region between the horizon and infinity, the mean field behavior found using a numerical approach is reproduced with a good precision, as can be seen following a very simple analytic approach proposed in [8] which we apply below for different models.

3. The Abelian Higgs Model in d Space Dimensions

Dynamics of the system is governed by the action

$$S = \int d^{d+1}x\sqrt{|g|}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \left|\nabla_\mu - iA_\mu\Psi\right|^2 - m^2|\Psi|^2\right). \tag{3.1}$$

The background metric is the standard AdS$_{d+1}$-Schwarschild black hole

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 dx_i dx^i, \quad i = 1,2,\ldots,d-1, \\ f(r) &= r^2\left(1 - \frac{r_h^d}{r^d}\right), \end{aligned} \tag{3.2}$$

with $i = 1, 2, \ldots, d-1$ and we have chosen the AdS radius to be unity. Different values for the mass term can be considered [9, 10]: $m^2 = 0, -2$ and $m^2 = m^2_{\mathrm{BF}} = -9/4$ in $d = 3$ and $m^2 = 0, -3, -4$ in $d = 4$, the last value corresponding to the Breitenlohner-Freedman bound

$m_{\mathrm{BF}}^2 < 0$ marking the boundary of stability for a scalar field in AdS [19]. The black hole temperature is given by

$$T = \frac{d}{4\pi} r_h. \tag{3.3}$$

In order to look for simple classical solutions for this model one can propose the following ansatz:

$$\Psi = |\Psi| = \psi(r), \qquad A_\mu = \phi(r)\delta_{\mu 0}. \tag{3.4}$$

Imposing regularity of the solution at the horizon ($r = r_h$) one gets

$$\begin{aligned} \psi(r_h) &= \frac{d r_h}{m^2}\psi'(r_h), \\ \phi(r_h) &= 0. \end{aligned} \tag{3.5}$$

Concerning the asymptotic behavior of the scalar potential ϕ and the scalar field ψ one has

$$\begin{aligned} \psi &= \frac{\psi_-}{r^{\lambda_-}} + \frac{\psi_+}{r^{\lambda_+}}, \\ \phi &= \mu - \frac{\rho}{r^{d-2}} + \cdots, \end{aligned} \tag{3.6}$$

with

$$\lambda_\pm = \frac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2}\right). \tag{3.7}$$

According to the gauge/gravity correspondence, μ corresponds to the chemical potential in the dual theory defined on the boundary and ρ to the charge density. Concerning the scalar field ψ, both falloffs are acceptable provided the following condition holds [20]:

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1, \tag{3.8}$$

One can then pick either $\psi_+ = 0$ or $\psi_- = 0$ leaving a one parameter family of solutions so that the condensate of the dual operator $\mathcal{O}^\sigma$ will be given by

$$\langle \mathcal{O}^\sigma \rangle = \psi_\sigma, \tag{3.9}$$

with the boundary condition written as

$$\varepsilon^{\sigma\rho}\psi_\rho = 0, \tag{3.10}$$

where $\sigma,\ \rho = +,\ -$ and $\varepsilon^{+-} = 1$. For definiteness we will impose the condition $\psi_- = 0$.

It will be convenient for the analysis that follows to change variables according to

$$z = \frac{r_h}{r}, \tag{3.11}$$

so that the horizon is fixed at $z = 1$ and the boundary at $z = 0$. In terms of this new variable z, the equations for the functions in ansatz (3.4) become

$$\psi'' - \frac{d-1+z^d}{z(1-z^d)}\psi' + \left(\frac{\phi^2}{r_h^2(1-z^d)^2} - \frac{m^2}{z^2(1-z^d)}\right)\psi = 0, \tag{3.12}$$

$$\phi'' - \frac{d-3}{z}\phi' - \frac{2\psi^2}{z^2(1-z^d)}\phi = 0, \tag{3.13}$$

where the prime denotes d/dz.

Conditions (3.5) and (3.6) now read

$$\begin{gathered}\psi'(1) = -\left(\frac{m^2}{d}\right)\psi(1) = 0,\\ \phi(1) = 0,\end{gathered} \tag{3.14}$$

$$\begin{gathered}\psi_B(z) \simeq_{z\to 0} D_+ z^{\lambda_+} + D_- z^{\lambda_-},\\ \phi_B(z) \simeq_{z\to 0} \mu - q z^{d-2},\end{gathered} \tag{3.15}$$

with $D_\pm = r_h^{-\lambda_\pm}\,\psi_\pm$ and $q = \rho / r_h^{d-2}$.

In view of (3.3), the system (3.12)-(3.13) only depends on the black-hole temperature T through the nonlinear $\phi^2\psi$ term in the scalar field equation (3.12): this describes the coupling of the scalar to the electric potential and gives an effective negative mass.

In the limit $T \to \infty$ the nonlinear term in (3.12) vanishes so that the equation becomes linear with no nontrivial soliton solutions. Now, as already pointed out in [11] the coupling of the scalar to the Maxwell field is responsible for producing a negative effective mass for ψ, and this effect becomes more important at low temperatures. This indicates that one should expect an instability taking place at some point towards forming scalar hair. At such point, the stabilizing effect of the m-term is overcome. If a nontrivial solution exists at low temperatures, with an asymptotic behavior associated with a nonzero v.e.v. of an order parameter $\langle O_\psi \rangle$ in the dual theory, it is natural to expect that a critical temperature T_c should exist such that $\langle O_\psi \rangle_{T>T_c} = 0$. This was indeed verified numerically for different values of the mass m (including the Breitenlohner-Freedman bound value) and various space-time dimensions [9, 10].

Now, (3.12) for fixed potential ϕ can be viewed as a Schrödinger equation with zero energy: in order to have nontrivial solutions, the resulting potential has to adjust itself for the existence of a unique zero eigenvalue for every temperature below the critical one. In the vicinity of the critical point, the variation of the potential is linear in the temperature variation, but quadratic in the normalization of the field through the effect of the nonlinear term in (3.13). The eigenvalue of the Schrödinger equation is, to first order, simply proportional to some integral of the potential as can be seen from its variational evaluation: with a suitable weight function $a(z)$, the integral of $a(z)\psi(z)$ times the left hand side of (3.12) gives the eigenvalue for a normalized ψ and its variations are of second order with respect to the variations of ψ. Fixing ψ to the eigenfunction for $T = T_c$, the two sources of the variation of this integral through the potential must compensate themselves, and one obtains a $\Delta T \propto \phi^2$ relation. It is precisely this type of behavior that has been found from the numerical solution to the differential equations on the gravity side.

To go further in the analysis without resorting to a numerical analysis, we will consider expansions of the fields in the bulk near $z = 1$ and $z = 0$. Imposing the conditions of continuity and smoothness of the solutions at a point z_m intermediate between the boundary ($z = 0$) and the horizon ($z = 1$) will give algebraic equations between the parameters of the solution.

For the solution near the horizon ($z = 1$) we have, up to order $(z-1)^2$,

$$\begin{aligned} \psi_H(z) &= \psi_0 + \psi_1(z-1) + \frac{1}{2}\psi_2(z-1)^2, \\ \phi_H(z) &= \phi_0 + \phi_1(z-1) + \frac{1}{2}\phi_2(z-1)^2, \end{aligned} \tag{3.16}$$

with $\psi_0,\ \psi_1,\ \psi_2,\ \phi_0,\ \phi_1,\ \phi_2$ constants. The boundary conditions (3.14) at $z = 1$ imply

$$\begin{gathered} \psi_1 = -\left(\frac{m^2}{d}\right)\psi_0, \\ \phi_0 = 0. \end{gathered} \tag{3.17}$$

Substituting these values in (3.16) and using the differential (3.12)-(3.13) we can obtain ϕ_2 and ψ_2 as a function of ϕ_1 and ψ_0. We get

$$\begin{aligned} \psi_H(z) &= \psi_0 + \frac{m^2}{d}\psi_0(1-z) + \frac{2dm^2r_h^2 + m^4r_h^2 - \phi_1^2}{4d^2r_h^2}\psi_0(1-z)^2, \\ \phi_H(z) &= -\phi_1(1-z) + \frac{1}{2}\left(d - 3 - \frac{2\psi_0^2}{d}\right)\phi_1(1-z)^2. \end{aligned} \tag{3.18}$$

As announced, imposing the conditions of continuity and smoothness at an intermediate point z_m allows to obtain a solution. Interestingly enough, as first observed in [8] the result of this crude approximation is quite stable with respect to the intermediate point, so we will consider the case $z_m = 1/2$. This can be understood from the fact that ψ is the ground state of a Schrödinger equation, so that it cannot have nodes: additional terms in the expansion of ψ should rapidly fade away.

We will first analyze the boundary condition $D_- = 0$, so we have the set of equations

$$\begin{aligned} \psi_H\left(\frac{1}{2}\right) &= \psi_B\left(\frac{1}{2}\right), \qquad \psi'_H\left(\frac{1}{2}\right) = \psi'_B\left(\frac{1}{2}\right), \\ \phi_H\left(\frac{1}{2}\right) &= \phi_B\left(\frac{1}{2}\right), \qquad \phi'_H\left(\frac{1}{2}\right) = \phi'_B\left(\frac{1}{2}\right), \end{aligned} \tag{3.19}$$

to be solved for ψ_0, ϕ_1, D_+, and μ. We obtain

$$\begin{aligned}
\psi_0^2 &= \frac{d}{2^{d+1}}\frac{16(d-2)Bq + 2^d(d-5)r_h A}{r_h A}, \\
D_+ &= \frac{2^{\lambda_+-1}(4d+m^2)}{dB^2}\psi_0, \\
\phi_1 &= -\frac{r_h A}{B}, \\
\mu &= \frac{1}{2^{d+2}}\frac{8dqB + 2^d r_h A}{B}
\end{aligned} \tag{3.20}$$

(for simplicity D_+ is written in terms of ψ_0). Here

$$\begin{aligned}
A &= \sqrt{16d^2\lambda_+ + m^4(\lambda_+ + 2) + 2dm^2(6+5\lambda_+)}, \\
B &= \sqrt{\lambda_+ + 2}.
\end{aligned} \tag{3.21}$$

Using the AdS/CFT dictionary (3.9), we can identify the v.e.v. $\langle\mathcal{O}\rangle$ of the operator $\mathcal{O}$ dual to the scalar field with the asymptotic coefficient ψ_+, $\langle\mathcal{O}\rangle \equiv \psi_+ = r_h^{\lambda_+}D_+$. Now, one can write D_+ (or ψ_0 since they are proportional) as a function of $T = (d/4\pi)r_h$. Remembering that $q = \rho/r_h^{d-2}$ one has

$$\psi_0^2 = \frac{d(5-d)}{2}\left(\frac{T_c}{T}\right)^{d-1}\left[1 - \left(\frac{T}{T_c}\right)^{d-1}\right], \tag{3.22}$$

where we have defined

$$T_c^{d-1} = \frac{2^{4-d}(d-2)}{(5-d)(4\pi/d)^{d-1}}\frac{B}{A}\rho. \tag{3.23}$$

We then have, for the order parameter

$$\langle\mathcal{O}\rangle = C_d\left(\frac{T}{T_c}\right)^{(\sqrt{d^2+4m^2}+1)/2}\sqrt{1 - \left(\frac{T}{T_c}\right)^{d-1}}, \tag{3.24}$$

with

$$C_d = \frac{(4d+m^2)}{(\lambda_+ + 2)}\left(\frac{d}{2\pi}\right)^{\lambda_+}\left(\frac{(5-d)}{8d}\right)^{1/2}. \tag{3.25}$$

One can see from (27) that for T close to T_c one has the typical second-order phase transition behavior $\langle\mathcal{O}\rangle \propto \sqrt{(1-T)/T_c}$. Note that $T_c \propto \rho^{1/(d-1)}$ in agreement with the change of dimensions of the charge density for different d's.

Our results for the critical temperature, as inferred from (3.23)–(3.25) in the cases $d = 3, 4$, $m^2 = -2$ (with $z_m = 1/2$) are $T_c = 0.15\rho^{1/2}$ and $T_c = 0.2\rho^{1/3}$, respectively. They can be compared with those obtained using a different analytical approximation based on perturbation theory near the critical temperature. For the case $m = m_{\mathrm{BF}}$, we obtain $T_c = 0.12\rho^{1/2}$ and $T_c = 0.25\rho^{1/3}$ [21, 22] in very good agreement with the exact numerical results, $T_c = 0.15\rho^{1/2}$ and $T_c = 0.25\rho^{1/3}$. One can conclude that there is a good quantitative agreement between the three sets, which is not much affected by the choice of the point z_m in the method. This last fact was already observed in [8] for the particular case $d = 3$ compared with the numerical results given in [9].

4. Non-Abelian Gauge Field in $d = 3$ Space Dimensions

We now consider the case of an SU(2) Yang-Mills theory in 3 + 1 dimensional Antide Sitter-Schwarzschild background, as a prototype for the gauge/gravity duality in the case of pure gauge theories [12, 13]. The action is

$$S = -\frac{1}{4}\int d^4x\sqrt{|g|}F^a_{\mu\nu}F^{a\mu\nu}, \tag{4.1}$$

with the field strength defined as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu, \quad a = 1, 2, 3. \tag{4.2}$$

Writing the gauge field as an isospin vector

$$\vec{A} = \left(A^1, A^2, A^3\right) = \left(A^a_\mu dx^\mu\right), \tag{4.3}$$

we consider the following ansatz for solving the equations of motion:

$$\vec{A} = J(r)dt\check{e}_3 - K(r)\rho d\varphi\check{e}_\rho + K(r)d\rho\check{e}_\varphi, \tag{4.4}$$

where r is the radial variable in spherical coordinates and

$$\begin{aligned} x &= \rho\cos\varphi, \\ y &= \rho\sin\varphi. \end{aligned} \tag{4.5}$$

The background metric is

$$ds^2 = -f(r)d^2t + \frac{1}{f(r)}d^2r + r^2\left(dx^2 + dy^2\right), \tag{4.6}$$

with

$$f(r) = r^2\left(1 - \frac{r_h^3}{r^3}\right). \tag{4.7}$$

In terms of the z variable defined in (3.11) the equations of motion read

$$\begin{aligned}\left(\left(1 - z^3\right)K'\right)' &= \frac{1}{r_h^2}\left(K^2 - \frac{J^2z^2}{1 - z^3}\right)K,\\ J''(z) &= \frac{2}{r_h^2}\frac{J(z)K^2(z)}{1 - z^3}.\end{aligned} \tag{4.8}$$

At the horizon $(z = 1)$, J must vanish, so we have the following expansions for the fields, up to order $(z - 1)^2$,

$$\begin{aligned}K_H(z) &= K_0 + K_1(z - 1) + \frac{1}{2}K_2(z - 1)^2,\\ J_H(z) &= J_1(z - 1) + \frac{1}{2}J_2(z - 1)^2.\end{aligned} \tag{4.9}$$

Using the equations of motion (4.8), the coefficients K_1, K_2, and J_2 can be written in terms of K_0 and J_1 as

$$\begin{aligned}K_H(z) &= K_0 - \frac{K_0^3}{3r_h^2}(z - 1) + \frac{-J_1^2K_0r_h^2 + 6K_0^3r_h^2 + 3K_0^5}{36r_h^4}(z - 1)^2,\\ J_H(z) &= J_1(z - 1) - \frac{J_1K_0^2}{3r_h^2}(z - 1)^2.\end{aligned} \tag{4.10}$$

At the $z = 0$ boundary one has the asymptotic expansions

$$\begin{aligned}K_B(z) &= \frac{C_1}{r_hz},\\ J_B(z) &= D_0 - \frac{D_1}{r_hz},\end{aligned} \tag{4.11}$$

where D_1 can be associated with the charge density. Coefficient C_1 in (4.11) should be identified with the order parameter $\langle \mathcal{O}_K \rangle$ for the theory on the border. Since the order parameter is related to a vector field (A_i), the associated theory on the border is a p-wave superconductor [13].

As in the scalar field case, we will match both solutions at an intermediate point which we again choose as $z_m = 1/2$,

$$K_H\left(\frac{1}{2}\right) = K_B\left(\frac{1}{2}\right), \qquad K'_H\left(\frac{1}{2}\right) = K'_B\left(\frac{1}{2}\right),$$
$$J_H\left(\frac{1}{2}\right) = J_B\left(\frac{1}{2}\right), \qquad J'_H\left(\frac{1}{2}\right) = J'_B\left(\frac{1}{2}\right), \tag{4.12}$$

and solve these equations for K_0, J_1, C_1, and D_0. From the first two identities (4.12), we obtain

$$C_1 = \frac{4}{3}r_h K_0 + \frac{K_0^3}{9r_h},$$
$$J_1 = \sqrt{-48\ r_h^2 + 22\ K_0^2 + 3\ (K_0^4/r_h^2)} \tag{4.13}$$

(we chose $J_1 < 0$ so $J(z) > 0$).

Substituting these values in (4.12), we can obtain K_0 as the root of the following polynomial in K_0^2:

$$\frac{3}{r_h^2}K_0^8 + 40K_0^6 + 207r_h^2K_0^4 + 486r_h^4K_0^2 - 9r_h^2\left(D_1^2 - 48r_h^4\right) = 0. \tag{4.14}$$

This equation implies that K_0^2 vanishes as

$$K_0^2 = \frac{D_1^2 - 48r_h^4}{54r_h^2} + O\left(D_1^2 - 48r_h^4\right)^2. \tag{4.15}$$

Finally, introducing the temperature $T = 3/(4\pi)r_h$, we can write

$$K_0^2 = \frac{128\pi^2}{81}\frac{T_c^4}{T^2}\left[1 - \left(\frac{T_c}{T}\right)^4\right] \quad \text{for } T \text{ near } T_c, \tag{4.16}$$

where

$$T_c^2 = \frac{3\sqrt{3}}{64\pi^2}D_1. \tag{4.17}$$

Then, for T close to T_c, we have the typical second-order phase transition behavior for $C_1 = \langle \mathcal{O}_K \rangle \propto \sqrt{(1-T)/T_c}$, in good agreement with [12, 13]. Our numerical value for T_c from (4.17), $T_c/\sqrt{D_1} = 0.091$, can be compared with the numerical value obtained in [13], $T_c/\sqrt{D_1} = 0.125$.

5. The $d = 3$ Scalar Case in the Presence of an Applied Magnetic Field

We will consider here a system with dynamics governed by the action (3.1) in the $d = 3$, $m^2 = -2$ case, when an external magnetic field H is applied. This corresponds to the case where the background metric is an AdS_{3+1} *magnetically* charged black hole [14, 15]:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 dx_i dx^i, \quad i = 1,2, \\ f(r) &= \frac{r^2}{L^2} - \frac{M}{r} + \frac{H^2}{r^2}. \end{aligned} \tag{5.1}$$

The $f(r) = 0$ condition for having event horizons gives a quartic algebraic equation with 4 roots which can be explicitly written in terms of surds. There are two complex conjugate roots which we will call r_1 and r_2 and two real roots $r_3 < r_4$. We will then call $r_h \equiv r_4$ the external black hole horizon. Taking $L = 1$ from here on, one has

$$r_h = \frac{1}{26^{1/3}}\left(B + \sqrt{\frac{12M}{B} - \mathrm{B}^2}\right), \tag{5.2}$$

where

$$B = \sqrt{\frac{83^{1/3}H^2}{\left(D + 9M^2\right)^{1/3}} + 2^{1/3}\left(D + 9M^2\right)^{1/3}}, \qquad D = \sqrt{81M^4 - 768H^6}. \tag{5.3}$$

The actual form of the other roots r_a $(a = 1,2,3)$ is not necessary since one will only need the standard relationships between roots and coefficients

$$\begin{aligned} r_1 + r_2 + r_3 &= -r_h, \\ r_1 r_2 + r_1 r_3 + r_2 r_3 &= r_h^2, \\ r_1 r_2 r_3 &= \frac{H^2}{r_h}, \end{aligned} \tag{5.4}$$

together with the relation

$$\frac{H^2}{r_h^4} - \frac{M}{r_h^3} + 1 = 0. \tag{5.5}$$

The Hawking temperature associated to the black hole is

$$T = \frac{f'(r_h)}{4\pi} = \frac{1}{4\pi}\left(3r_h - \frac{h^2}{r_h^3}\right). \tag{5.6}$$

In terms of the variable $z = r_h/r$, the equations of motion read

$$\begin{aligned}
&\tilde{\psi}''(z) + \frac{df(r_h/z)/dz}{f(r_h/z)}\,\tilde{\psi}'(z) + \frac{r_h^2}{z^4}\,\left(\frac{\tilde{\phi}^2(z)}{f(r_h/z)^2} + \frac{2}{f(r_h/z)}\right) = 0,\\
&\tilde{\phi}''(z) - 2\frac{\psi^2(z)}{f(r_h/z)}\tilde{\phi}(z)\left(\frac{r_h^2}{z^4}\right) = 0,
\end{aligned} \tag{5.7}$$

where $\tilde{\psi}(z) \equiv \psi(r_h/z)$ and $\tilde{\phi}(z) \equiv \phi(r_h/z)$ (but from now on, the tilde will be omitted). In terms of the new variables, we have

$$\begin{aligned}
f\left(\frac{r_h}{z}\right) &= \left(\frac{r_h}{z}\right)^2(1-z)(1-\tilde{r}_1 z)(1-\tilde{r}_2 z)(1-\tilde{r}_3 z),\\
\frac{df(r_h/z)/dz}{f(r_h/z)} &= \frac{-2 - M(z^3/r_h^3) + 2H^2(z^4/r_h^4)}{z(1-z)(1-\tilde{r}_1 z)(1-\tilde{r}_2 z)(1-\tilde{r}_3 z)},
\end{aligned} \tag{5.8}$$

where $\tilde{r}_a = r_a/r_h$, $a = 1, 2, 3$.

The boundary conditions for system (5.7) at the horizon are

$$\phi(1) = 0, \tag{5.9}$$

$$\psi'(1) = \frac{2r_h}{3r_h - (H^2/r_h^3)}\psi(1), \tag{5.10}$$

while asymptotically one has

$$\begin{aligned}
\psi(z) &= D_1 z + D_2 z^2,\\
\phi(z) &= \mu - \left(\frac{\rho}{r_h}\right)z.
\end{aligned} \tag{5.11}$$

For the solution of system (5.7) near the horizon, we have the expansions, up to order $(z-1)^2$,

$$\begin{aligned}
\psi_H(z) &= \psi_0 + \psi_1(z-1) + \frac{1}{2}\psi_2(z-1)^2,\\
\phi_H(z) &= \phi_0 + \phi_1(z-1) + \frac{1}{2}\phi_2(z-1)^2,
\end{aligned} \tag{5.12}$$

with ψ_0, ψ_1, ψ_2, ϕ_0, ϕ_1, and ϕ_2 constants. Using (5.10) we get

$$\begin{aligned}
\phi_0 &= 0,\\
\psi_1 &= \frac{2r_h}{3r_h - (H^2/r_h^3)}\ \psi_0.
\end{aligned} \tag{5.13}$$

Substituting these values in (5.12) and using the differential equations, we can obtain ϕ_2 and ψ_2 as functions of ϕ_1 and ψ_0. We get

$$\begin{aligned}\psi(z) &= \psi_0 + \frac{2r_h^4}{H^2 - 3r_h^4}\psi_0(1-z) - \frac{r_h^4\left(8r_h^4 + r_h^2\phi_1^2 - 12H^2\right)}{4\left(H^2 - 3r_h^4\right)^2}\psi_0(1-z)^2, \\ \phi(z) &= -\phi_1(1-z) + \frac{r_h^4\psi_0^2\phi_1}{H^2 - 3r_h^4}(1-z)^2.\end{aligned} \tag{5.14}$$

As before, we impose matching conditions at $z = 1/2$:

$$\begin{aligned}\psi_H\left(\frac{1}{2}\right) &= \psi_B\left(\frac{1}{2}\right), & \psi'_H\left(\frac{1}{2}\right) &= \psi'_B\left(\frac{1}{2}\right), \\ \phi_H\left(\frac{1}{2}\right) &= \phi_B\left(\frac{1}{2}\right), & \phi'_H\left(\frac{1}{2}\right) &= \phi'_B\left(\frac{1}{2}\right),\end{aligned} \tag{5.15}$$

and solve for ψ_0, ϕ_1, D_2, and μ. We look for solutions with $D_1 = 0$ and D_2 corresponding to the order parameter of the $2+1$ system defined on the boundary. We obtain

$$\begin{aligned}\phi_1 &= -2R(r_h, H), \\ \mu &= \frac{H^2\left((\rho/r_h) + 2R(r_h,H)\right) - r_h^4\left((3\rho/r_h) + R(r_h,H)\left(\psi_0^2 + 6\right)\right)}{2\left(H^2 - 3r_h^4\right)}, \\ \psi_0^2 &= \frac{\left(3r_h^4 - H^2\right)\left((\rho/r_h) - 2R(r_h,H)\right)}{2r_h^4 R(r_h,H)}, \\ D_2 &= \frac{\left(88r_h^8 - \phi_1^2 r_h^6 - 68H^2 r_h^4 + 16H^4\right)\psi_0}{4\left(H^2 - 3r_h^4\right)^2},\end{aligned} \tag{5.16}$$

where

$$R(r_h, H) = \sqrt{\frac{7r_h^8 - 6H^2r_h^4 + 2H^4}{r_h^6}}. \tag{5.17}$$

The equation for ψ_0^2 in terms of the dimensionless variable $u = r_h/\sqrt{H}$ takes the form

$$\psi_0^2 = \frac{\left(3u^4 - 1\right)\left((\rho/H) - 2\sqrt{(2/u^4) - 6 + 7u^4}\right)}{2u^4\sqrt{(2/u^4) - 6 + 7u^4}}, \tag{5.18}$$

with the temperature (5.6) given by

$$\frac{T}{\sqrt{H}} = \frac{1}{4\pi}\left(3u - \frac{1}{u^3}\right). \tag{5.19}$$

The minimum value that u can take is the one for which $T = 0$, $u = 3^{-1/4}$, and corresponds to the condition $3^3 M^4 = 2^7 H^6$. From this, we see that in order to have a nontrivial solution the following inequality should hold:

$$H \leq \frac{\rho}{2\sqrt{(2/u^4) - 6 + 7u^4}}. \tag{5.20}$$

The maximum of the r.h.s. is attained for $u_m = (2/7)^{1/8} > u_0$ so that there is a critical value H_c of the magnetic field beyond which no nontrivial solution exists,

$$H_c = 0.41\rho. \tag{5.21}$$

Using (5.18)-(5.19), one can determine the critical temperature T_c as a function of H. We give in Figure 1 the resulting $T_c = T_c(\mathrm{H})$ curve. Interestingly, we find that in the range

$$H_c > H > \frac{\rho}{2\sqrt{(2/u_0^4) - 6 + 7u_0^4}} = 0.327\rho, \tag{5.22}$$

the curve $T_c(H)$ becomes double valued so that a nontrivial solution only exists in the range $T_{c_2} > T > T_{c_1}$. According to the gauge/gravity duality D_2 should be identified with the order parameter, $\langle \mathcal{O}_\psi \rangle = D_2$. We obtain the following expression:

$$D_2 = \frac{5u^4 - 2}{6u^4 - 2}\frac{\sqrt{(\rho/H) - 2\sqrt{(2/u^4) - 6 + 7u^4}}}{u^2\left((2/u^4) - 6 + 7u^4\right)^{1/4}}. \tag{5.23}$$

Note that D_2 becomes negative for $u_0 < u < (2/5)^{1/4}$ or, equivalently, for $0 < T < 0.0316\sqrt{H}$. However, when $H \to 0$ the numerator and denominator coefficients of D_2 that multiply ψ_0 cancel out. This is consistent with the result of the nomagnetic field model. (In fact, we have checked that the results of both models, $H = 0$, $d = 3$, and $m^2 = -2$, are identical. We present in Figure 2 curves for D_2 as a function of temperature for different values of the external magnetic field.

The results are qualitatively in agreement with those described in [14, 15] where the reported curves obtained by numerically solving the equations of motion are similar to those in Figure 2.

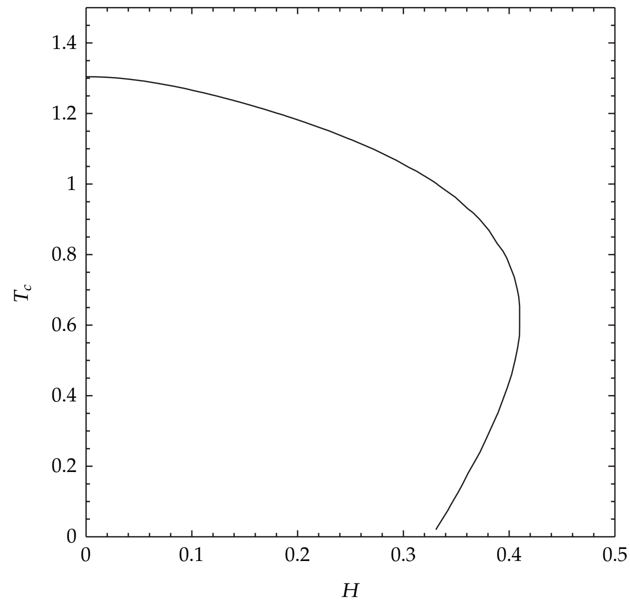

Figure 1: The phase diagram of T_c against the magnetic field H. The condensed phase ($D_2 \neq 0$) corresponds to the lower left part below the line. The critical temperature decreases as the magnetic field grow up to the critical value H_c.

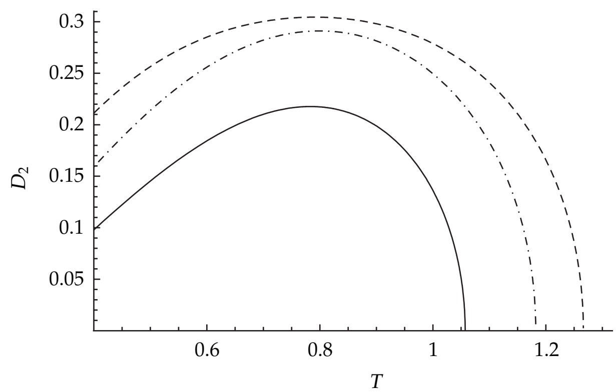

Figure 2: A plot of the order parameter D_2 as a function of temperature T. The charge density is $\rho = 1$. The dashed line corresponds to a magnetic field $H = 0.1$, the dash-dot one to $H = 0.2$, and the solid one to $H = 0.3$. (the critical value is $H_c = 0.6$).

6. Summary and Discussion

We have analyzed a number of models which have been proposed to study phase transition through the AdS/CFT correspondence. The common feature of all three models we discussed was that the space time bulk geometry was an Anti de Sitter black hole. Although the dynamical field content was very different—a charged scalar coupled to an electric potential, the same model in an external magnetic field, and a pure non-abelian gauge theory—the emerging scenarios are very similar and always include a second-order phase transition with mean-field critical exponents.

On general grounds, we were able to explain why the highly symmetric ansätze generally used produce the critical behaviors seen in mean field theory or the Landau approach. Founded on basic principles as the connection between the equations of motion and the Schrödinger equation, we clarify the similarity between several relevant quantities along a variety of models. In particular we showed that resorting to simple matching conditions, we obtain closed form solutions that significantly agree with the results obtained by numerically solving the exact set of equations of motion. This uncovers the important role played by analyticity to explain the universal behavior of certain physical constants.

The method seems to work very well near the critical temperature, though it deviates from the numerical results as we approach $T \rightarrow 0$. In this regime our approach should be refined.

Alternative analytic calculations have been recently presented in [23, 24] where the phase transition vicinity is studied solving the equations of motion in terms of a series expansion near the horizon. Although the approach in these works is close to the one proposed in [8] and applied here, the possibility in the latter of varying the intermediate point z_m at which the matching is performed allows to obtain better solutions at fixed order N in the expansion, as already pointed out in [24]. As we have seen that, in the matching approach, the problem reduces to find the solution of an algebraic equations system and this can be done, to the order we worked here, in a straightforward way. Increasing the order will of course complicate the algebraic system but in view of its main features, it can be handled by a simple computational software like Mathematica, at least for the next few orders. For a large-order expansion the method followed in references [23, 24] seems to be more appropriate.

Although the matching method works very well near the critical temperature, it deviates from the numerical results as T approaches 0. In this regime the method should be refined. In particular it is to be expected that taking into account the quantum fluctuations of the gravity theory, one should be able to go beyond mean field approximation results. Also, one should consider generalized Lagrangians (like the Stückelberg one considered in [16]) leading to various types of phase transitions (first or second order with both mean and nonmean field behavior) as parameters are changed. There is also the possibility that including fermions in the bulk model could substantially change the critical behavior of the theory in the bulk (see [25] and references therein). The simplicity of the approach presented here, not requiring refined numerical calculations, should be an asset when trying to explore these more complex situations.

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