The Existential Fragment of Third Order Logic and Third Order Relational Machines

Arroyuelo Jorge[1] and Turull-Torres José Maria[2]

[1] Depto. de Informática
Universidad Nacional de San Luis
bjarroyu@unsl.edu.ar

[2] Depto. de Ingeniería e Investigaciones Tecnológicas
Universidad Nacional de La Matanza
J.M.Turull@massey.ac.nz

Abstract. We introduce a new sub logic of third order logic (TO), the logic TO^{ω}, as a semantic restriction of TO. We focus on the existential fragment of TO^{ω}, which we denote $\Sigma_1^{2,\omega}$, and we study its relational complexity by introducing a variation of the non deterministic relational machine, which we denote 3-NRM, where we allow third order relations in the relational store of the machine. We then prove that $\Sigma_1^{2,\omega}$ characterizes exactly $\mathrm{NEXPTIME}_{3,r}$.

1 Introduction

With this article we initiate the study of a new sub logic of third order logic (TO), the logic TO^{ω}, under finite interpretations. We introduce TO^{ω} and we define it as a semantic restriction of TO where the (second order) relations which form the tuples in the third order relations that valuate the quantified third order variables are unions of complete FO^k types for r-tuples (i.e., the relations are *closed* under the relation $\equiv_k$ of equality of FO^k types in the set of r-tuples of the structure), for some constants $k \geq r \geq 1$, that depend on the quantifiers. In the sense of [FPT,10] these relations are *redundant* relations. This investigation is a natural continuation of the study of the logic SO^{ω} (in the context of Finite Model Theory, i.e., with sentences interpreted by finite relational structures or database instances - dbi's), a semantic restriction of second order logic (SO) where the valuating relations for the quantified second order variables are closed under $\equiv_k$ as above. SO^{ω} was introduced by A. Dawar ([Daw,98]), and studied by him and later by the second author jointly with F. Ferrarotti ([FT,08]). A. Dawar proved that its existential fragment $\Sigma_1^{1,\omega}$ is characterized by the non deterministic fixed point logic ($FO+NFP$), that extends First Order logic with the NFP quantifier. Previously, in [AVV,97], it was proved that ($FO+NFP$) characterizes the class of nondeterministic relational machines (NRM) working in $r$elational polynomial time (NP_r). This is a complexity class in $r$elational complexity, where the input structure is measured as the number of equivalence classes in the relation $\equiv_k$

mentioned above for the input dbi (called its $size_k$). Hence, it turned out that $\Sigma^{1,\omega}_1$ characterizes the class NP_r of relational complexity, analogously to the well known relationship $\Sigma^1_1 = NP$ ([Fag,74]). Continuing the analogy, the characterization of full SO$^\omega$ by the relational polynomial time hierarchy PH_r was stated without proof in [Daw,98], and proved in [FT,08]. Other known characterizations are $P_r = (FO + IFP)$, and $PSPACE_r = (FO + PFP)$ (see [AVV,97]), with the inflationary and partial fixed point quantifiers, respectively. In that context, we introduce the logic TO$^\omega$ looking for the logical characterization of higher relational time complexity classes. In the present article we focus on the existential fragment of TO$^\omega$, and we study its relational complexity. To that end, we introduce a variation of the non deterministic relational machine, which we denote 3-NRM (for third order NRM), where we allow *third order* relations in the relational store of the machine. We define the class NEXPTIME$_{3,r}$ as the class of 3-NRM's that work in time exponential in the $size_k$ of the input dbi. We then prove that the existential fragment of TO$^\omega$ characterizes exactly third order nondeterministic relational exponential time.

2 Preliminaries

We assume a basic knowledge of Logic and Model Theory (refer to [Lib,04]). We only consider vocabularies of the form $\sigma = \langle R_1, \ldots, R_s\rangle$ (i.e., *purely relational*), where the arities of the relation symbols are $r_1, \ldots, r_s \geq 1$, respectively. We assume that they also contain equality. And we consider only *finite* σ-structures, denoted as $\mathbf{A} = \langle A, R_1^{\mathbf{A}}, \ldots, R_s^{\mathbf{A}}\rangle$, where A is the domain, also denoted $dom(\mathbf{A})$, and $R_1^{\mathbf{A}}, \ldots, R_s^{\mathbf{A}}$ are (second order) relations in $A^{r_1}, \ldots, A^{r_s}$, respectively. If $\gamma(x_1, \ldots, x_l)$ is a formula of some logic with free FO variables $\{x_1, \ldots, x_l\}$, for some $l \geq 1$, with $\gamma^{\mathbf{A}}$ we denote the l-ary relation defined by γ in $\mathbf{A}$, i.e., the set $\{(a_1, \ldots, a_l) : a_1, \ldots, a_l \in A \wedge \mathbf{A} \models \gamma(x_1, \ldots, x_l)[a_1, \ldots, a_l]\}$. For any l-tuple $\bar{a} = (a_1, \ldots, a_l)$ of elements in A, with $1 \leq l \leq k$, we define the FO^k type of $\bar{a}$, denoted $Type_k(\mathbf{A}, \bar{a})$, to be the set of FO^k formulas $\varphi \in FO^k$ with free variables among $x_1, \ldots, x_l$, such that $\mathbf{A} \models \varphi[a_1, \ldots, a_l]$. If τ is an FO^k type, we say that the tuple $\bar{a}$ *realizes* τ in $\mathbf{A}$, if and only if, $\tau = Type_k(\mathbf{A}, \bar{a})$. Let $\mathbf{A}$ and $\mathbf{B}$ be σ-structures and let $\bar{a}$ and $\bar{b}$ be two l-tuples on $\mathbf{A}$ and $\mathbf{B}$ respectively, we write $(\mathbf{A}, \bar{a}) \equiv_k (\mathbf{B}, \bar{b})$, to denote that $Type_k(\mathbf{A}, \bar{a}) = Type_k(\mathbf{B}, \bar{b})$. If $\mathbf{A} = \mathbf{B}$, we also write $\bar{a} \equiv_k \bar{b}$. We denote as $size_k(\mathbf{A})$ the number of equivalence classes in $\equiv_k$ in $\mathbf{A}$. An l-ary relation R in $\mathbf{A}$ is *closed under* $\equiv_k$ if for any two l-tuples $\bar{a}, \bar{b}$ in A^l, $\bar{a} \in R \wedge \bar{a} \equiv_k \bar{b} \Rightarrow \bar{b} \in R$. Let S be a set, a binary relation R is a *pre-order* on S if it satisfies: 1) $\forall a \in S\ \ (a, a) \in R$ (reflexive). 2) $\forall a, b, c \in S\ \ (a, b) \in R \wedge (b, c) \in R \Rightarrow (a, c) \in R$ (transitive). 3) $\forall a, b \in S\ \ (a, b) \in R \vee (b, a) \in R$ (conex). A *pre-order* $\preceq$ on S induces an equivalence relation $\equiv$ on S (i.e., $a \equiv b \Leftrightarrow a \preceq b \wedge b \preceq a$), and also induces a total order over the set of equivalence classes of $\equiv$. When the equivalence classes induced by a pre-order on k-tuples from some structure $\mathbf{A}$ agree with the equivalence classes of $\equiv_k$, then the pre-order establishes a total order over the FO^k types for k-tuples which are realized on $\mathbf{A}$.

3 The Restricted Third-Order Logic $\mathbf{TO}^w$ and 3-NRM's

A *third order relation type* is a w-tuple $\tau = (r_1, \ldots, r_w)$ where $w, r_1, \ldots, r_w \geq 1$. In addition to the symbols of second-order logic, the alphabet of TO^ω contains for every $k \geq 1$, a *third-order quantifier* $\exists^k$, and for every relation type τ such that $r_1, \ldots, r_w \leq k$ a countably infinite set of third order variables, denoted as $\mathcal{X}^{\tau_1,k}, \mathcal{X}^{\tau_2,k}, \ldots$. We use upper case Roman letters $X_i^{r,k}$ for SO^ω variables (in this article we will often drop the superindex k, when it is clear from the context), where r is their arity, and lower case Roman letters for individual (i.e., FO) variables. Let σ be a relational vocabulary. A *third order atomic formula* of vocabulary σ, on the third order variable $\mathcal{X}^{\tau,k}$ is a formula of the form $\mathcal{X}^{\tau,k}(V_1, \ldots, V_w)$, where $V_1, \ldots, V_w$ are either second order variables of the form $X_i^{r_i,k}$, or relation symbols in σ, and whose arities are respectively $r_1, \ldots, r_w \leq k$. Note that all the relations that form a σ-structure are closed under $\equiv_k$, since k is $\geq$ than all the arities in σ (see above, and Fact 9 in [FT,08]). Let $m \geq 1$. We denote by $\Sigma_m^{2,\omega}[\sigma]$ the class of formulas of the form $\exists^{k_{3,11}} \mathcal{X}_{11}^{\tau_{11}} \ldots \exists^{k_{3,1s_1}} \mathcal{X}_{1s_1}^{\tau_{1s_1}} \forall^{k_{3,21}} \mathcal{X}_{21}^{\tau_{21}} \ldots \forall^{k_{3,2s_2}} \mathcal{X}_{2s_2}^{\tau_{2s_2}} \ldots Q^{k_{3,m1}} \mathcal{X}_{m1}^{\tau_{m1}} \ldots Q^{k_{3,ms_m}} \mathcal{X}_{ms_m}^{\tau_{ms_m}}(\psi)$, where for $i, j \geq 1$, with $\tau_{ij} = (r_{ij,1}, \ldots, r_{ij,w_{ij}})$, we have $r_{ij,1}, \ldots, r_{ij,w_{ij}} \leq k_{3,ij}$, Q is either $\exists^k$ or $\forall^k$, for some k, depending on whether m is odd or even, respectively, and ψ is an SO^ω formula with the addition of third order atomic formulas. As usual, $\forall^k \mathcal{X}(\psi)$ abbreviates $\neg\exists^k \mathcal{X}(\neg\psi)$. We define $\mathrm{TO}^\omega = \bigcup_{m\geq 1} \Sigma_m^{2,\omega}$.

A *third order relation* $\mathcal{R}$ of type τ on a σ structure $\mathbf{I}$ is a set of w tuples $(R_1, \ldots, R_w)$ of (second order) relations on $\mathbf{I}$ with respective arities $r_1, \ldots, r_w$. The third order quantifier $\exists^k$ has the following semantics: let $\mathbf{I}$ be a σ-structure; then $\mathbf{I} \models \exists^k \mathcal{X}^{\tau,k}\varphi$ if there is a third order relation $\mathcal{R}$ of type τ on I such that all the relations which form the tuples in $\mathcal{R}$ are closed under the relation $\equiv_k$ in $\mathbf{I}$, and $(\mathbf{I}, \mathcal{R}) \models \varphi$. Here $(\mathbf{I}, \mathcal{R})$ is the *third order* $(\sigma \cup \{\mathcal{X}^\tau\})$ structure expanding $\mathbf{I}$, in which $\mathcal{X}$ is interpreted as $\mathcal{R}$. Note that a *valuation* in this setting also assigns to each second order variable $X^{r,k}$ a (second order) relation on I of arity r that is closed under $\equiv_k$ in $\mathbf{I}$, and to each third order variable $\mathcal{X}^{\tau,k}$ a third order relation $\mathcal{R}$ on I of type τ, such that all the relations that form the tuples of $\mathcal{R}$ are closed under $\equiv_k$ in $\mathbf{I}$. We don't allow free second or third order variables in the logics SO^ω and TO^ω.

As an example of a non trivial query in $\Sigma_1^{2,\omega}$, consider the query "the graph G is undirected, connected, with $|V| \geq 2$, and its diameter is *even*". We give next a sketch of the formula. We quantify two third order relations $\mathcal{X}^{(2,2)}$ and $\preceq^{(2,2,2,2)}$, that form a totally ordered set of pairs of (second order) relations, where for $1 \leq i \leq m$, in the first component of the i-th pair (R_1, R_2), we have all the pairs of nodes (x, y) s. t. the minimum distance between them is i. Note that all these relations are closed under $\equiv_3$ since with 3 variables we can say in FO that there is a path of length d between two nodes, for every $d \geq 1$ (see [Lib,04,11.3]). Then, if two pairs of nodes are $\equiv_3$, either they are both in the relation or none of them are, which is correct since the set of distances of all the paths between the two pairs of nodes is the same. We use the second relation in each pair (S_2 and R_2) as Boolean flags, where $\emptyset$ means *off* and $V \times V$ means *on*. Then, along the sequence of pairs of relations we switch the flag *on* and *off*, starting in *on*. Note that if the position of the last pair of relations in the sequence is i (i.e., m), it means that the diameter of G is i, and if the flag is *off*, then the diameter is *even*: $\varphi \equiv \exists^3 \mathcal{X}^{(2,2)} \preceq^{(2,2,2,2)} \forall^3 R_1^2 R_2^2 S_1^2 S_2^2 [([$"$\preceq$ is a total order in $\mathcal{X}$"$]) \wedge ([$"the succesor of (S_1, S_2) is (R_1, R_2)"$] \rightarrow [$"R_2 is the complement of S_2" $\wedge$ "the pairs in R_1 are formed by extending the pairs in S_1 with the edges in E $]) \wedge ([$"$(S_1, S_2) \prec (R_1, R_2)$"$] \rightarrow [$"no pair in R_1 is in S_1, i.e., the distances at every stage are minimal"$]) \wedge ([$"(R_1, R_2) is the first pair in $\preceq$"$] \rightarrow [$"the flag is *on*" $\wedge$ "$R_1 = E$"$]) \wedge ([$"(R_1, R_2) is the last pair in $\preceq$"$] \rightarrow [$"the flag is *off*" $\wedge$ "the pairs in R_1 cannot be extended with edges in E, i.e., there are no minimum distances bigger than the ones in R_1"$]\ )] \wedge [$"G is connected"$] \wedge [$"G is undirected"$] \wedge [$"$|V| \geq 2$"$]$. Finally, note that "G is connected" is well known to be in P_r, and hence in Σ_1^{1w} (see [FT,08]).

A *third order non-deterministic relational machine*, denoted as 3-NRM, of *arity* k, for $k \geq 1$, is a 11-tuple $\langle Q, \Sigma, \delta, q_0, \mathfrak{b}, F, \sigma, \tau, R_l, \Omega, \Phi \rangle$ where: Q is the finite set of internal states; $q_0 \in Q$ is the initial state; Σ is the finite tape alphabet; $\mathfrak{b} \in \Sigma$ is the symbol denoting blank; $F \subseteq Q$ is the set of accepting states; τ is the finite vocabulary of the *rs* (its *relational store*), with finitely many third order relation symbols of any arbitrary type $\tau = (r_1, \ldots, r_w)$, with $1 \leq r_1, \ldots, r_w \leq k$, and finitely many second order relation symbols of arities $\leq k$; σ is the vocabulary of the input structure; Ω is a finite set of TO^ω formulas with up to k FO variables, with no second or third order quantifiers, and with no free variables of any order (i.e., all the SO^ω and TO^ω relation symbols are in τ); Φ is a finite set of TO^ω formulas with up to k FO variables, that are not sentences, with no second or third order quantifiers, and where the free variables are either all FO variables, or all SO$^\omega$ variables; $\delta : Q \times \Sigma \times \Omega \rightarrow \mathcal{P}(\Sigma \times Q \times \{R, L\} \times \Phi \times \tau)$ is the transition function. At any stage of the computation of a 3-NRM on an input σ-structure $\mathbf{I}$, there is one relation in its *rs* of the corresponding relation type (or arity) in $\mathbf{I}$ for each relation symbol in τ, so that in each transition there is a (finite) τ-structure $\mathbf{A}$ in the *rs*, which we can *query* and/or *update* through the formulas in Ω and Φ, respectively, and a finite Σ string in its tape, which we can access as in Turing machines. The concept of *computation* is analogous to that in the Turing machine. We define the complexity class NEXPTIME$_{3,r}$ as the class of the *relational languages* or *Boolean queries* (i.e., sets of finite structures of a given relational vocabulary, closed under isomorphisms) that are decidable by 3-NRM machines of *some* arity k', that work in non deterministic exponential time in the number of equivalence classes in $\equiv_{k'}$ of the input structure. In symbols: NEXPTIME$_{3,r} = \bigcup_{c \in N} NTIME_{3,r}(2^{c \cdot (size_k)})$ (as usual, this notation *does not mean* that the arity of the 3-NRM must be k).

4 TO$^\omega$ captures NEXPTIME$_{3,r}$

Theorem 1. *$NTIME_{3,r}(2^{c \cdot (size_k)}) \subseteq \Sigma_1^{2,\omega}$. That is, given a 3-NRM M in $NTIME_{3,r}(2^{c \cdot (size_k)})$, for some positive integer c, and with input vocabulary σ, that computes a Boolean query q, we can build a formula $\varphi_M \in \Sigma_1^{2,\omega}$ such that, for every σ-structure $\mathbf{I}$, M accepts $\mathbf{I}$ iff $\mathbf{I} \models \varphi_M$.*

Proof. We follow a similar strategy to the one used in the proof of Proposition 4 in [FT,08] (i.e., $NP_r \subseteq \Sigma_1^{1,\omega}$). Let $q : Str[\sigma] \rightarrow \{0,1\}$ be a Boolean query which is computed by a 3-NRM M in NEXPTIME$_{3,r}$, of arity k. Here, $Q = \{q_0, \ldots, q_m\}$; q_0 is the initial state; $\Sigma = \{0, 1, \mathfrak{b}\}$; $F = \{q_m\}$; $\tau = \{R_0, \ldots, R_l\}$; and $\sigma = \{R_0, \ldots, R_u\}$, where $u \leq l$. We assume that the relations $R_{u+1}, \ldots, R_{l_2}$ in the *rs* of M are second order *work* relations (i.e., not in the input structure), and the relations $R_{l_2+1}, \ldots, R_{l_2+1+l_3}$ are third order work relations, with $l_2 + 1 + l_3 = l$. For $0 \leq i \leq l_2$, the arity of the second order relation R_i is $r_{2,i}$, and the types of the third order relations $R_{l_2+1}, \ldots, R_{l_2+1+l_3}$, are, respectively, $(r_{3,01}, ..., r_{3,0w_0}), ..., (r_{3,l_31}, ..., r_{3,l_3w_{l_3}})$. Also, $\Omega = \{\alpha_0, \ldots, \alpha_v\}$, and $\Phi = \{\gamma_0, \ldots, \gamma_w\}$.

We build now the $\Sigma_1^{2,\omega}$ formula φ_M according to the definition of M. Since M runs in time $2^{c \cdot (size_k(\mathbf{I}))}$, and hence visits at most that many tape cells, we can *model time* as well as *position* in the tape by c-tuples of k-ary relations closed under $\equiv_k$ (we will respectively use the tuples of relations $\bar{T}$ and $\bar{P}$, see below). Note that there are $2^{size_k(\mathbf{I})}$ different such relations over $\mathbf{I}$, and $(2^{size_k(\mathbf{I})})^c = 2^{c \cdot (size_k(\mathbf{I}))}$ different c-tuples of such relations.

Then we build the following formula: $\exists^k \mathcal{O}^{\tau_{0,1}} \exists^k \mathcal{T}_0^{\tau_{0,2}} \exists^k \mathcal{T}_1^{\tau_{0,3}} \exists^k \mathcal{T}_{\mathfrak{b}}^{\tau_{0,4}} \exists^k \mathcal{H}_{q_0}^{\tau_{0,5}} \ldots \exists^k \mathcal{H}_{q_m}^{\tau_{0,5+m}} \exists^k \mathcal{S}_{30}^{\tau_{3,0}} \ldots \exists^k \mathcal{S}_{3l_3}^{\tau_{3,l_3}} \exists^k \mathcal{S}_{20}^{\tau_{2,0}} \ldots \exists^k \mathcal{S}_{2l_2}^{\tau_{2,l_2}} (\psi)$ where the types of the third order variables are as follows: $\tau_{0,1} = (k,k)$, $\tau_{0,2} = \tau_{0,3} = \tau_{0,4} = \tau_{0,5} = \ldots = \tau_{0,5+m} = (k, ..., k)$ with cardinality $2c$; $\tau_{3,0}, \ldots, \tau_{3,l_3}$ are the types of $R_{l_2+1}, \ldots, R_{l_2+1+l_3}$, augmented with c second order relations of arity k; and $\tau_{2,0} = (r_{2,0}, k, \ldots, k)$, $\ldots, \tau_{2,l_2} = (r_{2,l_2}, k, \ldots, k)$ with cardinality $c+1$, are the types of the third order variables that represent in φ_M $R_0, \ldots, R_{l_2}$ augmented with c (second order) relations of arity k; with ψ being an SO$^\omega$ formula (i.e., $\psi \in \Sigma_j^{1,\omega}$ for some $j \geq 1$) of vocabulary σ augmented with second order variables that we will quantify in the sub formulas below, and with free third order variables $\mathcal{O}^{\tau_{0,1}}, \mathcal{T}_0^{\tau_{0,2}}, \mathcal{T}_1^{\tau_{0,3}}, \mathcal{T}_{\mathfrak{b}}^{\tau_{0,4}}, \mathcal{H}_{q_0}^{\tau_{0,5}}, \ldots, \mathcal{H}_{q_m}^{\tau_{0,5+m}}, \mathcal{S}_{30}^{\tau_{3,0}}, \ldots, \mathcal{S}_{3l_3}^{\tau_{3,l_3}}, \mathcal{S}_{20}^{\tau_{2,0}}, \ldots, \mathcal{S}_{2l_2}^{\tau_{2,l_2}}$. Note that the third order variables $\mathcal{S}_{20}^{\tau_{2,0}}, \ldots, \mathcal{S}_{2l_2}^{\tau_{2,l_2}}$ represent the contents of the second order relations $R_0, \ldots, R_{l_2}$, and the third order variables $\mathcal{S}_{30}^{\tau_{3,0}}, \ldots, \mathcal{S}_{3l_3}^{\tau_{3,l_3}}$ represent the contents of the third order relations $R_{l_2+1}, \ldots, R_{l_2+1+l_3}$ (see below). Let $\bar{P} = (P_1, \ldots, P_c)$, $\bar{T} = (T_1, \ldots, T_c)$, and $\bar{X} = (X_1, \ldots, X_c)$, where $P_1, \ldots, P_c$, $T_1, \ldots, T_c$, $X_1, \ldots, X_c$ are k-ary second order relation variables, closed under $\equiv_k$. The intended interpretation of these relation symbols is as follows: $\mathcal{O}$ is a total order in the class of k-ary relations closed under $\equiv_k$; $\mathcal{T}_0$, $\mathcal{T}_1$, and $\mathcal{T}_{\mathfrak{b}}$ are *tape* relations: for $x \in \{0, 1, \mathfrak{b}\}$, $\mathcal{T}_x(\bar{P}, \bar{T})$ indicates that position $\bar{P}$ of the tape at time $\bar{T}$ contains symbol x; $\mathcal{H}_q$'s are *tape head* relations: for $q \in Q$, $\mathcal{H}_q(\bar{P}, \bar{T})$ indicates that at time $\bar{T}$, the machine M is in state q, and its tape head is in position $\bar{P}$; $\mathcal{S}_i$'s are *rs* relations: for $0 \leq i \leq l_3$, $\mathcal{S}_{3i}(\bar{A}, \bar{T})$ indicates that at time $\bar{T}$, the third-order relation R_{l_2+1+i} in the *rs contains* the w_i-tuple of second order relations of the corresponding arities closed under $\equiv_k$, $\bar{A}$; for $0 \leq j \leq l_2$, $\mathcal{S}_{2j}(B, \bar{T})$ indicates that at time $\bar{T}$, the $r_{2,j}$-ary second order relation R_j closed under $\equiv_k$, in the *rs is equal* to the $r_{2,j}$-ary, second order relation B. Note that as the arities of the relations in τ are $\leq k$ by definition of the 3-NRM, all the relations $R_0, \ldots, R_{l_2}$ are closed under $\equiv_k$, including the relations in σ (see Fact 9 in [FT,08]).

The sentence φ_M must now express that when M starts with an empty tape and an input σ-structure $\mathbf{I}$ in the designated relations of its rs, the relations $\mathcal{T}_x$'s, $\mathcal{H}_q$'s and $\mathcal{S}_i$'s encode its computation, and eventually M reaches an accepting state. We define φ_M to be the *conjunction* of the following sentences:

1): A formula expressing that $\mathcal{O}^{2k}$ defines a total order of k-ary relations closed under $\equiv_k$: $[\forall^k X(\mathcal{O}(X,X))] \wedge [\forall^k XYZ((\mathcal{O}(X,Y) \wedge \mathcal{O}(Y,Z)) \rightarrow \mathcal{O}(X,Z))] \wedge [\forall^k XY((\mathcal{O}(X,Y) \wedge \mathcal{O}(Y,X)) \rightarrow \text{"}X = Y\text{"})] \wedge [\forall^k XY(\mathcal{O}(X,Y) \vee \mathcal{O}(Y,X))]$.

2): A formula defining the initial configuration of M: $\exists^k \bar{T} \forall^k \bar{X} \forall^k \bar{P}[(\bar{T} \leq \bar{X}) \wedge \mathcal{H}_{q_0}^{2ck}(\bar{T},\bar{T}) \wedge \mathcal{T}_{\mathfrak{b}}^{2ck}(\bar{P},\bar{T})]$ (that says "at time 0, M is in state q_0, the head is in the left-most position of the tape, and the tape contains only blanks") $\wedge$ $\exists^k \bar{T}\, \forall^k \bar{X}\, \forall^k X_{01}^{r_{3,01}}, \ldots, X_{0w_0}^{r_{3,0w_0}}, \ldots, X_{l_3 1}^{r_{3,l_3 1}}, \ldots, X_{l_3 w_{l_3}}^{r_{3,l_3 w_{l_3}}} \exists^k Y_0^{r_{2,0}} \ldots \exists^k Y_u^{r_{2,u}} \exists^k Y_{u+1}^{r_{2,u+1}} \ldots \exists^k Y_{l_2}^{r_{2,l_2}} \Big((\bar{T} \leq \bar{X}) \wedge \neg \mathcal{S}_{30}^{\tau_{3,0}}(X_{01}^{r_{3,01}}, \ldots, X_{0w_0}^{r_{3,0w_0}}, \bar{T}) \wedge \ldots \wedge \neg \mathcal{S}_{3l_3}^{\tau_{3,l_3}}(X_{l_3 1}^{r_{3,l_3 1}}, \ldots, X_{l_3 w_{l_3}}^{r_{3,l_3 w_{l_3}}}, \bar{T}) \quad \wedge [(\mathcal{S}_{20}(Y_0,\bar{T}) \wedge \forall x_1 \ldots x_{r_{2,0}}(Y_0(x_1,\ldots,x_{r_{2,0}}) \leftrightarrow R_0(x_1,\ldots,x_{r_{2,0}}))) \wedge \ldots \wedge (\mathcal{S}_{2u}(Y_u,\bar{T}) \wedge \forall x_1 \ldots x_{r_{2,u}}(Y_u(x_1,\ldots,x_{r_{2,u}}) \leftrightarrow R_u(x_1,\ldots,x_{r_{2,u}})))] \wedge [(\mathcal{S}_{2,u+1}(Y_{u+1},\bar{T}) \wedge \forall x_1 \ldots x_{r_{2,u+1}}(\neg Y_{u+1}(x_1,\ldots,x_{r_{2,u+1}}))) \wedge ... \wedge (\mathcal{S}_{2,l_2}(Y_{l_2},\bar{T}) \wedge \forall x_1 \ldots x_{r_{2,l_2}}(\neg Y_{l_2}(x_1,\ldots,x_{r_{2,l_2}})))]\Big)$ (that says "the relations in the rs hold a τ-structure $\mathbf{A}$ which extends $\mathbf{I}$ with an empty relation for each relation symbol R_i in $\tau - \sigma$").

Here, we say that $\bar{X} \leq \bar{Y}$ iff $[\mathcal{O}(\bar{X}_1,\bar{Y}_1) \wedge \neg(\bar{Y}_1,\bar{X}_1)] \vee [\mathcal{O}(\bar{X}_1,\bar{Y}_1) \wedge \mathcal{O}(\bar{Y}_1,\bar{X}_1) \wedge \mathcal{O}(\bar{X}_2,\bar{Y}_2) \wedge \neg\mathcal{O}(\bar{Y}_2,\bar{X}_2)] \vee \ldots \vee [\mathcal{O}(\bar{X}_1,\bar{Y}_1) \wedge \mathcal{O}(\bar{Y}_1,\bar{X}_1) \wedge \ldots \wedge \mathcal{O}(\bar{X}_{c-1},\bar{Y}_{c-1}) \wedge \mathcal{O}(\bar{Y}_{c-1},\bar{X}_{c-1}) \wedge \mathcal{O}(\bar{X}_c,\bar{Y}_c) \wedge \neg\mathcal{O}(\bar{Y}_c,\bar{X}_c)] \vee [\bar{X} \sim^3 \bar{Y}]$, where $\bar{X} \sim^3 \bar{Y}$ is simply: $\mathcal{O}(\bar{X}_1,\bar{Y}_1) \wedge \mathcal{O}(\bar{Y}_1,\bar{X}_1) \wedge \ldots \wedge \mathcal{O}(\bar{X}_c,\bar{Y}_c) \wedge \mathcal{O}(\bar{Y}_c,\bar{X}_c)$.

3): A formula stating that in every configuration of M, each cell of the tape contains exactly one element of Σ : $\forall^k \bar{P}\bar{T}\big([\mathcal{T}_0(\bar{P},\bar{T}) \leftrightarrow (\neg\mathcal{T}_1(\bar{P},\bar{T}) \wedge \neg\mathcal{T}_{\mathfrak{b}}(\bar{P},\bar{T}))] \wedge [\mathcal{T}_1(\bar{P},\bar{T}) \leftrightarrow (\neg\mathcal{T}_0(\bar{P},\bar{T}) \wedge \neg\mathcal{T}_{\mathfrak{b}}(\bar{P},\bar{T}))] \wedge [\mathcal{T}_{\mathfrak{b}}(\bar{P},\bar{T}) \leftrightarrow (\neg\mathcal{T}_0(\bar{P},\bar{T}) \wedge \neg\mathcal{T}_1(\bar{P},\bar{T}))]\big)$.

4): A formula stating that at any time the machine is in exactly one state: $\forall^k \bar{T} \exists^k \bar{P} \forall^k \bar{P}'\bar{T}'\big[\big(\bigvee_{q \in Q} \mathcal{H}_q(\bar{P},\bar{T})\big) \wedge \big(\bigwedge_{q \neq q', q, q' \in Q} \big(\neg\mathcal{H}_q(\bar{P}',\bar{T}') \vee \neg\mathcal{H}_{q'}(\bar{P}',\bar{T}')\big)\big)\big]$.

5): A formula stating that the variables $\mathcal{S}_{2j}$'s, at any time hold at most one relation (and hence, by the second sub formula above, *exactly one* relation): $\forall^k \bar{T} A^{r_{2,0}} B^{r_{2,0}}[(\mathcal{S}_{20}(A,\bar{T}) \wedge \mathcal{S}_{20}(B,\bar{T})) \rightarrow \text{"}A = B\text{"}] \wedge \ldots \wedge \forall^k \bar{T} A^{r_{2,l_2}} B^{r_{2,l_2}}[(\mathcal{S}_{2l_2}(A,\bar{T}) \wedge \mathcal{S}_{2l_2}(B,\bar{T})) \rightarrow \text{"}A = B\text{"}]$.

6): A *conjunction* of formulas expressing that the relations $\mathcal{T}_i$'s, $\mathcal{H}_q$'s, $\mathcal{S}_{2j}$'s and $\mathcal{S}_{3j}$'s respect the transitions of M. For every $a \in \Sigma$, $q \in Q$ and $\alpha \in \Omega$ for which the transition function δ is defined, we have a sentence of the form: $\bigvee_{(b,q',m,\gamma,R) \in \delta(q,a,\alpha)} \chi(q,a,\alpha,b,q',m,\gamma,R)$, where $\chi(q,a,\alpha,b,q',m,\gamma,R)$ is the sentence describing the corresponding transition. Assume that $m = L$, $\mathcal{S}_{3j}$ is the relation variable which encodes R (a third order relation) if the free variables of γ are of second order (recall that, for $0 \leq j \leq l_3$, $\mathcal{S}_{3j}$ encodes R_{j+l_2+1} of the rs of M) and $\mathcal{S}_{2j}$ is the relation variable which encodes R (a second order relation) if the free variables of γ are individual. We write $\chi(q,a,\alpha,b,q',m,\gamma,R)$ as the *conjunction of two formulas*; the first one is as follows:

$$\forall^k \bar{P}\bar{P}_{-1}\bar{P}'\bar{T}_{+1}\bar{T}'\bar{T}\bar{X}\forall^k R_{h'_1}^{r_{2,h'_1}} \dots R_{h'_{t'}}^{r_{2,h'_{t'}}} \Big(\big[(\bar{P} \le \bar{X}) \vee \neg\mathcal{T}_a(\bar{P},\bar{T}) \vee \neg\mathcal{H}_q(\bar{P},\bar{T}) \vee$$
$$\neg[\mathcal{S}_{2h'_1}(R_{h'_1},\bar{T}) \wedge \dots \wedge \mathcal{S}_{2h'_{t'}}(R_{h'_{t'}},\bar{T}) \wedge \hat{\alpha}(\bar{T})]] \vee$$
$$\big[\neg((\bar{P}_{-1} \le \bar{P}) \wedge (\bar{P}_{-1} \le \bar{P}') \wedge (\bar{P}' \le \bar{P}) \wedge \neg(\bar{P}_{-1} \sim^3 \bar{P}) \wedge (\bar{P}_{-1} \sim^3 \bar{P}')) \vee$$
$$\neg((\bar{T} \le \bar{T}_{+1}) \wedge (\bar{T}' \le \bar{T}_{+1}) \wedge (\bar{T} \le \bar{T}') \wedge \neg(\bar{T}_{+1} \sim^3 \bar{T}) \wedge (\bar{T}_{+1} \sim^3 \bar{T}'))\big] \vee$$
$$\big[\mathcal{T}_b(\bar{P},\bar{T}_{+1}) \wedge \mathcal{H}_{q'}(\bar{P}_{-1},\bar{T}_{+1}) \wedge \big(\bigwedge_{0 \le i \le l_2} \forall^k Z_i^{r_{2,i}} (\mathcal{S}_{2i}(Z_i,\bar{T}) \leftrightarrow \mathcal{S}_{2i}(Z_i,\bar{T}_{+1}))\big)$$
$$\wedge \big(\bigwedge_{0 \le i \le l_3} \forall^k Z_{i1}^{r_{3,i1}} ... Z_{iw_i}^{r_{3,iw_i}} (\mathcal{S}_{3i}(Z_{i1},...,Z_{iw_i},\bar{T}) \leftrightarrow \mathcal{S}_{3i}(Z_{i1},...,Z_{iw_i},\bar{T}_{+1}))\big) \wedge \beta\big] \Big)$$

Lines 3 and 4 in the formula above say that $\bar{P}_{-1}$ and $\bar{T}_{+1}$ are the predecessor of $\bar{P}$, and the successor of $\bar{T}$ in $\le$, respectively. β is a sub formula which says that the relation $R^{\mathbf{A}}$ (i.e., the relation R in the *rs* of M) is replaced by $\gamma^{\mathbf{A}}$. In the formula φ_M, we represent that action by saying that the variable $\mathcal{S}$ that represents R at different times, at the time $\bar{T}_{+1}$ holds the relation defined by γ, where its relation symbols are interpreted with the contents of the corresponding variables $\mathcal{S}$, at time $\bar{T}$. The two sub formulas with the big conjunctions immediately before β, say that the relations *other than* R are not altered in the computation step from time $\bar{T}$ to time $\bar{T}_{+1}$. We have two different cases: **i):** the free variables in γ are only FO variables, and hence it defines a second order relation, so that R is a second order relation; **ii):** the free variables in γ are only second order variables, and hence it defines a third order relation, so that R is a third order relation. Note that in the two cases, we should modify γ (as well as α) in such a way that instead of using the different (second and third order) relation symbols that correspond to relations in the *rs* of M, it should use the third order variables that represent those relations, and in particular its contents at the time $\bar{T}$. For second order relation symbols, we do that by existentially quantifying a second order variable with the same name R (note that in this case $R \in \{R_0,\dots,R_{l_2}\}$), and then by saying that that relation is in the tuple for time $\bar{T}$ in the corresponding variable $\mathcal{S}$. In the formula below, let $R_{h_1},\dots,R_{h_t}$ be the second order relation symbols that appear in γ, and let $R_{h'_1},\dots,R_{h'_{t'}}$ be those that appear in α. For third order relation symbols, we do that by building the formula $\hat{\gamma}$ by replacing in γ every atomic formula of the form $R_i(\bar{W})$ (where $l_2+1 \le i \le l_2+1+l_3$) by $\mathcal{S}_{3(i-(l_2+1))}(\bar{W},\bar{T})$. For case **(i):** $\beta \equiv \exists^k V_j^{r_{2,j}} \big(\mathcal{S}_{2j}(V_j,\bar{T}_{+1}) \wedge \forall x_1...x_{r_{2,j}} \big(V_j(x_1,...,x_{r_{2,j}}) \leftrightarrow \exists^k R_{h_1}^{r_{2,h_1}} \dots R_{h_t}^{r_{2,h_t}}$ $[\mathcal{S}_{2h_1}(R_{h_1},\bar{T}) \wedge \dots \wedge \mathcal{S}_{2h_t}(R_{h_t},\bar{T}) \wedge \hat{\gamma}(x_1,...,x_{r_{2,j}},\bar{T})]\big)\big)$. Note that the inner $\exists^k$ block above can be safely moved out of the big parenthesis (and hence to the SO^ω quantifier prefix in φ_M), since we want a *single* tuple of relations $R_{h_1},\dots,R_{h_t}$ for the evaluation of *all* the possible tuples $x_1,...,x_{r_{2,j}}$. Also, in the *first* of the two sub formulas with the big conjunctions immediately before β, we must change the big conjunction from $\bigwedge_{0 \le i \le l_2}$ to $\bigwedge_{0 \le i \le l_2, i \ne j}$. For case **(ii):** $\beta \equiv \forall^k Z_1^{r_{3,j1}} ... Z_{w_j}^{r_{3,jw_j}} \big(\mathcal{S}_{3j}(Z_1,...,Z_{w_j},\bar{T}_{+1}) \leftrightarrow \exists^k R_{h_1}^{r_{2,h_1}} \dots R_{h_t}^{r_{2,h_t}} [\mathcal{S}_{2h_1}(R_{h_1},\bar{T}) \wedge \dots \wedge \mathcal{S}_{2h_t}(R_{h_t},\bar{T}) \wedge \hat{\gamma}(Z_1,...,Z_{w_j},\bar{T})]\big)$. Also, in the *second* of the two sub formulas with the big conjunctions immediately before β, we must change the big conjunction from $\bigwedge_{0 \le i \le l_3}$ to $\bigwedge_{0 \le i \le l_3, i \ne j}$. Up to this point, we have built the first formula for $\chi(q,a,\alpha,b,q',m,\gamma,R)$, which corresponds to the case where the tape head *is not* in the *first* position. To build the *second formula*, which corresponds to the case where the tape head *is* in the *first* position, we need to do only the following changes in the first formula: we replace "$\forall^k \bar{X}$" by "$\exists^k \bar{X}$", "$(\bar{P} \leq \bar{X})$" by "$(\bar{P} > \bar{X})$", and "$\mathcal{H}_{q'}(\bar{P}_{-1}, \bar{T}_{+1})$" by "$\mathcal{H}_{q'}(\bar{P}, \bar{T}_{+1})$". A final note for this sub formula is that, in order to be able to move all the SO^ω quantifiers to the prefix of φ_M for it to be a $\Sigma_1^{2,\omega}$ formula, we must have one set of second order variables $\{\bar{P}, \bar{P}_{-1}, \bar{P}', \bar{T}_{+1}, \bar{T}', \bar{T}, \bar{X}, V_j, Z_0, ..., Z_{l_2}, Z_{01}, ..., Z_{0w_0}, ..., Z_{l_3 1}, ..., Z_{l_3 w_{l_3}}\}$ for *each* sub formula $\chi(q, a, \alpha, b, q', m, \gamma, R)$. As for the variables $R_{h_1}, \ldots, R_{h_t}$ and $R_{h'_1}, \ldots, R_{h'_{t'}}$, technically, we must also rename each to have the sets that we need, which implies also the renaming of the respective symbols in all the formulas $\gamma \in \Phi$ and $\alpha \in \Omega$. The important point is that this is possible, because the definition of M is finite.

7): Finally, a formula stating that at some point, M is in an accepting final state: $\exists^k \bar{P}\bar{T}\big(\mathcal{H}_{q_m}^{2ck}(\bar{P}, \bar{T})\big)$.

By the construction of φ_M, we can see that M accepts a given σ-structure $\mathbf{I}$ iff there are relations closed under equivalence of FO^k-types of tuples as required by the TO^ω quantifiers in the prefix of φ_M, which assigned to the relation variables $\mathcal{O}$, $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_\mathfrak{b}$, $\mathcal{H}_{q_0}, \ldots, \mathcal{H}_{q_m}$, $\mathcal{S}_{30}, \ldots, \mathcal{S}_{3l_3}$, $\mathcal{S}_{20}, \ldots, \mathcal{S}_{2l_2}$, satisfy ψ. □

Theorem 2. *$\Sigma_1^{2,\omega} \subseteq NTIME_{3,r}(2^{c \cdot (size_k)})$. That is, every class of relational structures definable in $\Sigma_1^{2,\omega}$ is in* $\mathrm{NEXPTIME}_{3,r}$.

Proof. For a relational vocabulary σ, let φ be a $\Sigma_1^{2,\omega}[\sigma]$ sentence of the form $\exists^{k_{3,1}} \mathcal{X}_1^{\tau_1} \ldots \exists^{k_{3,s}} \mathcal{X}_s^{\tau_s}(\psi)$, where ψ is a $\Sigma_t^{1,\omega}$ formula, for some $t \geq 1$, with atomic TO^ω formulas formed with the TO^ω variables $\mathcal{X}_1, \ldots, \mathcal{X}_s$. To simplify the presentation we assume w.l.o.g. that for $1 \leq i \leq s$ the type of the relation $\mathcal{X}_i$ is $\tau_i = (r_{3,i}, ..., r_{3,i})$ of cardinality $r_{3,i}$, with $r_{3,i} \leq k_{3,i}$.

Let the $\Sigma_t^{1,\omega}$ formula ψ be of the form $\exists^{k_{2,11}} Y_{11}^{r_{2,11}} \ldots \exists^{k_{2,1l_1}} Y_{1l_1}^{r_{2,1l_1}} \forall^{k_{2,21}} Y_{21}^{r_{2,21}} \ldots \forall^{k_{2,2l_2}} Y_{2l_2}^{r_{2,2l_2}} \ldots Q^{k_{2,t1}} Y_{t1}^{r_{2,t1}} \ldots Q^{k_{2,tl_t}} Y_{tl_t}^{r_{2,tl_t}}(\phi)$, where the quantifiers $Q^{k_{2,t1}}, \ldots, Q^{k_{2,tl_t}}$ are $\forall^{k_{2,t1}}, \ldots, \forall^{k_{2,tl_t}}$, if t is even, or $\exists^{k_{2,t1}}, \ldots, \exists^{k_{2,tl_t}}$, if t is odd, ϕ is an FO formula in the vocabulary $\sigma \cup \{Y_{11}^{r_{2,11}}, \ldots, Y_{tl_t}^{r_{2,tl_t}}\}$, with atomic TO^ω formulas, and $r_{2,11} \leq k_{2,11}$, $\ldots$, $r_{2,tl_t} \leq k_{2,tl_t}$, respectively. We build a 3-NRM M_φ which accepts a given σ structure $\mathbf{I}$ iff $\mathbf{I} \models \varphi$. It is known that for every σ, and every $k \geq 1$, a formula $\gamma(\bar{x}, \bar{y})$ with $k'' \geq 2k$ variables of the fixpoint logic $(FO + LFP)$ can be built s. t. on any σ structure $\mathbf{J}$, γ defines a pre-order $\preceq_k$ in the set of k-tuples of $\mathbf{J}$, whose induced equivalence relation is $\equiv_k$ (see T.11.20 in [Lib,04]). On the other hand, it is known that $(FO + LFP)$ captures *relational polynomial time* P_r ([AVV,97]). Hence, an RM $M_{\preceq_k}$ of some arity $k' \geq 2k$ can be built, that constructs, on input $\mathbf{J}$, the pre-order $\preceq_k$ in $\mathbf{J}$, in time polynomial in $size_{k'}(\mathbf{J})$. We define the arity of M_φ as $k = max(\{k'_{3,1}, \ldots, k'_{3,s}, k'_{2,11}, \ldots, k'_{2,tl_t}\})$, where the k'_{ij}'s are the arities of the RM's $M_{\preceq_{k_{3,1}}}$, $\ldots$, $M_{\preceq_{k_{2,tl_t}}}$, respectively. Let $\mathbf{I}$ be the input structure. M_φ works as follows:

1): M_φ simulates the RM's $M_{\preceq_{k_{3,1}}}$, $\ldots$, $M_{\preceq_{k_{2,tl_t}}}$, to build the pre-orders $\preceq_{k_{3,1}}$, $\ldots$, $\preceq_{k_{2,tl_t}}$, respectively. M_φ builds those pre-orders in time polynomial in $size_{k'_{3,1}}(\mathbf{I}), \ldots, size_{k'_{2,tl_t}}(\mathbf{I})$, respectively. As all these arities are $\leq k$ (see above), that time is also polynomial in $size_k(\mathbf{I})$ (see [FT,08]). **2):** With those pre-orders M_φ computes the sizes $size_{k_{3,1}}(\mathbf{I}), \ldots, size_{k_{2,tl_t}}(\mathbf{I})$ in time polynomial in $size_k(\mathbf{I})$, by using the corresponding pre-orders as *clocks*, as in [FT,08] (recall that those pre-orders induce total orders in the equivalence classes of the corresponding equivalence relations $\equiv_{k_{3,1}}, \ldots, \equiv_{k_{2,tl_t}}$). **3):** M_φ *guesses* the TO^ω relations $\mathcal{S}_1^{\tau_1}, \ldots, \mathcal{S}_s^{\tau_s}$, as interpretations of the TO^ω variables $\mathcal{X}_1^{\tau_1}, \ldots, \mathcal{X}_s^{\tau_s}$, respectively. Each $\mathcal{S}_i^{\tau_i}$ is a set of $r_{3,i}$-tuples of $r_{3,i}$-ary (SO^ω) relations closed under $\equiv_{k_{3,i}}$. Then, to guess $\mathcal{S}_i^{\tau_i}$ we use three kinds of bit strings as follows: a) each bit string $b^3_{R^{r_{3,i}}}$ of size $size_{k_{3,i}}(\mathbf{I})$ represents one of the possible $r_{3,i}$-ary (SO^ω) relations on $\mathbf{I}$, closed under $\equiv_{k_{3,i}}$; note that each bit represents one equivalence class in $\equiv_{k_{3,i}}$; b) each bit string $b^2_{\bar{R}^{r_{3,i}}}$ of size $r_{3,i} \cdot size_{k_{3,i}}(\mathbf{I})$ represents one of the possible $r_{3,i}$-tuples of $r_{3,i}$-ary (SO^ω) relations on $\mathbf{I}$, closed under $\equiv_{k_{3,i}}$; c) each bit string $b^1_{\mathcal{S}_i^{\tau_i}}$ of size $2^{r_{3,i} \cdot size_{k_{3,i}}(\mathbf{I})}$ represents one of the possible sets of $r_{3,i}$-tuples of $r_{3,i}$-ary (SO^ω) relations on $\mathbf{I}$, closed under $\equiv_{k_{3,i}}$, i.e., one of the possible TO^ω relations on $\mathbf{I}$ of type τ_i, closed under $\equiv_{k_{3,i}}$. For all those strings we use the total orders induced by $\preceq_{k_{3,i}}$ in the equivalence classes of $\equiv_{k_{3,i}}$. Then, M_φ guesses each TO^ω relation $\mathcal{S}_i^{\tau_i}$ by first guessing the bit string $b^1_{\mathcal{S}_i^{\tau_i}}$, and then, by stepping in each equivalence class in $\equiv_{k_{3,i}}$ and choosing the class iff the corresponding bit is 1. This is done for every possible $r_{3,i}$-tuple of $r_{3,i}$-ary (SO^ω) relations closed under $\equiv_{k_{3,i}}$. Note that this is done in time $2^{c \cdot size_{k_{3,i}}(\mathbf{I})}$, and hence also in time $2^{c \cdot size_k(\mathbf{I})}$, since $k_{3,i} \leq k$ (see above), for some constant c. As before, we use the total orders induced by $\preceq_{k_{3,i}}$ in $\equiv_{k_{3,i}}$ as clocks, in the process of building the relations $\mathcal{S}_i^{\tau_i}$, so as to be able to guess each one of the possible such TO^ω relations. The details on how to do that are similar to the strategy used in [FT,08] to prove $\Sigma_1^{1,\omega} \subseteq NTIME_r((size_k)^c)$. **4):** Regarding the SO^ω variables quantified in the $\Sigma_t^{1,\omega}$ formula ψ, to interpret each of them we *generate all* the possible SO^ω relations of the corresponding arity and closed under the corresponding equivalence class, by stepping in the classes according to the corresponding pre-order. Note that we can afford to do that because for each variable $Y_{ij}^{r_{2,ij}}$ the number of such relations is bounded by $2^{d \cdot size_{k_{2,ij}}(\mathbf{I})}$, and hence also by $2^{d \cdot size_k(\mathbf{I})}$, since $k_{2,ij} \leq k$ (see above), for some constant d that depends on the arity. Then, for each SO^ω variable we will require that either for all the generated relations, or for at least one of them, depending on the corresponding quantifier being $\forall$ or $\exists$, respectively, the formula ϕ is true. Note that as ϕ is an FO formula with atomic TO^ω formulas, it can be used in the finite control of M_φ, and hence is evaluated in *one* step.

Then, M_φ accepts $\mathbf{I}$ iff $\mathbf{I} \models \varphi$, and works in $\text{NEXPTIME}_{3,r}$. □

5 Conclusions

It is well known that RM's with no time restrictions can compute exactly the class of (recursive) queries that are expressible in the infinitary logic with finitely many variables $\mathcal{L}^\omega_{\infty\omega}$ (see [Tur,06], among other sources). This logic extends FO with conjunctions and disjunctions of sets of formulas of arbitrary (infinite) cardinality, while restricting the number of variables in each (infinitary) formula to be finite. This is a very important logic in Descriptive Complexity, in which among other properties, equivalence is characterized by pebble (Ehrenfeucht-Fraissee) games, and on ordered dbi's it can express all computable queries (see [Lib,04], among others). We are currently working on a simulation of the 3-NRM with a standard (i.e., a second order) NRM, aiming to prove that the logic $\Sigma_1^{2,\omega}$ is included in $\mathcal{L}^{\omega}_{\infty\omega}$. Also, and more generally, we aim to characterize all the fragments $\Sigma_j^{2,\omega}$ of TO^{ω} with third order relational complexity classes. Beyond the natural theoretical relevance in creating and studying new logics as computation models, and thus getting information on new aspects of the problems that can be expressed in them, one important application to Complexity Theory is the separation of NP complete problems (or NEXPTIME complete, etc) that are not distinguished by classical computational complexity techniques. As an example of this, in [Daw,98] it was shown that two particular NP complete problems are expressible in $\Sigma_1^{1,\omega}$, and that 3-colorability is not. On the other hand, in [GFT,12] it was shown that a certain NP complete problem is expressible in the existential fragment $\Sigma_1^{1,F}$ of another logic (denoted as SO^F and introduced in [GT,10]), and that that problem is not expressible in $\mathcal{L}^{\omega}_{\infty\omega}$, and hence neither in $\Sigma_1^{1,\omega}$, since $\Sigma_1^{1,\omega} \subset \mathcal{L}^{\omega}_{\infty\omega}$.

References

1. [AVV,97] Abiteboul, S., Vardi, M. Y., Vianu, V., "Fixpoint logics, relational machines, and computational complexity", JACM 44 (1997) 30-56.
2. [Daw,98] Dawar, A., "A restricted second order logic for finite structures". Information and Computation 143 (1998) 154-174.
3. [Fag,74] Fagin, R., "Generalized First-Order Spectra and Polynomial-Time Recognizable Sets", in "Complexity of Computations", edited by R. Karp, SIAM-AMS Proc., American Mathematical Society, Providence, RI, pp. 27–41, 1974.
4. [FPT,10] F. A. Ferrarotti, A. L. Paoletti, J. M. Turull-Torres, "Redundant Relations in Relational Databases: A Model Theoretic Perspective", Journal of Universal Computer Science, Vol. 16, No. 20, pp. 2934-2955, 2010. http://www.jucs.org/jucs_16_20/redundant_relations_in_relational
5. [FT,08] Ferrarotti, F. A., Turull-Torres, J. M., "The Relational Polynomial-Time Hierarchy and Second-Order Logic", invited for "Semantics in Databases", edited by K-D. Schewe and B. Thalheim, Springer LNCS 4925, 29 pages, 2008.
6. [GFT,12] Ferrarotti, F. A., Grosso, A., Turull-Torres, J. M., "Semantic Restrictions over Second-Order Logic", in "Semantics in Databases and Knowledge bases", Springer, LNCS series, edited by K-D. Schewe and B. Thalheim, 23 pages, 2012.
7. [GT,10] Grosso, A. L., Turul-Torres J. M., "A Second-Order Logic in which Variables Range over Relations with Complete First-Order Types", 2010 XXIX International Conference of the Chilean Computer Science Society (SCCC) IEEE, p. 270-279, 2010.
8. [Lib,04] Libkin, L., "Elements of Finite Model Theory", Springer, 2004.
9. [Tur,06] J. M. Turull-Torres, "Relational Databases and Homogeneity in Logics with Counting", Acta Cybernetica, Vol 17, number 3, pp. 485-511, 2006.