Real State as Housing and as Financial Investment: A First Assessment for Argentina

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Documento de Trabajo Nro. 95

Noviembre 2012

ISSN 1853-3930
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A First Assessment for Argentina

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Abstract

There exists a widespread perception that Argentine households channel a large share of their savings into real state. However, no hard evidence has produced to date to measure how important this saving behavior is for the determination of housing prices vis-à-vis the traditional housing motive to buy property. In this light, this paper assesses for the first time whether housing prices in Argentina are mostly driven by housing or by investment motives. To this end, we devised a simple empirical test taking the form of an Equilibrium Correction model of apartment prices in Buenos Aires City on four explanatory variables that separately capture the housing motive (affordability and mortgage loans) and the investment motive (private bank deposits and income). It is found that private bank deposits and income have strong long and short run effects on housing prices behavior. Affordability only shows a short run effect, while mortgage loans turn out to be non-significant. These findings suggest, in line with the popular view on this topic, that real state fulfills a prominent role as a financial investment in Argentina.

JEL classifications: C32, G21, R21
Keywords: Housing demand; Financial investment; Equilibrium Correction model.

*The views expressed herein are solely our own and should not be interpreted as those of the Central Bank of Argentina.
1. Motivation and Hypothesis

Real estate purchases are typically assumed to meet the household demand for housing. However, real estate may also fulfill an additional role as a financial investment vehicle. Offsetting its relative illiquidity, real estate has the advantage of being a safer, less volatile asset than most capital market instruments. In the Argentine case, the latter appeal is heightened by the recurrence of financial crises that have been accompanied by restricted convertibility and partial expropriation of bank deposits as well as the collapse of the bond and stock markets. The ensuing lack of confidence in the domestic financial system has fed a thirst for alternative assets, such as foreign currency and real estate, which are perceived as more immune to crisis-related losses.

Although the acquisition of housing units as a financial investment is not uncommon around the world, its extent in Argentina seems to stand out relative to other countries. With this in mind, our brief note constitutes a first attempt at producing hard evidence on the following question: Are housing prices in Argentina mostly driven by housing or by investment motives? To this end, we devised a simple empirical test taking the form of a regression of apartment prices in Buenos Aires City on four explanatory variables. Ideally, one should directly survey real state buyers as to whether they intend it for one motive or the other. Or we may look at individual housing wealth, considering more than one house as financial investment. However, neither of these information sources is publicly disclosed, so other empirical proxies should be employed. Our procedure incorporates four variables that confidently capture the effects to be tested and that are available for our econometric exercises.¹

On one hand, if household demand for housing exerts a noticeable impact on the real state market, the affordability ratio (per capita income to average real state prices) and the flow of personal mortgage loans (as a measure of credit availability to buy a house) should enter positive and significantly into the above price regression.

On the other hand, if the appetite for real state as a financial investment proves to have some empirical grounds, the flow of private bank deposits and the growth of average income should display some explanatory power in the above price equation. The flow of private bank deposits is an easily available and sound indicator of the substitution between intermediated and non-intermediated saving, under the premise that at times of undermined confidence in the financial system, saving flees the banking system into safer heavens, such as the real state market. Likewise, income growth represents a proxy for the volume of private saving, part of which investors might channel into this market. Take into account that (a) the choice of the latter variable is mainly dictated by

¹ Our model focuses on demand factors affecting housing prices. Since supply factors (such as Tobin’s q) are unlikely to be correlated with the included variables, this omission is not expected to cause any bias in our estimates. We acknowledge this observation by an anonymous referee appointed by the Center of Latin American Monetary Studies (CEMLA).
the lack of monthly saving data, and by the evidence that income growth and saving are highly correlated in Argentina and worldwide (see Bebczuk (2002)), (b) Income effects on the demand for housing are already captured by the affordability ratio, and (c) The two hypotheses are not mutually exclusive, once both the housing and the investment motives may be significant -the relative importance of each one is indeed the central empirical question to be addressed here.

The paper is organized as follows: Section 2 reviews the literature. In section 3 a brief description of the data is presented, along with the econometric methodology and results, while Section 4 concludes.

2. Literature Review

There exists a profuse literature on housing and real state. Nevertheless, the research work on the housing versus investment goals is comparatively underdeveloped, and even more so is the analysis of housing finance in Argentina. We will concisely summarize in what follows the existing body of papers.

In regard to the consumption versus investment reasons to buy real state, a handful of papers have addressed the issue at the theoretical level -see Cocco (2005), Yao and Zhang (2004), Brueckner (1997), Grossman and Laroque (1990) and Henderson and Ioannides (1983). Cocco (2005), for instance, finds that there is a trade off between stock risk and housing risk: due to investment in housing, younger and poorer investors have limited financial wealth to invest in stocks, which reduces the benefits of equity market participation. When housing is incorporated to the portfolio analysis, limited equity market participation is obtained at lower cost of participation. A related study is Yao and Zhang (2004), who also study the effects of housing on the portfolio allocation of liquid wealth among stocks and bonds. They find that when investors are indifferent between renting and owning a house, they choose substantially different portfolio allocations when owning a house versus when renting housing services. When owning a house, investors substitute home equity for risky stocks, but hold a higher equity proportion in their liquid financial portfolio (bonds and stocks).

Concerning Argentina, the housing finance literature has more or less the same negligible size as its mortgage market. Coremberg (2000) analyzing the determinants of housing prices in the long run, find a positive relationship with construction cost and a

\[ \text{Footnote:} \text{This model can rationalize Heaton and Lucas (2000) empirical result that higher mortgage leads to higher stock holdings: due to the consumption dimension of housing, investors who have more human capital acquire more expensive houses and borrow more. At the same time, human capital, although risky, resembles Treasury bills more closely, inducing a tilt in the financial portfolio toward stocks, as in Heaton and Lucas (1997) and Viceira (2001).} \]
negative one with real deposit interest rates and the stock market index over the period 1980-1998. Kiguel and Podjarny (2007) describe the evolution of the mortgage market for Argentina, Chile and Uruguay. Banzas and Fernández (2007) and Cristini and Moya (2004) focus more on public housing. Agarwal et al (2005), who study the 2001-2002 crises, use a loan-level dataset to empirically assess the impact of the currency devaluation and the policy measures taken afterwards. Auguste, Bebczuk and Moya (2011) conduct a household survey that, along with some aggregate data, suggests that demand factors have also played a major role in explaining the anemic situation of the mortgage market in Argentina.

3. Data, methodology and results

3.1 Data

As a first step in the analysis, monthly summary statistics are presented for the period under study (2002-2009) in Figures 1 through 4. Regarding the time series used in the paper, these are: (1) The price of apartments in Buenos Aires City (index expressed in dollars), (2) The flow of mortgage loans for housing in real terms, (3) The flow of private deposits in real terms, (4) The ratio of the Industrial Production Index to Real State Price Index (in pesos) defined above as a proxy for affordability, and (5) The percentage change in the Industrial Production Index as a proxy for income growth. More details and sources appear in Appendix 1.

Figures 1-4 below display clear trends over 2002-2009 for all the variables involved, namely, (a) Fast growing real state prices, (b) Declining volume of mortgage loans, (c) Diminishing affordability ratio, (d) Declining private bank deposits, and (e) Growing income. In light of the observed booming housing prices, the trajectory of the explanatory variables gives preliminary support to the hypothesis that a potent investment motive (financial disintermediation, income growth) is at play, fully offsetting a weak housing motive (reduced affordability and mortgage loan availability). In consequence, the investment motive should not be disregarded as an influential factor on determining real market prices.
Figure 1

Figure 2
3.2 Empirical Approach

In this section, the econometric methodology is described intuitively. In the Appendix 2 the methodology is presented formally.

In order to analyze the empirical results it is important to remember some econometric concepts. The estimations reported in Table 2 correspond to an Equilibrium Correction model. In the applied econometric literature the Equilibrium Correction model has become one of the most used dynamic specifications given that it models jointly short run and long run relations of economic variables. The long run relationship of economic variables is obtained following cointegration methodologies. By the end of the eighties the idea of econometric modeling of integrated variables was associated to the concept of cointegration developed by Engle and Granger (1987): two variables $y_t$ and $x_t$ integrated of order 1 are cointegrated if exists a linear combination of the variables that is stationary. Engle and Granger note that even though economic series may wander through time, economic theory often provides a rationale why certain variables should obey certain equilibrium constraints. That is, there may exist some linear combination of the variables that, over time, converges to equilibrium. The methodological strategy followed by Engle and Granger (1987) and by the General to Particular methodology followed by Hendry and his co-authors: (i) uses the Equilibrium Correction model, (ii) estimate the coefficients following the OLS estimation methodology and (iii) assumes a valid conditional model of $y_t$ on $x_t$. The OLS estimates in practice will differ according to the arbitrary normalization implicit in the selection of the left-hand-side variable for the regression equation. Moreover, different arbitrary normalizations can alter the Engle and Granger results. One of the main critics made towards these methodologies is that the single equation has been estimated selecting previously which is the endogenous variable and which are the exogenous ones. The advantage of the Johansen (1988), Johansen and Juselius (1990) and Juselius (2006) methodology is that it allows modeling short run and long run relations jointly and permits to model the variables in a system without determine if they are endogenous or exogenous.

It is important to recognize that is natural to build an empirical model previously selecting the variables that are going to be treated as endogenous and exogenous. However, the concept of exogeneity is the tool that modern econometrics uses to solve problems associated with the specification forms and the selection of exogenous variables. The weak exogeneity conditions -according to Engle, Hendry and Richard (1983) definition- allow valid statistical inferences in an econometric model. An exogenous variable is defined as one that is determined from the outside of the analyzed system without loss of relevant sample information. Intuitively, weak exogeneity of the variables located on the right-hand-side of the equation means that there is no loss of relevant information conditioning endogenous variables on these exogenous ones. Estimation and testing of cointegrating systems, following Johansen
(1988) and Johansen and Juselius (1990) approaches allow testing weak exogeneity evaluating which of the variables of the system responds to the long run disequilibrium (see Johansen, 1992, Urbain, 1992, Ericsson, 1994, Juselius, 2006 and Ahumada and Garegnani, 2005 and Garegnani, 2008, for the Argentine case). The test of weak exogeneity evaluates the significance of the Equilibrium Correction term in each equation of the system. If the null hypothesis is rejected, the variable is endogenous because, in this methodology, the variables that react to the deviations of the long run relationship (represented by the Equilibrium Correction term) are considered as endogenous.

3.3 Results

Following the Johansen’s approach, Table 1 reports the estimation of the unrestricted system including housing prices, the flow of mortgage loans, private deposits and income and affordability measured as the ratio previously defined in section 3.1. Only one cointegration relationship was identified. In the long run, private deposits and income -the proxies capturing the investment motive- are the only significant determinants of housing prices. The tests reported in the low part of the Table correspond to the weak exogeneity tests. The hypothesis evaluated is $\alpha_i=0$ vs. $\alpha_i\neq 0$, with $i=1,2,3,4,5$. The $\alpha$'s represent the response of each variable to the deviations of the long run relationship. In this case the hypothesis $\alpha_1=0$, the response of housing prices to the disequilibrium equals to zero, is rejected but the hypotheses $\alpha_2=0$, $\alpha_3=0$, $\alpha_4=0$ and $\alpha_5=0$ are not rejected (see the respective p-values in Table between [ ]). Weak exogeneity results indicate that the only variable that reacts to the deviations from the cointegration relationship is housing prices.
Table 1

<table>
<thead>
<tr>
<th>Housing Prices</th>
<th>Long run Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Income</td>
<td>0.9332</td>
</tr>
<tr>
<td></td>
<td>[0.0000]**</td>
</tr>
<tr>
<td>Private Deposits</td>
<td>-0.0797</td>
</tr>
<tr>
<td></td>
<td>[0.0017]**</td>
</tr>
<tr>
<td>Mortgage Loans</td>
<td>-0.0044</td>
</tr>
<tr>
<td></td>
<td>[0.4452]</td>
</tr>
<tr>
<td>Affordability</td>
<td>0.0271</td>
</tr>
<tr>
<td></td>
<td>[0.1299]</td>
</tr>
<tr>
<td>Weak Exogeneity test</td>
<td>Ho: α_1=0;[0.0000]**</td>
</tr>
<tr>
<td></td>
<td>Ho: α_2=0;[0.1315]</td>
</tr>
<tr>
<td></td>
<td>Ho: α_3=0;[0.7355]</td>
</tr>
<tr>
<td></td>
<td>Ho: α_4=0;[0.2292]</td>
</tr>
<tr>
<td></td>
<td>Ho: α_5=0;[0.2852]</td>
</tr>
</tbody>
</table>

**significance at 1%
p-value between []

These results demonstrate the validity of the conditional model of housing prices as a function of the other variables of the system. In this case the long run relationship becomes:

\[
\text{Housing Prices} = 0.93 \text{ Income} - 0.08 \text{ Private Deposits}
\]

Variables are measured in logs, which mean that each coefficient represents an elasticity. The results imply that a 1% income increase leads to a 0.93% housing prices increase in the long run. By the same token, when private deposits increase by 1%, housing prices decrease by 0.08% in the long run. While not particularly strong, these effects lend indisputable support to the investment motive as determinants of housing prices in Argentina.

Therefore, the relationship between these three variables could be model as a simple version of an Equilibrium Correction model. Thus the econometric analysis continued with a “general” model that included an Equilibrium Correction term of the long run
relationship and the determinants of housing prices that did not enter the long run relationship but could be part of the dynamics: affordability and mortgages loans. A model with twelve lags for each variable and with seasonal dummy variables was estimated. Using the General to Particular methodology, the following step consists on simplify this initially general model that adequately characterizes the empirical evidence. After the simplification based on eliminating the variables that satisfy the selection (i.e., simplification) criteria of not being individually significant at 5%, the final model is reported in Table 2. Tests reported in the low part of Table 2 presents diagnostic statistics for testing residual autocorrelation (AR), autoregressive conditional heteroscedasticity (ARCH), skewness and excess kurtosis (Normality), heteroscedasticity (Xi^2, which uses squares of the original regressors and Xi*Xj which uses squares of the original regressors and cross-products) and RESET (RESET) (See Doornik and Hendry (2009a) for details and references). The tests show homoscedastic white-noise and normal residuals according to the above reported statistics and a correct specification of the model.

Table 2

\[ \Delta \text{Housing prices}_t = -0.02574 + 0.6728 \Delta \text{Housing prices}_{t-1} - 0.03953 \text{EqC}_{t-1} \]
\[ [\text{HCSE}] \quad [0.0113]^* \quad [0.0571]^{**} \quad [0.0015]^{**} \]
\[ -0.0027 \Delta \text{Private Deposits}_t - 0.4948 \Delta \Delta \text{Affordability}_t \]
\[ [0.0013]^* \quad [0.0804]^{**} \]
\[ + 0.4886 \Delta \text{Income}_t + \text{Seasonal} + \text{dummy variables} \]
\[ [0.0748]^{**} \]

\[ R^2 = 0.863693 \quad F(10, 82) = 51.958 \quad [0.0000] \quad \sigma = 0.013 \quad DW = 2.31 \]

* **significance at 1%
* *significance at 5%
HC Standard Errors between [ ]

Residual and Specification Tests

\[ \text{AR 1-6} \quad F(6, 76) = 1.4452 \quad [0.2087] \]
\[ \text{ARCH 6} \quad F(6, 70) = 1.4224 \quad [0.2184] \]
\[ \text{Normality Chi}^2(2) = 2.8423 \quad [0.2414] \]
\[ \text{Xi}^2 \quad F(15, 66) = 0.5967 \quad [0.8671] \]
\[ \text{Xi*Xj} \quad F(25, 56) = 1.0745 \quad [0.3996] \]
\[ \text{RESET} \quad F(1, 81) = 0.6305 \quad [0.4295] \]
In the model presented in Table 2 the Equilibrium Correction term (EqCₜ₋₁) is significant and about 0.04 of the disequilibrium is corrected in the first month in order to adjust the long run relationship among housing prices, private deposits and income. The dependent variable lagged one month has a positive impact of 0.67, which means that the rate of growth of housing prices has the inertia shown in the Figures of section 3.1. Income and private deposits have not only a long run effect but also a short run effect on housing prices. An increase of 1% in the growth rate of private deposits has a negative effect of 0.002% in the monthly growth rate of housing prices. In the case of income and affordability the restricted model shows that the rate of growth of each variable and its first lag resulted significant with the same coefficient but inverse sign. The restriction of the sum of estimated coefficients of ΔIncomeₜ and ΔIncomeₜ₋₁ equals to zero is not rejected at traditional significance levels meaning that is (ΔIncomeₜ - ΔIncomeₜ₋₁) or the acceleration rate of income which has a short run effect on housing prices. In the case of affordability the dynamics specification also reflects that (ΔAffordabilityₜ - ΔAffordabilityₜ₋₁) has an impact effect on housing prices. The short run effect of income acceleration is about 0.49%. The affordability although did not enter in the long run, it has a significant and negative impact in the short run. When the affordability variable is measured as its acceleration rate, it has a significant and negative effect on housing prices of approximately -0.49.

Figure 5 shows the goodness of fit of the previous estimated Equilibrium Correction model.
Figure 6 shows the predictive performance only of the variables related to investment motive. The Equilibrium Correction model is estimated eliminating the acceleration of affordability. The performance is quite similar to that of the original model of Table 2, which adequately characterizes the rate of growth of housing prices during the sample period, indicating that housing prices in Argentina are mostly driven by investment motives.
4. Conclusions

Although the acquisition of housing units as a financial investment is not uncommon around the world, its extent in Argentina seems to stand out relative to other countries. The research work on the housing versus investment goals is comparatively underdeveloped, and even more so is the analysis of housing finance in Argentina.

This paper is a first attempt at producing hard evidence on the following question: Are housing prices in Argentina mostly driven by housing or by investment motives? The two hypotheses are not mutually exclusive, once both the housing and the investment motives may be significant - the relative importance of each one is indeed the central empirical question to be addressed here. To this end, we devised a simple empirical test taking the form of an Equilibrium Correction model of apartment prices in Buenos Aires City on four explanatory variables: affordability, mortgages loans, private bank deposits and income.
On one hand, if household demand for housing exerts a noticeable impact on the real state market, the affordability ratio and the flow of personal mortgage loans should enter positively and significantly into the above price regression. On the other hand, if the appetite for real state as a financial investment proves to have some empirical grounds, the flow of private bank deposits and the growth of average income should display some explanatory power in the price equation.

The results suggest that housing prices in Argentina are mostly driven by investment motives, as opposed to the traditional housing motive, as the evolution of private bank deposits and income have short and long run explanatory power on housing prices, unlike affordability and mortgage loans. The acceleration of affordability, although highly significant, only has a short run impact effect on housing prices. These findings suggest, in line with the popular view on this topic, that real state fulfills a prominent role as a financial investment in Argentina.
References


Appendix 1

Housing prices: Apartment prices in Buenos Aires City. Instituto de Economía FACE – UADE.

Mortgage loans: Total mortgages loans to private sector in real terms. BCRA.

Private Deposits: Private Deposits in real terms. BCRA.

Industrial Production Index: IPI. F.I.E.L.
Appendix 2

It is important to remember that a variable is said to be integrated of order \( d \) (denoted \( I(d) \)) if the series would be stationary after being first-differenced \( d \) times. For example, a random walk is integrated of order 1, because the series should be first-differenced once in order to be stationary. A stationary process is denoted as \( I(0) \). By the end of the eighties the idea of econometric modeling integrated variables was associated to the concept of cointegration developed by Engle and Granger (1987): two variables \( y_t, x_t \) \( I(1) \) are cointegrated if exists a linear combination of the variables that is \( I(0) \). Engle and Granger note that even though economic series may wander through time, economic theory often provides a rationale why certain variables should obey certain equilibrium constraints. That is, there may exist some linear combination of the variables that, over time, converges to equilibrium. The methodological strategy followed by Engle and Granger differs from the General to Particular methodology followed by Hendry and his co-authors, which take into account the time series properties of the data and models jointly short and long run relations. However, both methodologies: (i) uses the Equilibrium Correction model, (ii) estimate the coefficients following the OLS estimation methodology and (iii) assumes a valid conditional model of \( y_t \) on \( x_t \). The OLS estimates in practice will differ according to the arbitrary normalization implicit in the selection of the left-hand-side variable for the regression equation. Moreover, different arbitrary normalizations can alter the Engle and Granger results.

It is important to recognize that is natural to build an empirical model previously selecting the variables that are going to be treated as endogenous and exogenous. However, the concept of exogeneity is the tool that modern econometrics uses to solve problems associated with the specification forms and the selection of exogenous variables. The weak exogeneity conditions allow valid statistical inferences in an econometric model. An exogenous variable is defined as one that is determined from the outside of the analyzed system without loss of relevant sample information. Intuitively, weak exogeneity of the variables located on the right hand side of the equation means that there is no loss of relevant information conditioning endogenous variables on these exogeneous ones.

Engle, Hendry and Richard (1983) define three concepts of exogeneity, weak, strong and super, depending on the purpose of the model, inference, forecasting and policy analysis and the parameters of interest. Weak exogeneity is essential, as it is required for the other concepts (along with Granger non-causality for strong and invariance for super). Weak exogeneity allows validating a conditional model of \( y_t \) on \( x_t \) and \( x_t \) may be treated “as if” it were determined outside the (conditional) model under study making the analysis simpler and more robust. The relevance of this definition of weak exogeneity is the following; many of the exogeneity definitions have been formulated based on orthogonality conditions among the observed variables and the non observed
errors of the models with Gaussian distributions. The weak exogeneity definition described here can be applied in a more general way to non-normal and non-linear processes. Basically the concept of weak exogeneity allows to model a group of variables (the endogenous) without necessarily specify how the second group of variables (the exogenous) is determined.

Formally, if the regression of $y_t$ on $x_t$ is considered as a model representation of the conditional density function of "$y$" given "$x$" ($D_{y/x}$), and if $D_{yx}(y_t, x_t; I_{t-1}, \phi)$ is the joint density function, the latter can be factorized as the conditional density times the marginal density of "$x$":

$$D_{yx}(y_t, x_t; I_{t-1}, \phi) = D_{y/x}(y_t, x_t; I_{t-1}, \phi_1) \cdot D_{x}(x_t; I_{t-1}, \phi_2)$$

where $I_{t-1}$ is the information set previous to "$t" y \phi$ represents the parameters of interest$^3$ that characterizes $D_{yx}$. Considering this, "$x$" is weakly exogenous for the parameters of interest if inference concerning $\phi$ from the joint density will be equivalent to that from the conditional density so that the latter may be used without loss of relevant information. This occurs: i) if the parameters $\phi$ can be partitioned in $\phi_1$ and $\phi_2$, without loss of relevant information known as "sequential cut" and $\phi_1$ and $\phi_2$ are "variation free", which means that should not be subject to "cross-restrictions" and ii) the parameters of interest are only a function of $\phi_1$ can be uniquely determined from the parameters of the conditional model.

In the case of integrated variables, estimation and testing of cointegrating systems, following Johansen (1988) and Johansen and Juselius (1990) approaches allow evaluating a necessary condition for weak exogeneity. The cointegrated VAR models can be seen as a general framework within which one can describe economic behavior in terms of the short run and the long run. This reduced form allows testing weak exogeneity evaluating which of the variables of the system responds to the long run disequilibrium (see Johansen, 1992, Urbain, 1992, Ericsson, 1994, Juselius 2006 and Ahumada and Garegnani, 2005 and Garegnani, 2008, for the Argentine case).

This methodology can be analyzed for the simplest system with two variables and one lag (Ericsson, 1994). Let be (1a) and (1b) the VAR representation (joint system) of $y_t$ and $x_t$ (which can also be interpreted as a reduced form),

---

$^3$ Parameters may be of interest, e.g. because they are directly related to theories the model user wishes to test concerning the structure for the economy. Equally in seeking empirical econometric relationships, which are constant over the sample period and hopefully over the forecast period, parameters that are structurally invariant are typically of interest (See Engle, Hendry and Richard (1983)).
\[ y_t = \pi_{11} y_{t-1} + \pi_{12} x_{t-1} + \varepsilon_{yt} \quad (1a) \]
\[ x_t = \pi_{21} y_{t-1} + \pi_{22} x_{t-1} + \varepsilon_{xt} \quad \varepsilon_{i} \sim \mathcal{N}(0, \Omega) \quad (1b) \]

where \[ \varepsilon_{i} = \begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix} ; \Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \]

It is under this parameterization (or in differences of the original variables if the variables are I(1)) that Granger non-causality (H_0: \pi_{12} = 0 \text{ y } H_0: \pi_{21} = 0) is usually tested. However, for I(1) variables, a re-parameterization of the system allows testing both the presence of cointegration (the long-run relationship between \( y_t \) and \( x_t \) if integrated) and the weak exogeneity (for the parameters of interest to study) given cointegration.

The number of cointegration relationships can be evaluated through the rank of a matrix closely related to \( \Pi \), which is obtained by rewriting equations (1) in the next form,

\[ \Delta y_t = \pi_{11} \cdot y_{t-1} + \pi_{12} \cdot x_{t-1} + \varepsilon_{1t} \quad (2a) \]
\[ \Delta x_t = \pi_{21} \cdot y_{t-1} + \pi_{22} \cdot x_{t-1} + \varepsilon_{2t} \quad (2b) \]

where \( \Pi = \{ \pi_{ij} \} \) and \( \pi_{jj} = \pi_{jj} \cdot 1 \text{ if } i=j \) and \( \pi_{ij} = \pi_{ij} \text{ if } i \neq j \).

If the rank of matrix \( \Pi \) is zero the variables would be two independent random walks. If the matrix is full rank, the variables would be stationary and we could model the VAR in levels. If \( \Pi \) is not the full rank it can be factorized as the outer product of \( \alpha \) and \( \beta \), \( \Pi = \alpha \cdot \beta' \) where \( \beta' \) are “cointegrating vectors” (the long-run relationships) and \( \alpha \) the weights that each relationship enters each equation. Note that this factorization is not unique since: \( \Pi = \alpha \cdot \beta' = \alpha \cdot P \cdot P^{-1} \cdot \beta' = \alpha' \cdot \beta' \), and it is common to normalize for one of the variables. For the bivariate case, finding cointegration implies \( r=1 \) and \( \alpha \) and \( \beta' \) vectors 2.1 and 1.2, respectively: \( \alpha = (\alpha_1, \alpha_2) \) y \( \beta' = (\beta_1, \beta_2) \). The second vector can be written without loss of generality as \( \beta' = (1, -\delta) \), (normalizing the coefficient of \( y_t \)) and then the system of equations (2) expressed in a cointegrating vector form becomes,

\[ \Delta y_t = \alpha_1 \cdot (y_{t-1} - \delta \cdot x_{t-1}) + \varepsilon_{yt} \quad (3a) \]
\[ \Delta x_t = \alpha_2 \cdot (y_{t-1} - \delta \cdot x_{t-1}) + \varepsilon_{xt} \quad (3b) \]

where \( \alpha_1 = \pi_{11} \cdot ; \alpha_2 = \pi_{21} \cdot ; \delta = -\pi_{12} \cdot / \pi_{11} \cdot = -\pi_{22} \cdot / \pi_{21} \cdot . \)

This is a joint model of \( y_t \) and \( x_t \) (conditional on their past) expressed as a Vector Equilibrium Correction. This system can be expressed (factorized) in terms of a conditional and a marginal model as follows,
\[ \Delta y_t = \gamma_1 \Delta x_t + \gamma_2 (y_{t-1} - \delta x_{t-1}) + \nu_t \]  
(4a)

\[ \Delta x_t = \alpha_2 (y_{t-1} - \delta x_{t-1}) + \varepsilon_t \]  
(4b)

where \( \gamma_1 = \omega_{22} / \omega_{21} \) and \( \gamma_2 = \alpha_1 - (\omega_{22} / \omega_{21}) \alpha_2 \).

Equation (4a) is also an Equilibrium Correction (EqC) model. In this representation weak exogeneity requires \( \omega_2 = 0 \) in order to fulfill the weak exogeneity condition (Urbain, 1992) and therefore (4a) is a valid conditional model. The parameters of the conditional model can be estimated from (4a) alone. Note that in this case the single equation model of \( y_t \) on \( x_t \) is validated and can be estimated using the OLS method.

For a multivariate setting, the Johansen and Juselius methodology consider:

\[ X_t = \Pi_1 X_{t-1} + \ldots + \Pi_k X_{t-k} + \varepsilon_t \ (t = 1, 2, \ldots, T) \]

where \( X_t \) is a sequence of random vectors with components \( (X_{1t}, X_{2t}, \ldots, X_{pt}) \) and the innovations to this process are drawn from a Gaussian distribution.

Johansen and Juselius suggest writing the previous equation in the equivalent form:

\[ \Delta X_t = \Gamma_1 \Delta X_{t-1} + \ldots + \Gamma_{k-1} \Delta X_{t-k+1} - \Pi X_{t-k} + \varepsilon_t \]

where

\[ \Gamma_i = -I + \Pi_1 + \ldots + \Pi_i \quad (i = 1, \ldots, k - 1) \]

\[ \Pi = I - \Pi_1 - \ldots - \Pi_k \]

The only difference between a standard first-difference version of a VAR is the term \( \Pi X_{t-k} \); this \( \Pi \) matrix incorporates all the information about the long run relationship between the \( X \) variables.

As it was explained for the two-variables case, the cointegration can be detected by examine the rank of the \( \Pi \) (pxp) matrix: if \( \Pi \) has rank 0 then all the elements of \( X_t \) have unit roots and the estimation in first differences might be recommended, if \( \Pi \) is of full rank \( p \), then all elements of \( X_t \) are stationary in levels. The interesting case in this study is when \( 0 < \text{rank}(\Pi) = r < p \). In this case there are \( r \) cointegrating relations among the elements of \( X_t \). If \( \Pi \) has rank \( r < p \), this imply \( \Pi = \alpha \beta' \), matrices pxr. \( \beta \) is the matrix of cointegrating vectors and \( \alpha \) is a matrix of equilibrium correction parameters.

Johansen and Juselius (1990) demonstrate that \( \beta \) can be estimated as the eigenvector associated with the \( r \) largest, statistically significant eigenvalues found by solving:
\[ |\lambda S_{kk} - S_{ko} S_{ok}^{-1} S_{ok}| = 0 \]

where \( S_{oo} \) represents the residual moment matrix from the least squares regression of :
\[ \Delta X_t, \Delta X_{t-1}, \ldots, \Delta X_{t-k+1}, S_{kk} \] is the residual moment matrix from a least square regression of :
\[ X_{t-k} \Delta X_{t-1}, \ldots, \Delta X_{t-k+1} \text{ and } S_{ok} \text{ is the cross - product moment matrix.} \]

Using these eigenvalues, one may test the hypothesis that there are at most \( r \) cointegrating vectors by calculating the likelihood test statistic

\[ \lambda_{\pi(r)} = -T \sum_{i=1}^{p-r} \ln(1 - \hat{\lambda}_i) \]

where \( \hat{\lambda}_i \) are the \( p - r \) smallest eigenvalues.

Johansen and Juselius (1990) procedure provides a framework in which alternative hypotheses about elements of \( \alpha \) and \( \beta \) matrixes can be tested. For example, within the context of consumption function analysis, tests of \( \beta \) include testing the hypothesis that the long run income elasticity is unity and tests of \( \alpha \) include the weak exogeneity test of \( \alpha \) equals to zero.

To test these hypotheses, one compares a model incorporating the test restriction to an unrestricted model. The relevant likelihood ratio test statistic is:

\[ (-2) \ln(Q) = -T \sum_{i=1}^{r} \ln\left\{ (1 - \hat{\lambda}_i^*) / (1 - \hat{\lambda}_i) \right\} \]

where \( \hat{\lambda}_i^* \) and \( \hat{\lambda}_i \) are the \( r \) largest eigenvalues when the restrictions were imposed and under no restrictions.

The test statistic is distributed as \( \chi^2 \) with \( r(p - s) \) degrees of freedom.