B_1-EPG GRAPHS ARE 4-CLIQUE COLORABLE

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Abstract: We consider the problem of clique coloring, that is, coloring the vertices of a given graph such that no (maximal) clique of size at least two is monocolored. It is known that interval graphs are 2-clique colorable. In this work we prove that B_1-EPG graphs (edge intersection graphs of paths on a grid, where each path has at most one bend) are 4-clique colorable. Moreover, given a B_1-EPG representation of a graph, we provide a linear time algorithm that constructs a 4-clique coloring of it.

Keywords: *clique coloring, edge intersection graphs, paths on grids, polynomial time algorithm.*
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1 INTRODUCTION

An *EPG representation* of a graph G, $\langle \mathcal{P}, \mathcal{G} \rangle$, is a collection of paths $\mathcal{P}$ of the two-dimensional grid $\mathcal{G}$ where the paths represent the vertices of G in such a way that two vertices of G are adjacent in G if and only if the corresponding paths share at least one edge of the grid. A graph which has an EPG representation is called an *EPG graph* (EPG stands for Edge-intersection of Paths on a Grid). In this work, we consider the subclass B_1-EPG.

A B_1*-EPG representation* of G is an EPG representation in which each path in the representation has at most one *bend* (turn on a grid point). Recognizing B_1-EPG graphs is an NP-complete problem (see [9]).

EPG graphs have a practical use, for example, in the context of circuit layout setting, which may be modelled as paths (wires) on a grid. In the knock-knee layout model, two wires may either cross or bend (turn) at a common point grid, but are not allowed to share a grid edge; that is, overlap of wires is not allowed. In this context, some of the classical optimization graph problems are relevant, for example, maximum independent set and coloring. More precisely, the layout of a circuit may have multiple layers, each of which contains no overlapping paths. Referring to a corresponding EPG graph, then each layer is an independent set and a valid partitioning into layers corresponds to a proper coloring.

In this work, we consider the problem of *clique coloring*, that is, coloring the vertices of a given graph such that no (maximal) clique of size at least two is monocolored. We prove that B_1-EPG graphs are 4-clique colorable. Moreover, given a B_1-EPG representation of a graph, we provide a polynomial time algorithm that constructs a 4-clique coloring of it.

2 PRELIMINARIES

In this work all graphs are connected, finite and simple. Notation we use is that used by Bondy and Murty [2]. The *vertex set* of a graph G is the set of all the vertices of G, denoted by $V(G)$, and the *edge set* of G is the set of all the edges of G, denoted by $E(G)$.

A *complete* is a set of pairwise adjacent vertices. A *clique* is a complete which is not properly contained in another complete.

A *k-coloring* of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that adjacent vertices have different labels, i.e., no complete of size 2 is monocolored. The *chromatic number* of a graph G is the smallest positive integer k such that G has a k-coloring, denoted by $\chi(G)$. A *k-clique coloring* of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such hat no clique of G with size at least two is monocolored. A graph G is *k-clique colorable* if G has a k-clique coloring. The *clique chromatic number* of G is the smallest k such that G has a k-clique coloring, denoted by $\chi_c(G)$.

Clique coloring has some similarities with usual coloring, for example, any coloring is also a clique coloring, and optimal colorings and clique colorings coincide in the case of triangle-free graphs. But there are also essential differences, for example, a clique coloring of a graph may not be a clique coloring for its subgraphs. Subgraphs may have a greater clique chromatic number than the original graph. Another difference is that even a 2-clique colorable graph can contain an arbitrarily large clique. It is known that the 2-clique coloring problem is NP-complete, even under different constraints [1, 10].

Many families of graphs are 3-clique colorable, for example, comparability graphs, co-comparability graphs, circular arc graphs and the k-powers of cycles [3, 4, 6, 7]. In [1], Bacso et al. proved that almost all perfect graphs are 3-clique colorable and conjectured that all perfect graphs are 3-clique colorable. On the other hand, some families of graphs have unbounded clique chromatic numbers, for example, triangle-free graphs, UE graphs and line graphs [1, 5, 11]. It has been proved that chordal graphs, and in particular interval graphs, are 2-clique colorable [12].

Moreover, every chordal graph admits a 2-clique coloring in which one of the color classes is an independent set. Such a coloring is obtained by the following algorithm: let $v_1, \ldots, v_n$ be a perfect elimination ordering of the vertices of a chordal graph G, i.e., for each i, $N[v_i]$ is a clique of $G[\{v_i, \ldots, v_n\}]$; color the vertices from v_n to v_1 with colors a and b in such a way that v_n gets color a and v_i gets color b if and only if all of its neighbors that are already colored got color a.

3 B_1-EPG GRAPHS ARE 4-CLIQUE COLORABLE

In this Section, we prove that B_1-EPG graphs are 4-clique colorable. In order to prove the main result of this work we need the following definitions and theorem.

The *claw* is a star with three edges (see Figure 1). The edges of the star are called the *legs*.

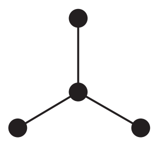

Figure 1: The claw.

Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a B_1-EPG representation of a graph G. For any edge e in the grid $\mathcal{G}$, let $\mathsf{P}[e] = \{P \in \mathcal{P} / e \in P\}$. For any claw H in $\mathcal{G}$, let $\mathsf{P}[H]$= $\{P \in \mathcal{P} /$ P contains two legs of the claw $H\}$. If the collection $\mathsf{P}[e]$ corresponds to a clique C in G, then C is called an *edge clique*. Similarly, if $\mathsf{P}[H]$ corresponds to a clique C in G, then C is called a *claw clique* (it holds if every pair of legs of H is contained in a path P of $\mathcal{P}$). We say that a claw clique is *centered at* x if the corresponding claw is a star with center x.

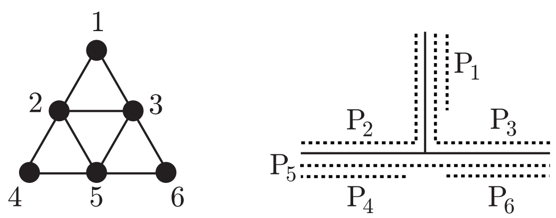

Figure 2: An EPG representation of the 3-sun. The central triangle $\{2, 3, 5\}$ is a claw clique; the other three triangles are edge cliques.

Theorem 1 [8] *Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a B_1-EPG representation of a graph G. Every clique in G corresponds to either an edge clique or a claw clique in $\langle \mathcal{P}, \mathcal{G} \rangle$.*

The proof of the following theorem have been omitted for lack of space.

Theorem 2 *Let G be a B_1-EPG graph. Then, G is 4-clique colorable. Moreover, a 4-clique coloring of G can be obtained in linear time on the number of vertices and edges, given a B_1-EPG representation of G.*

4 Conclusion

In this work we have proved that B_1-EPG graphs are 4-clique colorable, and that such coloring can be obtained in polynomial time, given a B_1-EPG representation. We conjecture that indeed B_1-EPG graphs are 3-clique colorable. An example of a B_1-EPG graph that requires 3 colors for a clique coloring is the chordless cycle on 5 vertices.

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