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Parabolic Partial Differential Equations with Border Conditions of Dirichlet as Inverse Moments Problem

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Abstract

We considerer parabolic partial differential equations: $w_t-(w_x)_x=r(x,t)$ under the conditions $w(a_1,t)=k_1(t)$ $w(b_1,t)=k_2(t)$, $w(x,a_2)=h_1(t)$ on a region $E=(a_1,b_1)\times(a_2,b_2)$; $b_2=\infty$. We will see that an approximate solution can be found using the techniques of generalized inverse moments problem and also bounds for the error of estimated solution. First we transform the parabolic partial differential equation to the integral equation $\int_{a_1}^{b_1}\int_{a_2}^{b_2} e^{-m(x+t)}\left(w_x(x,t)-w_t(x,t)\right)\mathrm{d}t\mathrm{d}x=\varphi_1(m)$. Using the inverse moments problem techniques we obtain an approximate solution $p_n(x,t)$ of $w_x(x,t)-w_t(x,t)$. Then we find a numerical approximation of $w(x,t)$ when solving the integral equation $\int_{a_1}^{b_1}\int_{a_2}^{b_2} e^{-(m+1)(x+1)-(z+1)(t+1)}\left(w(x,t)(t-x)\right)\mathrm{d}t\mathrm{d}x=\varphi_2(m,z)$, because solving the previous integral equation is equivalent to solving the equation $w_x(x,t)-w_t(x,t)=p_n(x,t)$.

Keywords

Parabolic PDEs, Integral Equations, Generalized Moment Problem

1. Introduction

We considerer parabolic partial differential equation of the form:

$$w_t-(w_x)_x=r(x,t) \tag{1}$$

where the unknown function $w(x,t)$ is defined in $E=(a_1,b_1)\times(a_2,\infty)$ and $r(x,t)$ is known function. Under the conditions

$$w(a_1,t)=k_1(t) \qquad w(b_1,t)=k_2(t) \tag{2}$$

$$w(x,a_2)=h_1(t) \tag{3}$$

This problem was studied under conditions of Cauchy in [1] and under conditions of Neumann in [2].

Parabolic differential equations are commonly used in the fields of engineering and science for simulating physical processes. These equations describe various processes in viscous fluid flow, filtration of liquids, gas dynamics, heat conduction, elasticity, biological species, chemical reactions, environmental pollution, etc. [3] [4].

In a variety of cases, approximations are used to convert parabolic PDEs to ordinary differential equations or even to algebraic equations. The existence and uniqueness properties of this problem are presented in literature. Several numerical methods have been proposed for the solution of this problem [5] [6] [7].

Parabolic partial differential equations have been numerically solved by using a variety of techniques [8] [9] [10] [11].

The finite element method for the numerical solution of partial differential equations is a general method covering all the three main types of equations: elliptic, parabolic and hyperbolic equations [12].

Some meshless schemes to solve differential partial equations are the diffuse element method [13], the partition of unity method [14], the element-free Galerkin method [15], the reproducing kernel particle method [16], the finite point method [17], the meshless local Petrov-Galerkin method [18], the use of radial basis functions [19] and the general finite difference method [20].

The d-dimensional generalized moment problem [21] [22] [23] [24] [25] can be posed as follows: find a function f on a domain $\Omega\subset \boldsymbol{R}^d$ satisfying the sequence of equations

$$\int_\Omega f(x)\,g_i(x)\,\mathrm{d}x=\mu_i \qquad i\in \boldsymbol{N} \tag{4}$$

where (g_i) is a given sequence of functions lying in $\boldsymbol{L}^2(\Omega)$ linearly independent, and the sequence of real numbers $\{\mu_i\}_{i\in N}$ is the known data.

The moments problem of Hausdorff is a classic example of moments problem, and is to find a function $f(x)$ in (a,b) such that

$$\mu_i=\int_a^b x^i f(x)\,\mathrm{d}x \qquad i\epsilon N$$

In this case the functions $g_i(x)=x^i\ i\epsilon N$. If the interval of integration is $(0,\infty)$ we have the problem of moments of Stieltjes; if the interval of integration is $(-\infty,\infty)$ we have the problem of moments of Hamburger.

Moment problem is usually ill-posed in the sense that there may be no solution and if there is no continuous dependence on the given data. There are various methods of constructing regularized solutions, that is, approximate solutions stable with respect to the given data. One of them is the *method of truncated expansion*.

The *method of truncated expansion* consists in approximating (4) by finite moment problems

$$\int_{\Omega} f(x) g_i(x) \mathrm{d}x = \mu_i \qquad i = 1, 2, \cdots, n \tag{5}$$

and consider as an approximate solution of $f(x)$ to $p_n(x) = \sum_{i=0}^{n} \lambda_i \varphi_i(x)$. The $\varphi_i(x)$ result from orthonormalize $\langle g_1, g_2, \cdots, g_n \rangle$ and λ_i are coefficients as a function of the μ_i.

Solved in the subspace $\langle g_1, g_2, \cdots, g_n \rangle$ generated by $g_1, g_2, \cdots, g_n$ (5) is stable. Considering the case where the data $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ are inexact, convergence theorems and error estimates for the regularized solutions they are applied.

In this paper we consider a different way to numerically solve the problem given by Equation (1) with conditions (2) and (3): we first transform it into an integral equation which we then handle as a bidimensional moment problem. This approach was already suggested by Ang [25] in relation with the heat conduction equation.

The work is organized as follows: in Section 2 first we transform the parabolic partial differential equation to the integral equation

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} e^{-m(x+t)} \left(w_x(x,t) - w_t(x,t) \right) \mathrm{d}t \mathrm{d}x = \varphi_1(m)$$

Using the inverse moments problem techniques we obtain an approximate solution $p_n(x,t)$ of $w_x(x,t) - w_t(x,t)$. Then we find a numerical approximation of $w(x,t)$ when solving the integral equation

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} e^{-(m+1)(x+1)-(z+1)(t+1)} \left(w(x,t)(t-x) \right) \mathrm{d}t \mathrm{d}x = \varphi_2(m,z)$$

In Section 3 the method is illustrated with examples.

2. Resolution of the Parabolic Partial Differential Equations

Let $F(w(x,t)) = r(x,t)$ be a partial differential equations such as (1). The solution $w(x,t)$ is defined on the region $E = (a_1, b_1) \times (a_2, b_2)$, $b_2 = \infty$ and verifies on the boundary $C = \partial E$:

$$w(a_1, t) = k_1(t) \qquad w(b_1, t) = k_2(t)$$

$$w(x, a_2) = h_1(t)$$

We apply the technique used in [2]. Let $F^* = (F_1(w), F_2(w))$ be a vectorial field such that w verifies $\operatorname{div}(F^*) = h^*(w)$ with h^* a known function and, reciprocally, if w verifies $\operatorname{div}(F^*) = h^*(w)$ then $F(w(x,t)) = r(x,t)$.

Specifically in this case $F(w(x,t)) = w_t - (w_x)_x$ and we take

$$F^* = (F_1(w), F_2(w)) = (w_x, -w)$$

$$h^*(w) = -r(x,t)$$

Let $u(m,z,x,t)$ be the auxiliary function

$$u(m,z,x,t) = e^{-(m+1)x-(z+1)t}$$

Since

$$u\,\mathrm{div}\left(F^*\right) = uh^*\left(w\right)$$

we have

$$\iint_E u\,\mathrm{div}\left(F^*\right)\mathrm{d}A = \iint_E uh^*\left(w\right)\mathrm{d}A$$

Moreover, as

$$u\,\mathrm{div}\left(F^*\right) = \mathrm{div}\left(uF^*\right) - F^*\cdot\nabla u$$

$$\iint_E u\,\mathrm{div}\left(F^*\right)\mathrm{d}A = \iint_E \mathrm{div}\left(uF^*\right)\mathrm{d}A - \iint_E F^*\nabla u\,\mathrm{d}A \tag{6}$$

where $\nabla u = \left(u_x, u_t\right)$ besides

$$\begin{aligned}\iint_E \mathrm{div}\left(uF^*\right)\mathrm{d}A &= \iint_E \left(\left(uw_x\right)_x - \left(uw\right)_t\right)\mathrm{d}A = \\ &= \iint_E u\,\mathrm{div}\left(F^*\right)\mathrm{d}A + \iint_E \left(\left(u_x w_x\right) - \left(u_t w\right)\right)\mathrm{d}A\end{aligned} \tag{7}$$

Then of (6) and (7)

$$\iint_E \left(u_x w_x - u_t w\right)\mathrm{d}A = \iint_E F^*\nabla u\,\mathrm{d}A \tag{8}$$

On the other hand it can be proved that, after several calculations, (8) is written as

$$\begin{aligned}&\int_{a_1}^{b_1} w\left(x,a_2\right)u\left(m,z,x,a_2\right)\mathrm{d}x + \frac{z+1}{m+1}\int_{a_2}^{b_2}\left(w\left(b_1,t\right)u\left(m,z,b_1,t\right) - w\left(a_1,t\right)u\left(m,z,a_1,t\right)\right)\mathrm{d}t \\ &= \frac{z+1}{m+1}\int_{a_1}^{b_1}\int_{a_2}^{b_2} w_x\left(x,t\right)u\,\mathrm{d}t\mathrm{d}x - \int_{a_1}^{b_1}\int_{a_2}^{b_2} w_t\left(x,t\right)u\,\mathrm{d}t\mathrm{d}x\end{aligned}$$

and if $z = m$ then

$$\begin{aligned}&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left(w_x\left(x,t\right) - w_t\left(x,t\right)\right)u\left(m,m,x,t\right)\mathrm{d}t\mathrm{d}x \\ &= \int_{a_1}^{b_1} w\left(x,a_2\right)u\left(m,m,x,a_2\right)\mathrm{d}x \\ &\quad + \int_{a_2}^{b_2}\left(w\left(b_1,t\right)u\left(m,m,b_1,t\right) - w\left(a_1,t\right)u\left(m,m,a_1,t\right)\right)\mathrm{d}t = \varphi_1\left(m\right)\end{aligned}$$

We take a base $\left\{\psi_i\left(m\right)\right\}_i$ of $L^2\left(a_2,b_2\right)$ and then the above equation can be transformed into a generalized moment problem

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left(w_x\left(x,t\right) - w_t\left(x,t\right)\right)H_i\left(x,t\right)\mathrm{d}t\mathrm{d}x = \mu_i \tag{9}$$

where

$$H_i\left(x,t\right) = \int_{a_2}^{b_2} u\left(m,n,x,t\right)\psi_i\left(m\right)\mathrm{d}m$$

and

$$\mu_i = \int_{a_2}^{b_2}\varphi_1\left(m\right)\psi_i\left(m\right)\mathrm{d}m$$

We can apply the truncated expansion method detailed in [24] and generalized in [25] [26] to find an approximation $p_n\left(x,t\right)$ for $w_x\left(x,t\right) - w_t\left(x,t\right)$ for the corresponding finite problem with $i = 0,1,\cdots,n$ where n is the number of moments μ_i. We consider the base $\phi_i\left(x,t\right)$ $i = 0,1,2,\cdots$ obtained by applying the Gram-Schmidt orthonormalization process on $H_i\left(x,t\right)$ $i = 0,1,2,\cdots$ and adding to the resulting set the necessary functions until reaching an orthonormal basis.

We approach the solution $w_x(x,t)-w_t(x,t)$ with [25] [26]:

$$p_n(x,t)=\sum_{i=0}^{n}\lambda_i\phi_i(x,t) \quad \text{where} \quad \lambda_i=\sum_{j=0}^{i}C_{ij}\mu_j \quad i=0,1,2,\cdots,n$$

And the coefficients C_{ij} verifies

$$C_{ij}=\left(\sum_{k=j}^{i-1}(-1)\frac{\langle H_i(x,t)|\phi_k(x,t)\rangle}{\|\phi_k(x,t)\|^2}C_{kj}\right)\cdot\|\phi_i(x,t)\|^{-1} \quad 1<i\le n;1\le j<i$$

The terms of the diagonal are

$$C_{ii}=\|\phi_i(x,t)\|^{-1} \qquad i=0,1,\cdots,n.$$

The proof of the following theorem is in [27] [28]. In [28] he proof is done for b_2 finite. If $b_2=\infty$ instead of taking polynomials the Legendre are taken polynomials of Laguerre. In [2] the demonstration is done for the one-dimensional case.

Theorem. *Let* $\{\mu_i\}_{i=0}^{n}$ *be a set of real numbers and suppose that* $f(x,t)$ *verify for some* ε *and* E (*two positive numbers*)

$$\sum_{i=0}^{n}\left|\int_{a_2}^{b_2}\int_{a_1}^{b_1}H_i(x,t)f(x,t)\,\mathrm{d}x\mathrm{d}t-\mu_i\right|^2\le\varepsilon^2$$

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}\left(xf_x^2+tf_t^2\right)e^{x+t}\mathrm{d}x\mathrm{d}t\le E^2$$

then

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}|f(x,t)|^2\,\mathrm{d}x\mathrm{d}t\le\min_i\left\{\|CC^{\mathrm{T}}\|\varepsilon^2+\frac{E^2}{8(i+1)^2};i=0,1,\cdots,n\right\}$$

where $\boldsymbol{C}$ is the triangular matrix with elements C_{ij} $(1<i\le n;1\le j<i)$, and

$$\int_{a_2}^{b_2}\int_{a_1}^{b_1}|p_n(x,t)-f(x,t)|^2\,\mathrm{d}x\mathrm{d}t\le\|CC^{\mathrm{T}}\|\varepsilon^2+\frac{E^2}{8(n+1)^2}$$

It must be fulfilled that

$$t^i f(x,t)\to 0 \quad \text{if} \quad t\to\infty \quad \forall i\epsilon N$$

If we apply the truncated expansion method to solve Equation (9) we obtain an approximation $p_n(x,t)$ for $f(x,t)=w_x(x,t)-w_t(x,t)$. Then we have an equation in first order partial derivatives of the form

$$w_x(x,t)-w_t(x,t)=p_n(x,t)$$

It is solved as in [28], we can prove that solving this equation is equivalent to solving the integral equation

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}u(m,z,x,t)w(x,t)(t-x)\,\mathrm{d}t\mathrm{d}x=\varphi_2(m,z)$$

where

$$\varphi_2(m,z)=\int_{a_2}^{b_2}\left[u(m,z,b_1,t)w(b_1,t)-u(m,n,a_1,t)w(a_1,t)\right]\mathrm{d}t+$$
$$+\int_{a_1}^{b_1}\left[u(m,z,x,b_2)w(x,b_2)-u(m,n,x,a_2)w(x,a_2)\right]\mathrm{d}x-\int_{a_1}^{b_1}\int_{a_2}^{b_2}up_n\mathrm{d}t\mathrm{d}x$$

and $u(m,z,x,t)=e^{-(m+1)(x+1)-(z+1)(t+1)}$. Again we take a base $\{\psi_{ij}^*(m,z)\}_{ij}$ of $L^2(E)$ and then the above equation can be transformed into a generalized moment problem

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} w(x,t)\,H_{ij}^*(x,t)\,\mathrm{d}t\mathrm{d}x=\mu_{ij}^*$$

where

$$H_{ij}^*(x,t)=\int_{a_1}^{b_1}\int_{a_2}^{b_2} u(m,n,x,t)(t-x)\psi_{ij}^*(m,z)\,\mathrm{d}z\mathrm{d}m$$

and

$$\mu_{ij}^*=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\varphi_2(m,z)\psi_{ij}^*(m,z)\,\mathrm{d}z\mathrm{d}m$$

Applying again the techniques of generalized moments problem we found an approximate solution for $w(x,t)$.

3. Numerical Examples

3.1. Example 1

We consider the equation

$$w_t=w_{xx}-\frac{1+e^{-x-t}\left(1+t^2\left(2+xe^x\right)\right)}{1+t^2}\quad \text{in}\quad E=(0,1)\times(0,\infty)$$

and conditions

$$w(0,t)=e^{-t};\quad w(1,t)=\left(e^{-1}+1\right)e^{-t};\quad w(x,0)=x+e^{-x}$$

The solution is $w(x,t)=e^{-t}\left(e^{-x}+x\right)$

First step: approximates $f(x,t)=w_x(x,t)-w_t(x,t)$

We take the base $\psi_i(m)=m^{i-1}e^{-m}\quad i=1,2,\cdots,5$

Accuracy is $\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left|f(x,t)-p_5(x,t)\right|^2\mathrm{d}t\mathrm{d}x\right)^{\frac{1}{2}}=0.409962$.

In **Figure 1(a)** the exact solution and the approximate solution are compared

Second step: approximates $w(x,t)$

We take the base $\psi_{ij}(m,z)=m^{i-1}z^{j-1}e^{-m-z}\quad i=1,2,3\quad j=1,2,3$

Accuracy is $\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left|w(x,t)-p_9(x,t)\right|^2\mathrm{d}t\mathrm{d}x\right)^{\frac{1}{2}}=0.131406$.

In **Figure 1(b)** the exact solution and the approximate solution are compared.

3.2. Example 2

We consider the equation

$$w_t=w_{xx}\quad \text{in}\quad E=(0,1)\times(0,\infty)$$

and conditions

$$w(0,t)=\mathrm{Exp}\left(-\frac{\pi^2}{4}t\right);\quad w(1,t)=0;\quad w(x,0)=\mathrm{Cos}\left(\frac{\pi}{2}x\right)$$

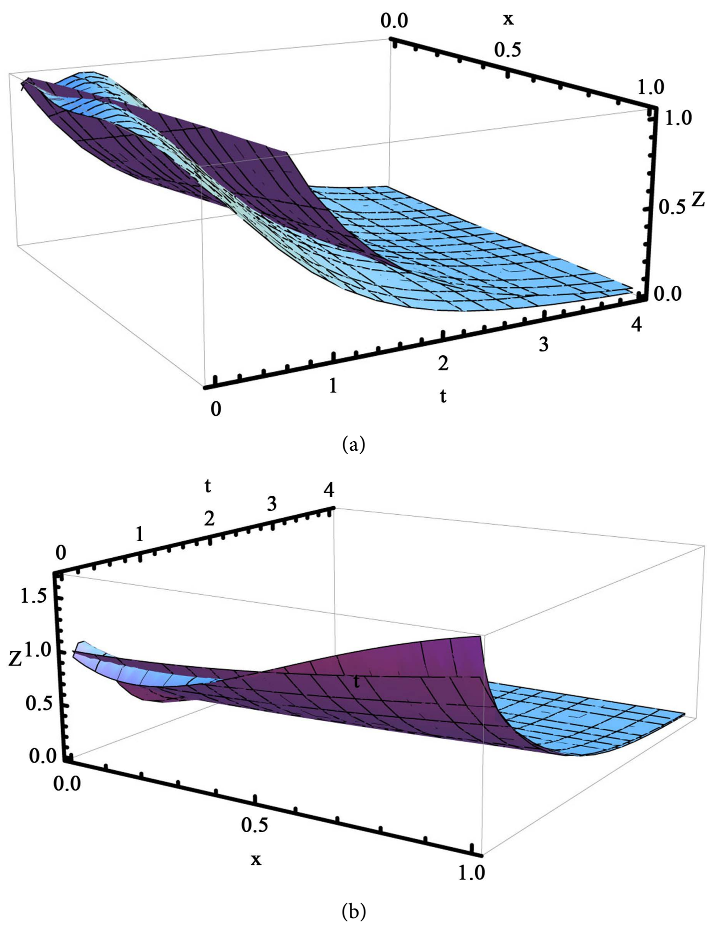


Figure 1. (a) $w_x(x,t) - w_t(x,t)$ and $p_5(x,t)$; (b) $w(x,t)$ and $p_9(x,t)$.

The solution is $w(x,t) = \mathrm{Exp}\left(-\frac{\pi^2}{4}t\right)\mathrm{Cos}\left(\frac{\pi}{2}x\right)$

First step: approximates $f(x,t) = w_x(x,t) - w_t(x,t)$

We take the base $\psi_i(m) = m^{i-1}e^{-m}$ $i = 1,2,\cdots,5$

Accuracy is $\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left|f(x,t) - p_5(x,t)\right|^2 \mathrm{d}t\mathrm{d}x\right)^{\frac{1}{2}} = 0.294629$.

In **Figure 2(a)** the exact solution and the approximate solution are compared

Second step: approximates $w(x,t)$

We take the base $\psi_{ij}(m,z) = m^{i-1}z^{j-1}e^{-m-z}$ $i = 1,2,3$ $j = 1,2,3$

Accuracy is $\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left|w(x,t) - p_9(x,t)\right|^2 \mathrm{d}t\mathrm{d}x\right)^{\frac{1}{2}} = 0.0673978$.

In **Figure 2(b)** the exact solution and the approximate solution are compared.

3.3. Example 3

We consider the equation

$$w_{xx} - w_t = p(t)w_x + e^{-x-3t}(5+t) \quad \text{in} \quad E = (0,1)\times(0,\infty)$$

where $p(t)$ is unknown. And conditions

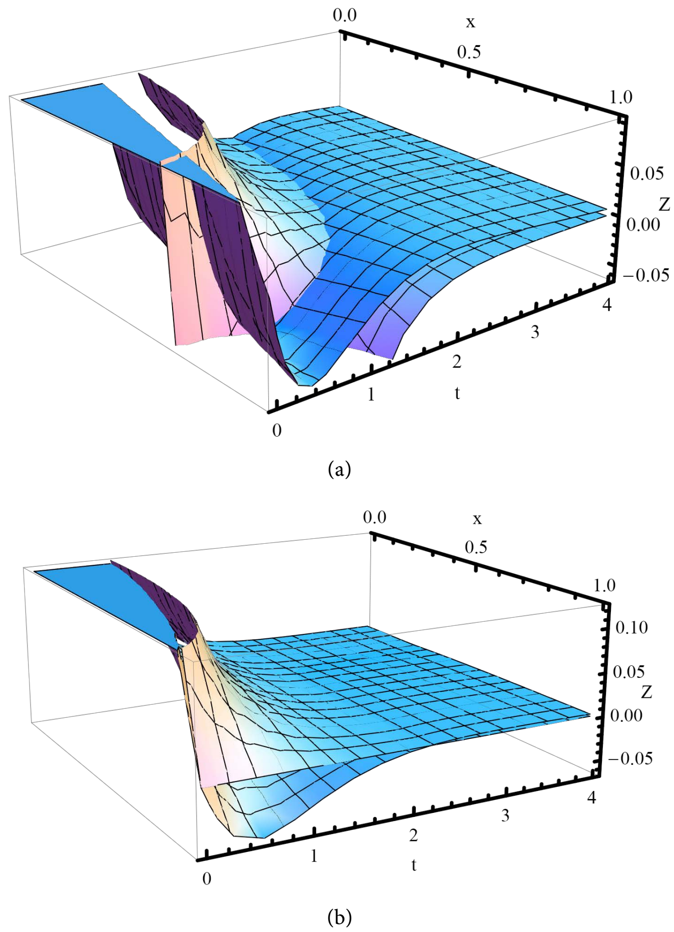


Figure 2. (a) $w_x(x,t)-w_t(x,t)$ and $p_5(x,t)$; (b) $w(x,t)$ and $p_9(x,t)$.

$$w(0,t)=e^{-3t};\quad w(1,t)=e^{-1-3t};\quad w(x,0)=e^{-x}$$

The solution is $w(x,t)=e^{-x-3t}$ if $p(t)=1+t$

First step: approximates $f(x,t)=w_x(x,t)-w_t(x,t)$

We take the base $\psi_i(m)=m^{i-1}e^{-m}$ $i=1,2,\cdots,5$

Accuracy is $\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left|f(x,t)-p_5(x,t)\right|^2 \mathrm{d}t\mathrm{d}x\right)^{\frac{1}{2}}=0.160504$.

In Figure 3(a) the exact solution and the approximate solution are compared

Second step: approximates $w(x,t)$

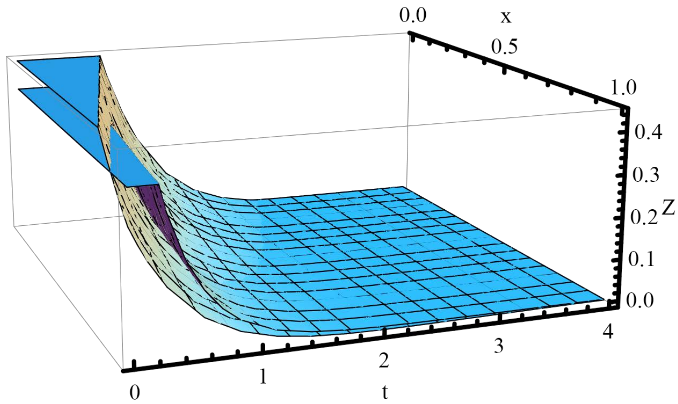
We take the base $\psi_{ij}(m,z)=m^{i-1}z^{j-1}e^{-m-z}$ $i=1,2,3$ $j=1,2,3$

Accuracy is $\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left|w(x,t)-p_9(x,t)\right|^2 \mathrm{d}t\mathrm{d}x=\right)^{\frac{1}{2}}=0.0388971$.

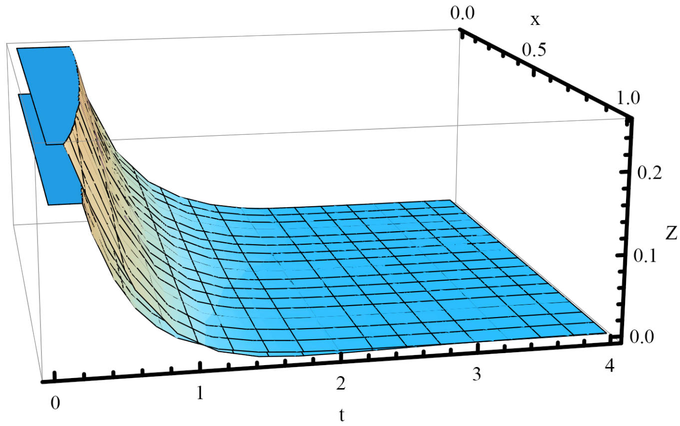
In Figure 3(b) the exact solution and the approximate solution are compared.

4. Conclusions

An equation in parabolic partial derivatives of the form $w_t-(w_x)_x=r(x,t)$ where the unknown function $w(x,t)$ is defined in $E=(a_1,b_1)\times(a_2,\infty)$ under the conditions



(a)



(b)

Figure 3. (a) $w_x(x,t)-w_t(x,t)$ and $p_5(x,t)$; (b) $w(x,t)$ and $p_9(x,t)$.

$$w(a_1,t)=k_1(t) \quad w(b_1,t)=k_2(t) \quad w(x,a_2)=h_1(t)$$

can be solved numerically by applying inverse moments problem techniques in two steps: first consider the integral equation

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} e^{-m(x+t)}\left(w_x(x,t)-w_t(x,t)\right)\mathrm{d}t\mathrm{d}x=\varphi_1(m)$$

and we can solve it numerically as an inverse moments problem, and get an approximate solution for $w_x(x,t)-w_t(x,t)$. Then in a second step we consider the integral equation

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} e^{-(m+1)(x+1)-(z+1)(t+1)}\left(w(x,t)(t-x)\right)\mathrm{d}t\mathrm{d}x=\varphi_2(m,z)$$

and again we can solve it numerically as an inverse moments problem, and get an approximate solution for $w(x,t)$. It is observed that the function $r(x,t)$ is not used in calculations, but it is implicitly considered in the boundary conditions.

In this way it would be possible to solve, for example, the problem of finding $w(x,t)$ that satisfies

$$w_t-(w_x)_x=p(t)w(x,t)+\Phi(x,t) \quad \text{in } E=(0,1)\times(0,\infty)$$

under the conditions

$$w(0,t)=k_1(t) \quad w(1,t)=k_2(t) \quad w(x,0)=h_1(t)$$

with unknown $p(t)$ and known $\Phi(x,t)$.

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