Differential Geometry and its Applications 28 (2010) 469–479

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Metric geometry in infinite dimensional Stiefel manifolds

Eduardo Chiumiento

Departamento de Matemática, FCE-UNLP, Calles 50 y 115, (1900) La Plata, Argentina

ARTICLE INFO

Article history:
Received 6 August 2009
Received in revised form 29 October 2009
Available online 23 December 2009
Communicated by T.S. Ratiu

MSC:
primary 22E65
secondary 47B10, 58B20

Keywords:
Partial isometry
Banach ideal
Finsler metric
Minimal curves

ABSTRACT

Let $\mathfrak{J}$ be a separable Banach ideal in the space of bounded operators acting in a Hilbert space $\mathcal{H}$ and $\mathcal{I}$ the set of partial isometries in $\mathcal{H}$. Fix $v \in \mathcal{I}$. In this paper we study metric properties of the $\mathfrak{J}$-Stiefel manifold associated to v, namely

$$\mathcal{S}t_{\mathfrak{J}}(v) = \{v_0 \in \mathcal{I}\colon\ v - v_0 \in \mathfrak{J},\ j(v_0^* v_0, v^* v) = 0\},$$

where $j(.,.)$ is the Fredholm index of a pair of projections. Let $\mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ be the Banach–Lie group of unitary operators which are perturbations of the identity by elements in $\mathfrak{J}$. Then $\mathcal{S}t_{\mathfrak{J}}(v)$ coincides with the orbit of v under the action of $\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ on $\mathcal{I}$ given by $(u, w) \cdot v_0 = u v_0 w^*$, $u, w \in \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ and $v_0 \in \mathcal{S}t_{\mathfrak{J}}(v)$. We endow $\mathcal{S}t_{\mathfrak{J}}(v)$ with a quotient Finsler metric by means of the Banach quotient norm of the Lie algebra of $\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ by the Lie algebra of the isotropy group. We give a characterization of the rectifiable distance induced by this metric. In fact, we show that the rectifiable distance coincides with the quotient distance of $\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ by the isotropy group. Hence this metric defines the quotient topology in $\mathcal{S}t_{\mathfrak{J}}(v)$.
The other results concern with minimal curves in $\mathfrak{J}$-Stiefel manifolds when the ideal $\mathfrak{J}$ is fixed as the compact operators in $\mathcal{H}$. The initial value problem is solved when the partial isometry v has finite rank. In addition, we use a length-reducing map into the Grassmannian to find some special partial isometries that can be joined with a curve of minimal length.

1. Introduction

In this paper we study the rectifiable distance and minimal curves in infinite dimensional Stiefel manifolds endowed with a quotient metric. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators acting in $\mathcal{H}$. We will denote by $\|.\|$ the spectral norm of operators. By a Banach ideal we mean a two-sided ideal $\mathfrak{J}$ of $\mathcal{B}(\mathcal{H})$ equipped with a norm $\|.\|_{\mathfrak{J}}$ satisfying $\|x\| \leqslant \|x\|_{\mathfrak{J}} = \|x^*\|_{\mathfrak{J}}$ and $\|axb\|_{\mathfrak{J}} \leqslant \|a\|\,\|x\|_{\mathfrak{J}}\|b\|$ whenever $a, b \in \mathcal{B}(\mathcal{H})$. In the sequel, $\mathfrak{J}$ stands for a separable Banach ideal.

Let $\mathcal{I}$ denote the set of partial isometries. Fix $v \in \mathcal{I}$. The *$\mathfrak{J}$-Stiefel manifold associated with v* is defined by

$$\mathcal{S}t_{\mathfrak{J}}(v) = \{v_0 \in \mathcal{I}\colon\ v - v_0 \in \mathfrak{J},\ j(v_0^* v_0, v^* v) = 0\},$$

where $j(.,.)$ is the index of a pair of orthogonal projections. The index of a pair of orthogonal projections (p, q) is the Fredholm index of $qp : p(\mathcal{H}) \longrightarrow q(\mathcal{H})$, when this operator is Fredholm (see for instance [5]). In a former article [7] the author has established the geometric facts of the $\mathfrak{J}$-Stiefel manifolds mentioned below.

E-mail address: eduardo@mate.unlp.edu.ar.

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2009.12.003

Denote by $\mathcal{U}(\mathcal{H})$ the group of unitary operators in $\mathcal{H}$, and $\mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ the group of unitaries which are perturbations of the identity by an operator in $\mathfrak{J}$, i.e.

$$\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) = \{u \in \mathcal{U}(\mathcal{H}): u - 1 \in \mathfrak{J}\}.$$

It is a real Banach–Lie group with the topology defined by the metric $(u_1, u_2) \mapsto \|u_1 - u_2\|_{\mathfrak{J}}$ (see [6]). We point out that $\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ acts transitively on $\mathcal{S}t_{\mathfrak{J}}(v)$. The action is giving by moving the initial and final subspace of the isometries, $(u, w) \cdot v_0 = uv_0w^*$, where $u, w \in \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$, $v_0 \in \mathcal{S}t_{\mathfrak{J}}(v)$. Also recall that the map

$$\pi_{v_0} : \mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \longrightarrow \mathcal{S}t_{\mathfrak{J}}(v), \qquad \pi_{v_0}((u, w)) = uv_0w^*$$

is a real analytic submersion. Moreover, $\mathcal{S}t_{\mathfrak{J}}(v)$ is a real analytic submanifold of $v + \mathfrak{J}$ and a homogeneous reductive space of $\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ (see [13]). Therefore the tangent space $(T\mathcal{S}t_{\mathfrak{J}}(v))_{v_0}$ at $v_0 \in \mathcal{S}t_{\mathfrak{J}}(v)$ can be identified with

$$\left(T\mathcal{S}t_{\mathfrak{J}}(v)\right)_{v_0} = \{xv_0 - v_0y: x, y \in \mathfrak{J}_{ah}\},$$

where $\mathfrak{J}_{ah}$ is the Lie algebra of $\mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ giving by the skew-adjoint elements of $\mathfrak{J}$. The isotropy group at $v_0 \in \mathcal{S}t_{\mathfrak{J}}(v)$ of the above action can be computed

$$G_{v_0} = \{(u, w) \in \mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H}): uv_0 = v_0w\}.$$

The Lie algebra of G_{v_0} is

$$\mathcal{G}_{v_0} = \{(a, b) \in \mathfrak{J}_{ah} \times \mathfrak{J}_{ah}: av_0 = v_0b\}.$$

By means of the quotient Banach norm of $(\mathfrak{J}_{ah} \times \mathfrak{J}_{ah})/\mathcal{G}_v$ one can define a Finsler metric on $\mathcal{S}t_{\mathfrak{J}}(v)$. Indeed, for $xv_0 - v_0y \in (T\mathcal{S}t_{\mathfrak{J}}(v))_{v_0}$,

$$\|xv_0 - v_0y\|_{v_0} = \inf\{\|(x+a, y+b)\|: (a, b) \in \mathcal{G}_{v_0}\}. \tag{1.1}$$

Here the norm of a pair is $\|(x+a, y+b)\| = \Phi(\|x+a\|_{\mathfrak{J}}, \|y+b\|_{\mathfrak{J}})$, where Φ is any symmetric norming function. A standard computation shows that this metric is invariant under the action. We can define the rectifiable distance induced by this metric in the usual fashion, i.e.

$$d(v_0, v_1) = \inf\left\{\int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)}\,dt: \gamma \subset \mathcal{S}t_{\mathfrak{J}}(v),\ \gamma(0) = v_0,\ \gamma(1) = v_1\right\}, \tag{1.2}$$

where the curves γ considered are piecewise smooth.

Metric geometry in homogeneous spaces in the setting of operator theory is an area of current research. The metric (1.1) was introduced in the remarkable work [9] of C. Durán, L. Mata Lorenzo and L. Recht, where they studied minimal curves with the quotient metric induced by the operator norm homogeneous spaces of the unitary group of C^*-algebras. When the quotient metric is induced by the p-norms, several interesting metric properties of abstract homogeneous spaces of the p-Schatten unitary groups were proved in [3]. On the other hand, we can cite the articles [4,11,14] and the references therein concerning geometrical and topological properties of partial isometries. In addition, we mention that background information on Finsler structures on Banach manifolds can be found in the book [18].

The contents of the paper are the following. In Section 2 we show that the rectifiable distance metricates the quotient topology of groups in $(\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H}))/G_v \cong \mathcal{S}t_{\mathfrak{J}}(v)$. This fact is proved by giving an alternative description of the rectifiable distance in terms of the metric distance of the quotient of groups. In Section 3 we fix the ideal $\mathcal{K}(\mathcal{H})$ of the compact operators. We focus on the study of minimal curves in the $\mathcal{K}(\mathcal{H})$-Stiefel manifold which we denote by $\mathcal{S}t_c(v)$. The initial value problem is solved when v is a partial isometry of finite rank. This means that for $v_0 \in \mathcal{S}t_c(v)$ and a tangent vector $xv_0 - v_0y$ given, there exists a curve δ in $\mathcal{S}t_c(v)$ satisfying $\delta(0) = v_0$, $\dot{\delta}(0) = xv_0 - v_0y$ and being of minimal length up to a critical value of t. Then we prove that $\mathcal{S}t_c(v)$ can be mapped into a product of Grassmannians with a length-reducing map. As a corollary of this simple fact some specials isometries in $\mathcal{S}t_c(v)$ can be joined by curves of minimal length.

2. The rectifiable distance in $\mathcal{S}t_{\mathfrak{J}}(v)$

In this section we give a characterization of the rectifiable distance in $\mathcal{S}t_{\mathfrak{J}}(v)$ as a quotient distance of groups. First we need to set some definitions and notations about $\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$. We endow $\mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ with the ambient Finsler metric which is defined by $\|(x, y)\| = \Phi(\|x\|_{\mathfrak{J}}, \|y\|_{\mathfrak{J}})$ for $(x, y) \in u_1\mathfrak{J}_{ah} \times u_2\mathfrak{J}_{ah} = (T(\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H})))_{(u_1,u_2)}$. The function Φ is a symmetric norming function in $\mathbb{R}^2$. This means that it is invariant under permutations, only depends on the absolute values of the coordinates and satisfies $\Phi(1, 0) = 1$. We measure the length of a piecewise C^1 curve $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t))$, $t \in [0, 1]$, as follows

$$L_{\mathfrak{I}}(\Gamma) = \int_0^1 \|(\dot\Gamma_1, \dot\Gamma_2)\|\, dt.$$

Therefore $\mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{I}}(\mathcal{H})$ has a rectifiable distance defined by

$$d_{\mathfrak{I}}\big((u_0, w_0), (u_1, w_1)\big) = \inf\big\{L_{\mathfrak{I}}(\Gamma):\ \Gamma \subset \mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{I}}(\mathcal{H}),\ \Gamma(0) = (u_0, w_0),\ \Gamma(1) = (u_1, w_1)\big\}.$$

On the other hand, the length of a C^1 curve $\gamma(t)$, $t \in [0,1]$, in $\mathcal{S}t_{\mathfrak{I}}(v)$ with the metric (1.1) is denoted by

$$L(\gamma) = \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}\, dt.$$

The rectifiable distance d is defined accordingly as Eq. (1.2) shows. Actually this provides a pseudo distance in general. However it is easy to show that this is in fact a distance in $\mathcal{S}t_{\mathfrak{I}}(v)$ (see [7]). Also $d_{\mathfrak{I}}$ is a distance in $\mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{I}}(\mathcal{H})$ by the estimates of Lemma 2.4.

The following result proved that the rectifiable metric in $\mathcal{S}t_{\mathfrak{I}}(v)$ can be approximated by lifting curves to the product group $\mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{I}}(\mathcal{H})$. It is adapted from [2] with the difference that we use any Banach norm and the assumption that the quotient metric is attained is dropped here.

Lemma 2.1. *Let $v_0, v_1 \in \mathcal{S}t_{\mathfrak{I}}(v)$. Then*

$$d(v_0, v_1) = \inf\big\{L_{\mathfrak{I}}(\Gamma):\ \Gamma \subset \mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{I}}(\mathcal{H}),\ \pi_{v_0}\big(\Gamma(0)\big) = v_0,\ \pi_{v_0}\big(\Gamma(1)\big) = v_1\big\},$$

where the curves Γ considered are continuous and piecewise C^1.

Proof. Consider any piecewise C^1 curve Γ in $\mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{I}}(\mathcal{H})$ satisfying $\pi_{v_0}(\Gamma(0)) = v_0$ and $\pi_{v_0}(\Gamma(1)) = v_1$. Let us point out that since the map

$$\pi_{v_0} : \mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \longrightarrow \mathcal{S}t_{\mathfrak{I}}(v), \qquad \pi_{v_0}\big((u, w)\big) = u v_0 w^*$$

is a real analytic submersion there exists such kind of curves. Then, note that the above map reduces length of curves with the previously defined metrics on each space. Since the action is isometric, it suffices to check that the differential map at the identity

$$\delta_{v_0} : \mathfrak{I}_{ah} \times \mathfrak{I}_{ah} \longrightarrow \big(T\mathcal{S}t_{\mathfrak{I}}(v)\big)_{v_0}, \qquad \delta_{v_0}\big((x, y)\big) = x v_0 - v_0 y$$

is contractive. But this follows trivially by the definition of the quotient metric in $\mathcal{S}t_{\mathfrak{I}}(v)$. Hence we have $d(v_0, v_1) \leqslant L(\pi_{v_0}(\Gamma)) \leqslant L_{\mathfrak{I}}(\Gamma)$.

To finish, we must prove that given γ in $\mathcal{S}t_{\mathfrak{I}}(v)$ one can approximate $L(\gamma)$ with lengths of curves in $\mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{I}}(\mathcal{H})$ joining the fibres of v_0 and v_1. Fix $\epsilon > 0$. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a uniform partition of $[0,1]$ ($\Delta t_i = t_i - t_{i-1} = 1/n$) such that the following hold:

1. $\|\dot\gamma(s) - \dot\gamma(s')\|_{\mathfrak{I}} < \epsilon/4$ if s, s' lie in the same interval $[t_{i-1}, t_i]$.
2. $|L(\gamma) - \sum_{i=0}^{n-1} \|\dot\gamma(t_i)\|_{\gamma(t_i)} \Delta t_i| < \epsilon/2$.

On the other hand, for each $i = 0, \ldots, n-1$, there exist $x_i, y_i \in \mathfrak{I}_{ah}$ such that $\delta_{\gamma(t_i)}((x_i, y_i)) = \dot\gamma(t_i)$ and $\|(x_i, y_i)\| \leqslant \|\dot\gamma(t_i)\|_{\gamma(t_i)} + \epsilon/2$.

Consider the following curve Γ in $\mathcal{U}_{\mathfrak{I}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{I}}(\mathcal{H})$:

$$\Gamma(t) = \begin{cases} (e^{t x_0}, e^{t y_0}), & t \in [0, t_1), \\ (e^{(t-t_1)x_1} e^{t_1 x_0}, e^{(t-t_1)y_1} e^{t_1 y_0}), & t \in [t_1, t_2), \\ \vdots & \vdots \\ (e^{(t-t_{n-1})x_{n-1}} \ldots e^{(t_2-t_1)x_1} e^{t_1 x_0}, e^{(t-t_{n-1})y_{n-1}} \ldots e^{(t_2-t_1)y_1} e^{t_1 y_0}), & t \in [t_{n-1}, 1]. \end{cases}$$

Then Γ is continuous and piecewise smooth, $\Gamma(0) = (1, 1)$ and

$$L_{\mathfrak{I}}(\Gamma) = \sum_{i=0}^{n-1} \big\|(x_i, y_i)\big\| \Delta t_i \leqslant \sum_{i=0}^{n-1} \big\|\dot\gamma(t_i)\big\|_{\gamma(t_i)} \Delta t_i + \epsilon/2 \leqslant L(\gamma) + \epsilon.$$

We claim that $\pi_{v_0}(\Gamma(1))$ lies close to v_0. Note that if we denote by $\alpha(t) = \pi_{v_0}(e^{tx_0}, e^{ty_0}) - \gamma(t)$, then $\alpha(0) = 0$, and using the mean value theorem in Banach spaces,

$$\left\|\pi_{v_0}\left(e^{t_1x_0}, e^{t_1y_0}\right) - \gamma(t_1)\right\|_{\mathfrak{J}} = \left\|\alpha(t_1) - \alpha(0)\right\|_{\mathfrak{J}} \leqslant \left\|\dot{\alpha}(s_1)\right\|_{\mathfrak{J}} \Delta t_1,$$

for some $s_1 \in [0, t_1]$. Explicitly,

$$\left\|\pi_{v_0}\left(e^{t_1x_0}, e^{t_1y_0}\right) - \gamma(t_1)\right\|_{\mathfrak{J}} \leqslant \left\|e^{s_1x_0}\delta_{v_0}\left((x_0, y_0)\right)e^{-s_1y_0} - \dot{\gamma}(s_1)\right\|_{\mathfrak{J}} \Delta t_1.$$

Note that $\delta_{v_0}((x_0, y_0)) = \dot{\gamma}(0)$, and that

$$\left\|e^{s_1x_0}\dot{\gamma}(0)e^{-s_1y_0} - \dot{\gamma}(s_1)\right\|_{\mathfrak{J}} \leqslant \left\|e^{s_1x_0}\dot{\gamma}(0)e^{-s_1y_0} - \dot{\gamma}(0)\right\|_{\mathfrak{J}} + \left\|\dot{\gamma}(0) - \dot{\gamma}(s_1)\right\|_{\mathfrak{J}}.$$

The second summand is bounded by $\epsilon/4$. The first summand can be bounded as follows

$$\begin{aligned}\left\|e^{s_1x_0}\dot{\gamma}(0)e^{-s_1y_0} - \dot{\gamma}(0)\right\|_{\mathfrak{J}} &= \left\|\left(e^{s_1x_0} - 1\right)\dot{\gamma}(0) - \dot{\gamma}(0)\left(e^{-s_1y_0} - 1\right)\right\|_{\mathfrak{J}} \\ &\leqslant \left\|\dot{\gamma}(0)\right\|_{\mathfrak{J}}\left\|\left(e^{s_1x_0} - 1, e^{-s_1y_0} - 1\right)\right\| \leqslant M\Delta t_1,\end{aligned}$$

where $M := \max_{t\in[0,1]} \|\dot{\gamma}(t)\|_{\mathfrak{J}}$. Thus,

$$\left\|\pi_{v_0}\left(e^{t_1x_0}, e^{t_1y_0}\right) - \gamma(t_1)\right\|_{\mathfrak{J}} \leqslant (M\Delta t_1 + \epsilon/4)\Delta t_1.$$

Next we estimate $\|\pi_{v_0}((e^{(t_2-t_1)x_1}e^{t_1x_0}, e^{(t_2-t_1)y_1}e^{t_1y_0})) - \gamma(t_2)\|_{\mathfrak{J}}$ which is less or equal than

$$\left\|e^{(t_2-t_1)x_1}e^{t_1x_0}v_0e^{-t_1y_0}e^{-(t_2-t_1)y_1} - e^{(t_2-t_1)x_1}\gamma(t_1)e^{-(t_2-t_1)y_1}\right\|_{\mathfrak{J}} + \left\|e^{(t_2-t_1)x_1}\gamma(t_1)e^{-(t_2-t_1)y_1} - \gamma(t_2)\right\|_{\mathfrak{J}}.$$

The first summand can be bounded by

$$\left\|e^{(t_2-t_1)x_1}\left(e^{t_1x_0}v_0e^{-t_1y_0} - \gamma(t_1)\right)e^{-(t_2-t_1)y_1}\right\|_{\mathfrak{J}} = \left\|e^{t_1x_0}v_0e^{-t_1y_0} - \gamma(t_1)\right\|_{\mathfrak{J}} \leqslant (M\Delta t_1 + \epsilon/4)\Delta t_1.$$

The second difference can be treated analogously as the first difference above,

$$\left\|e^{(t_2-t_1)x_1}\gamma(t_1)e^{-(t_2-t_1)y_1} - \gamma(t_2)\right\|_{\mathfrak{J}} \leqslant (M\Delta t_2 + \epsilon/4)\Delta t_2 = (M/n + \epsilon/4)/n.$$

Hence we obtain

$$\left\|\pi_{v_0}\left(\Gamma(t_2)\right) - \gamma(t_2)\right\|_{\mathfrak{J}} \leqslant 2(M/n + \epsilon/4)/n.$$

Then by induction we have that

$$\left\|\pi_{v_0}\left(\Gamma(t_{n-1})\right) - v_1\right\|_{\mathfrak{J}} \leqslant M/n + \epsilon/4 < \epsilon/2,$$

choosing n big enough. The proof follows since the map π_{v_0} has local continuous cross sections, then one can connect $\Gamma(t_{n-1})$ with the fibre of v_1 by a curve of arbitrary small length. □

We shall need the following result about metric groups from Takesaki's book [17, p. 109].

Lemma 2.2. *Let H be a metrizable topological group, and G be a closed subgroup. If d is a complete distance function on H inducing the topology of H, and if d is invariant under right translation by G, i.e. $d(xg, yg) = d(x, y)$ for any $x, y \in H$ and $g \in G$, then the left coset space $H/G = \{xG\colon x \in H\}$ is a complete metric space under the metric $\dot{d}$ given by*

$$\dot{d}(xG, yG) = \inf\left\{d(xg_1, yg_2)\colon g_1, g_2 \in G\right\}.$$

Actually, the distance $\dot{d}$ metricates the quotient topology of groups. Let us observe how Lemma 2.2 applies to our situation. We shall take $G = G_v$, $H = \mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$, and $d_{\mathfrak{J}}$ the rectifiable distance in $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$.

Remark 2.3. Te exponential map $\exp : \mathfrak{J}_{ah} \longrightarrow \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$, $\exp(z) = e^z$ is surjective. Moreover, we have that

$$\exp\left(\left\{z \in \mathfrak{J}_{ah}\colon \|z\| \leqslant \pi\right\}\right) = \mathcal{U}_{\mathfrak{J}}(\mathcal{H}).$$

Briefly we include an argument to prove our affirmation. Let $u \in \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$, then there is a well-known fact that using the Borel functional calculus is possible to find $x = x^*$ such that $\|x\| \leqslant \pi$ and $e^{ix} = u$. Any two-sided ideal is contained in the ideal $\mathcal{K}(\mathcal{H})$ of compact operators, then $e^{ix} = 1 + a$, $a \in \mathfrak{J} \subset \mathcal{K}(\mathcal{H})$. Note that $x \in \mathcal{K}(\mathcal{H})$ because $ix = \log(1 + a) = \sum_{j=1}^{\infty} \frac{a^j}{j}$, so x is the norm limit of compact operators. Therefore the spectrum of x consists of countable many nonzero eigenvalues of finite multiplicity and zero.

On the other hand, we have the elementary estimate,

$$\left(1-\frac{\pi^2}{12}\right)^{1/2}|t| \leqslant \left|e^{it}-1\right|, \tag{2.1}$$

for $t \in [-\pi, \pi]$. Since the functional calculus is positive, we have

$$\left(1-\frac{\pi^2}{12}\right)^{1/2}|x| \leqslant \left|e^{ix}-1\right|.$$

Therefore, if $s_j(.)$ denotes the singular values of an operator, we obtain the corresponding inequality for the singular values $(1-\frac{\pi^2}{12})^{1/2}s_j(x) \leqslant s_j(e^{ix}-1)$, for $j \in \mathbb{N}$. By the dominance property (see [10, p. 82]), and the fact that $e^{ix}-1 \in \mathfrak{I}$, we can conclude $x \in \mathfrak{I}$.

Lemma 2.4. *Let $u_0, u_1, w_0, w_1 \in \mathcal{U}_\mathfrak{I}(\mathcal{H})$, then*

$$\left(1-\frac{\pi^2}{12}\right)^{1/2} d_\mathfrak{I}\big((u_0,u_1),(w_0,w_1)\big) \leqslant \Phi\big(\|u_0-w_0\|_\mathfrak{I}, \|u_1-w_1\|_\mathfrak{I}\big) \leqslant d_\mathfrak{I}\big((u_0,u_1),(w_0,w_1)\big).$$

In particular, $(\mathcal{U}_\mathfrak{I}(\mathcal{H}) \times \mathcal{U}_\mathfrak{I}(\mathcal{H}), d_\mathfrak{I})$ is a complete metric space and G_v is a $d_\mathfrak{I}$-closed subgroup.

Proof. We can suppose that $u_0 = u_1 = 1$ since multiplication is isometric for each metric. Given $\epsilon > 0$, there exists $\Gamma = (\Gamma_0, \Gamma_1) \subset \mathcal{U}_\mathfrak{I}(\mathcal{H}) \times \mathcal{U}_\mathfrak{I}(\mathcal{H})$ such that $\Gamma(0) = (1,1)$ and $\Gamma(1) = (w_0, w_1)$ and

$$L_\mathfrak{I}(\Gamma) < d_\mathfrak{I}\big((1,1),(w_0,w_1)\big) + \epsilon.$$

Then, since straight are shortest curves in any vectorial space, we have

$$\Phi\big(\|1-w_0\|_\mathfrak{I}, \|1-w_1\|_\mathfrak{I}\big) \leqslant \int_0^1 \Phi\big(\|\dot{\Gamma}_0\|_\mathfrak{I}, \|\dot{\Gamma}_1\|_\mathfrak{I}\big)\,dt = L_\mathfrak{I}(\Gamma) < d_\mathfrak{I}\big((1,1),(w_0,w_1)\big) + \epsilon.$$

The inequality follows since ϵ is arbitrary. In order to prove the reversed inequality, consider $x_0, x_1 \in \mathfrak{I}_{ah}$ with $\|x_j\| \leqslant \pi$ satisfying $e^{x_0} = w_0$ and $e^{x_1} = w_1$. Note that this is possible by Remark 2.3. The curve $\Gamma(t) = (e^{tx_0}, e^{tx_1})$, $t \in [0,1]$, joins $(1,1)$ and (w_0, w_1). Then,

$$d_\mathfrak{I}\big((1,1),(w_0,w_1)\big) \leqslant L_\mathfrak{I}(\Gamma) = \Phi\big(\|x_0\|_\mathfrak{I}, \|x_1\|_\mathfrak{I}\big).$$

Now applying the estimate in (2.1), and passing to the corresponding inequality of the singular values we have

$$d_\mathfrak{I}\big((1,1),(w_0,w_1)\big) \leqslant \Phi\big(\|x_0\|_\mathfrak{I}, \|x_1\|_\mathfrak{I}\big) \leqslant \left(1-\frac{\pi^2}{12}\right)^{-1/2} \Phi\big(\|1-w_0\|_\mathfrak{I}, \|1-w_1\|_\mathfrak{I}\big),$$

which gives the desired inequality.

The completeness of $(\mathcal{U}_\mathfrak{I}(\mathcal{H}) \times \mathcal{U}_\mathfrak{I}(\mathcal{H}), d_\mathfrak{I})$ follows easy from the estimates. In fact, if $(u_n, w_n)_n$ is a Cauchy sequence with the distance $d_\mathfrak{I}$, it is also a Cauchy sequence with $\Phi(\|.\|_\mathfrak{I}, \|.\|_\mathfrak{I})$. Since $\mathcal{U}_\mathfrak{I}(\mathcal{H})$ is complete with the norm $\|.\|_\mathfrak{I}$, then there exist $u_0, w_0 \in \mathcal{U}_\mathfrak{I}(\mathcal{H})$ such that $\|u_n - u_0\|_\mathfrak{I} \to 0$ and $\|w_n - w_0\|_\mathfrak{I} \to 0$. It is apparent from the estimates above that (u_0, w_0) is the limit of $(u_n, w_n)_n$ with the distance $d_\mathfrak{I}$.

Finally, the fact that the isotropy group G_v is $d_\mathfrak{I}$-closed follows from the estimates and from G_v being a closed subgroup in the ideal norm. □

We give the main result of this section. Recall that the completeness of the metric space $(\mathcal{S}t_\mathfrak{I}(v), d)$ was proved in [7] by different methods. Also it is worthwhile noting that a similar statement was proved in [1] for homogeneous spaces in finite von Neumann algebras with the p-norm induced by the trace.

Theorem 2.5. *Let v be a partial isometry, $u_0, w_0, u_1, w_1 \in \mathcal{U}_\mathfrak{I}(\mathcal{H})$, and let*

$$\dot{d}_\mathfrak{I}\big(u_0 v w_0^*, u_1 v w_1^*\big) = \inf\big\{d_\mathfrak{I}\big((u_0,w_0),(u_1 u, w_1 w)\big):\ (u,w) \in G_v\big\}.$$

Then $\dot{d}_\mathfrak{I} = d$, where d is the rectifiable distance in $\mathcal{S}t_\mathfrak{I}(v)$. In particular, $(\mathcal{S}t_\mathfrak{I}(v), d)$ is a complete metric space and d metricates the quotient topology.

Proof. The quotient distance $\dot{d}_{\mathfrak{J}}$ is well defined because G_v is $d_{\mathfrak{J}}$-closed in $\mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$. Moreover, since multiplying by unitaries is isometric, it can be computed as

$$\dot{d}_{\mathfrak{J}}\big(u_0 v w_0^*, u_1 v w_1^*\big) = \inf\big\{d_{\mathfrak{J}}\big((u_0, w_0), (u_1 u, w_1 w)\big): (u, w) \in G_v\big\}.$$

In order to prove the equality between the distances fix $\epsilon > 0$. By Lemma 2.1 there exists a curve Γ in $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ satisfying

1. $\Gamma(0) = (u_0, w_0)$, $\Gamma(1) = (u_1 u, w_1 w)$, with $(u, w) \in G_v$.
2. $L_{\mathfrak{J}}(\Gamma) < d(u_0 v w_0^*, u_1 v w_1^*) + \epsilon$.

Therefore,

$$\dot{d}_{\mathfrak{J}}\big(u_0 v w_0^*, u_1 v w_1^*\big) \leqslant d_{\mathfrak{J}}\big((u_0, w_0), (u_1 u, w_1 w)\big) \leqslant L_{\mathfrak{J}}(\Gamma) < d\big(u_0 v w_0^*, u_1 v w_1^*\big) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have one inequality.

To prove the reversed inequality note that for any $\epsilon > 0$, there exists $(u, w) \in G_v$ satisfying $d_{\mathfrak{J}}((u_0, w_0), (u_1 u, w_1 w)) < \dot{d}_{\mathfrak{J}}(u_0 v w_0^*, u_1 v w_1^*) + \epsilon$. Then there is a curve $\Gamma \subset \mathcal{U}_{\mathfrak{J}}(\mathcal{H}) \times \mathcal{U}_{\mathfrak{J}}(\mathcal{H})$ such that $\Gamma(0) = (u_0, w_0)$, $\Gamma(1) = (u_1 u, w_1 w)$ and $L_{\mathfrak{J}}(\Gamma) < d_{\mathfrak{J}}((u_0, w_0), (u_1 u, w_1 w)) + \epsilon$. Therefore we have

$$d\big(u_0 v w_0^*, u_1 v w_1^*\big) \leqslant L_{\mathfrak{J}}(\Gamma) < d_{\mathfrak{J}}\big((u_0, w_0), (u_1 u, w_1 w)\big) + \epsilon < \dot{d}_{\infty}\big(u_0 v w_0^*, u_1 v w_1^*\big) + 2\epsilon.$$

Hence the equality $\dot{d}_{\mathfrak{J}} = d$ holds. The completeness of $(\mathcal{S}t_{\mathfrak{J}}(v), d)$ and the fact that d metricates the quotient topology are consequences of Lemma 2.2. □

3. Minimal curves in the $\mathcal{K}(\mathcal{H})$-Stiefel manifold

The problem of finding minimal curves in the $\mathfrak{J}$-Stiefel manifold clearly depends on the norm of the ideal $\mathfrak{J}$. The initial value problem was solved in [3] in the general setting of homogeneous spaces of the p-Schatten (p even integer) unitary groups. Since the Stiefel manifolds associated to the p-Schatten unitary groups fit in the context of those homogeneous spaces, we have that the initial value problem is already understood. Among of all the possible problems concerning minimal curves with the different available norms that remain unsolved, in this section we are interested in minimal curves in the $\mathcal{K}(\mathcal{H})$-Stiefel manifold, where $\mathcal{K}(\mathcal{H})$ is the ideal of compact operators. We denote the $\mathcal{K}(\mathcal{H})$-Stiefel manifold associated to a partial isometry v by $\mathcal{S}t_c(v)$. Recall from the Introduction that

$$\mathcal{S}t_c(v) = \big\{v_0 \in \mathcal{I}:\ v - v_0 \in \mathcal{K}(\mathcal{H}),\ j\big(v_0^* v_0, v^* v\big) = 0\big\}.$$

The isotropy group at $v_0 \in \mathcal{S}t_c(v)$ and its Lie algebra $\mathcal{G}_{v_0}$ are computed in the Introduction for any Banach ideal, in particular for the ideal of compact operators. The quotient Finsler metric of a tangent vector $x v_0 - v_0 y \in (T\mathcal{S}t_c(v))_{v_0}$ is given by

$$\|x v_0 - v_0 y\|_{v_0} = \inf\big\{\big\|(x + a, y + b)\big\|:\ a v_0 = v_0 b,\ a, b \in \mathcal{K}(\mathcal{H})_{ah}\big\}.$$

Here we take as product norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$, where $\|.\|$ is the operator norm. Recall that $\mathcal{S}t_c(v)$ is a real analytic submanifold of $v + \mathcal{K}(\mathcal{H})$ and a homogeneous space of the product of the unitary Fredholm group, which is denoted by

$$\mathcal{U}_c(\mathcal{H}) = \big\{u \in \mathcal{U}(\mathcal{H}):\ u - 1 \in \mathcal{K}(\mathcal{H})\big\}.$$

We refer the reader to [2] for the metric properties of this group. The length of a curve $\Gamma(t)$, $t \in [0, 1]$, in $\mathcal{U}_c(\mathcal{H})$ is measured with the Finsler metric given by the operator norm as follows

$$L_{\infty}(\Gamma) = \int_0^1 \big\|\dot{\Gamma}(t)\big\|\, dt.$$

The curves $\Gamma(t) = u e^{tz}$, where $u \in \mathcal{U}_c(\mathcal{H})$ and $z \in \mathcal{K}(\mathcal{H})_{ah}$ such that $\|z\| \leqslant \pi$ are geodesics of minimal length along their paths. We put in $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ the product metric induced by the product operator norm: $\|(x_0, x_1)\| = \max\{\|x_0\|, \|x_1\|\}$, for $(x_0, x_1) \in u_0\mathcal{K}(\mathcal{H})_{ah} \times u_1\mathcal{K}(\mathcal{H})_{ah}$. The length functional in $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ of a curve $\Gamma = (\Gamma_1, \Gamma_2)$ is also denoted by $L_{\infty}(\Gamma)$ since by the context no confusion will arise with the length functional defined in $\mathcal{U}_c(\mathcal{H})$. Then we note that if Γ_1, Γ_2 are minimal geodesics in $\mathcal{U}_c(\mathcal{H})$, it is apparent that $\Gamma = (\Gamma_1, \Gamma_2)$ is a geodesic of minimal length in $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$, when one measures lengths with the product metric. The corresponding rectifiable distances in $\mathcal{U}_c(\mathcal{H})$ and $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ are both denoted by d_{∞}.

3.1. The initial value problem

In [9] the initial value problem for homogeneous spaces of the unitary group with the quotient metric is solved in the category of unital von Neumann algebras. In our case, we cannot apply the same techniques since $\mathcal{G}_v$ does not consist in general of skew-adjoint operators of a unital von Neumann algebra. The set of compact pairs (a, b) such that $av = vb$ is neither self-adjoint nor closed in the weak operator topology. However, we can adapt the convexity argument given in [2] to find minimal curves when the partial isometry v has finite rank.

Let v be a partial isometry and $v_0 \in \mathcal{S}t_c(v)$. Any pair $(x_1, y_1) \in \mathcal{K}(\mathcal{H})_{ah} \times \mathcal{K}(\mathcal{H})_{ah}$, such that $\|(x_1, y_1)\| = \|xv_0 - v_0 y\|_{v_0}$ is called a *minimal lifting* for the tangent vector $xv_0 - v_0 y$. We will show that minimal liftings are relevant because they give minimal curves. The next result proves the existence of minimal liftings when v has finite rank.

Proposition 3.1. *Let $v_0 \in \mathcal{S}t_c(v)$ with v a partial isometry of finite rank. Let $x, y \in \mathcal{K}(\mathcal{H})_{ah}$, then there exists $(a, b) \in \mathcal{G}_{v_0}$ satisfying $\|xv_0 - v_0 y\|_{v_0} = \|(x + a, y + b)\|$.*

Proof. Since the action of $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ is isometric we can assume that $v = v_0$. We can argue as in Theorem 6.1 in [9] to find a sequence $((a_n, b_n))_n$ in $\mathcal{G}_v$ such that $(a_n, b_n) \to (a, b)$ in the weak operator topology and $\|(x+a, y+b)\| = \|xv - vy\|_v$. Note that $a, b \in \mathcal{B}(\mathcal{H})_{ah}$ and $av = vb$, but a, b may fail to be compact operators.

We denote the final projection of v by $p = vv^*$ and $q = v^*v$ the initial projection of v. In order to obtain compact operators attaining the quotient norm, note that $av = vb$ if and only if $ap = pa$, $bq = qb$ and $qbq = v^*av$. Therefore the pair (a, b) can be written as follows

$$a = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}_p, \qquad b = \begin{pmatrix} v^* a_{11} v & 0 \\ 0 & b_{22} \end{pmatrix}_q.$$

Here the subscripts $p = vv^*$ and $q = v^*v$ indicate that the matrices are regarded with respect to these projections and will be omitted from now on.

Recall Krein's solution to the extension problem for a self-adjoint operator (see [12]): Given an incomplete 2×2 self-adjoint block operator matrix of the form

$$\begin{pmatrix} X & Y \\ Y^* & ? \end{pmatrix}$$

find a self-adjoint operator Z in order that the complete matrix has minimal norm. Krein proved that there is always a solution, and that it may not be unique. More recently, Davis, Kahan and Weinberger [8] gave explicit formulas for Z. In particular, they showed that if the incomplete matrix has compact blocks, then there exists a compact solution Z.

Since v has finite rank, the operator a_{11} also has finite rank. Therefore, according to the extension problem, we can add a compact antihermitic operator $a'_{22} : p(\mathcal{H})^\perp \longrightarrow p(\mathcal{H})^\perp$ such that

$$\left\| \begin{pmatrix} x_{11} + a_{11} & x_{12} \\ -x_{12}^* & x_{22} + a'_{22} \end{pmatrix} \right\| \leqslant \left\| \begin{pmatrix} x_{11} + a_{11} & x_{12} \\ -x_{12}^* & x_{22} + a_{22} \end{pmatrix} \right\|.$$

We repeat this argument to find a compact antihermitic operator $b'_{22} : q(\mathcal{H})^\perp \longrightarrow q(\mathcal{H})^\perp$ satisfying

$$\left\| \begin{pmatrix} y_{11} + v^* a_{11} v & y_{12} \\ -y_{12}^* & y_{22} + b'_{22} \end{pmatrix} \right\| \leqslant \left\| \begin{pmatrix} y_{11} + v^* a_{11} v & y_{12} \\ -y_{12}^* & y_{22} + b_{22} \end{pmatrix} \right\|.$$

Let us set

$$a' = \begin{pmatrix} a_{11} & 0 \\ 0 & a'_{22} \end{pmatrix}, \qquad b' = \begin{pmatrix} v^* a_{11} v & 0 \\ 0 & b'_{22} \end{pmatrix}.$$

Then it follows that

$$\left\| (x + a', y + b') \right\| \leqslant \left\| (x + a, y + b) \right\| = \|xv - vy\|_v.$$

Hence, we finally obtain $\|(x + a', y + b')\| = \|xv - vy\|_v$, with $(a', b') \in \mathcal{G}_v$. $\square$

Our solution to the initial value problem relies on the following result about the convexity of the rectifiable distance in $\mathcal{U}_c(\mathcal{H})$.

Lemma 3.2. *(Theorem 2.7 in [2].) Let $u \in \mathcal{U}_c(\mathcal{H})$, $\beta : [0, 1] \longrightarrow \mathcal{U}_c(\mathcal{H})$ a geodesic such that $d_\infty(u, \beta) < \pi/2$. Then $g(s) = d_\infty(u, \beta(s))$, $s \in [0, 1]$ is a convex function.*

We shall use it with a minor change. We need to state a version for $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$.

Lemma 3.3. *Let $u, w \in \mathcal{U}_c(\mathcal{H})$, $\beta : [0,1] \longrightarrow \mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ a geodesic with $d_\infty((u,w),\beta) < \pi/2$. Then $g(s) = d_\infty((u,w),\beta(s))$, $s \in [0,1]$ is a convex function.*

Proof. We know the minimal curves of $\mathcal{U}_c(\mathcal{H})$, then we can construct $\beta = (\beta_1, \beta_2)$ a geodesic of minimal length in $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$. It also has minimal length in $\mathcal{U}_c(\mathcal{H} \oplus \mathcal{H})$, namely the unitary Fredholm group of $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Therefore given $(u_i, w_i) \in \mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$, $i = 1, 2$, we have that the rectifiable distance $d_\infty((u_1, w_1), (u_2, w_2))$ in $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ equals the rectifiable distance $d_\infty((u_1, w_1), (u_2, w_2))$ in $\mathcal{U}_c(\mathcal{H} \oplus \mathcal{H})$. Hence our lemma follows applying Lemma 3.2 to $\mathcal{U}_c(\mathcal{H} \oplus \mathcal{H})$. □

Now we state the main result of this section.

Theorem 3.4. *Let v be a partial isometry with finite rank, $v_0 \in \mathcal{S}t_c(v)$ and $xv_0 - v_0 y \in (T\mathcal{S}t_c(v))_{v_0}$ such that $\|xv_0 - v_0 y\|_{v_0} \leqslant \pi/2$. If (z_1, z_2) is a minimal lifting of $xv_0 - v_0 y$, then the curve $\delta(t) = e^{tz_1} v e^{-tz_2}$ has minimal length up to $|t| \leqslant 1$.*

Proof. Clearly we may assume $v_0 = v$. By Lemma 2.1 it suffices to compare the lengths of $\Delta(t) = (e^{tz_1}, e^{tz_2})$ and Γ, where Γ is a piecewise smooth curve in $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ joining $(1,1)$ and a unitary in the fibre $\delta(1)$. Observe that Δ lifts δ and satisfies

$$L_\infty(\Gamma) = L(\delta) = \|(z_1, z_2)\| < \pi/2.$$

If $L_\infty(\Gamma) \geqslant \pi/2$ there is nothing to prove. Otherwise, we have $\Gamma(1) = (e^{a_1}, e^{a_2})$, with $a_1, a_2 \in \mathcal{K}(\mathcal{H})_{ah}$ and $\|(a_1, a_2)\| < \pi/2$. Note that Γ and Δ may have different endpoints, however they satisfy

$$e^{a_1} v e^{-a_2} = e^{z_1} v e^{-z_2}.$$

Therefore, we obtain $e^{a_i} = e^{z_i} e^{b_i}$, $i = 1, 2$, where $(b_1, b_2) \in \mathcal{G}_v$. Since we suppose $\|(a_1, a_2)\| < \pi/2$ and $\|(z_1, z_2)\| < \pi/2$, it is apparent that $\|(b_1, b_2)\| \leqslant \pi$. Hence the curve $\beta(t) = (e^{z_1} e^{tb_1}, e^{z_2} e^{tb_2})$ is a geodesic of minimal length joining (e^{z_1}, e^{z_2}) and (e^{a_1}, e^{a_2}). Consider the following function

$$f(t) = d_\infty\big((1,1), \beta(t)\big) = \big\|\big(\log(e^{z_1} e^{tb_1}), \log(e^{z_2} e^{tb_2})\big)\big\|, \quad t \in [-1, 1].$$

Claim. *f has a minimum at $t = 0$.*

Since we now that f is convex by Lemma 3.3, it suffices to analyze the lateral derivatives at this point. We may suppose $\|(z_1, z_2)\| = \|z_1\|$. By continuity we have $\|\log(e^{z_1} e^{tb_1})\| \geqslant \|\log(e^{z_2} e^{tb_2})\|$ for t small. Therefore to compute the right derivative $\partial^+ f(0)$ of f at $t = 0$ it suffices to consider

$$\lim_{t \to 0+} \frac{1}{t} \big\{\big\|\log(e^{z_1} e^{tb_1})\big\| - \|z_1\|\big\}.$$

By the Baker–Campbell–Hausdorff formula we have the following linear approximation

$$\log(e^{z_1} e^{tb_1}) = z_1 + tb_1 + R(z_1, tb_1),$$

where $\lim_{t \to 0} \frac{\|R(z_1, tb_1)\|}{t} = 0$. Then,

$$\|z_i + tb_i\| - \big\|R_i(z_i, tb_i)\big\| \leqslant \big\|\log(e^{z_i} e^{tb_i})\big\| \leqslant \|z_i + tb_i\| + \big\|R_i(z_i, tb_i)\big\|.$$

For $t > 0$, we have

$$\begin{aligned} \frac{1}{t}\big\{\|z_1 + tb_1\| - \|z_1\|\big\} - \frac{1}{t}\big\|R(z_1, tb_1)\big\| &\leqslant \frac{1}{t}\big\{\big\|\log(e^{z_1} e^{tb_1})\big\| - \|z_1\|\big\} \\ &\leqslant \frac{1}{t}\big\{\|z_1 + tb_1\| - \|z_1\|\big\} + \frac{1}{t}\big\|R(z_1, tb_1)\big\|. \end{aligned}$$

If we take limit $t \to 0^+$, we obtain

$$\partial^+ f(0) = \lim_{t \to 0^+} \frac{1}{t}\big\{\|z_1 + tb_1\| - \|z_1\|\big\}.$$

Note that the above right derivative exists due to the convexity of the norm (see [15] for instance). Since (z_1, z_2) is a minimal lifting, it follows $\|z_1 + tb_1\| \geqslant \|z_1\|$, for t small enough, then $\partial^+ f(0) \geqslant 0$. Analogously one proves the corresponding statement for the left derivative, i.e. $\partial^- f(0) \leqslant 0$. Hence our claim follows.

Thus $f(0) \leqslant f(t)$, for all $t \in [0, 1]$. In particular,

$$L(\Delta) = \big\|(z_1, z_2)\big\| = f(0) \leqslant f(1) = \big\|(a_1, a_2)\big\| \leqslant L_\infty(\Gamma). \qquad \square$$

Remark 3.5. The proof only uses the assumption on the range of v to guarantee the existence of minimal liftings. We do not know if there exist minimal liftings when the partial isometry has no restrictions on its range. A positive answer to this problem would lead to a general solution of the initial value problem.

Remark 3.6. Let v be an isometry. The orbit given by left multiplication by unitaries of $\mathcal{B}(\mathcal{H})$, i.e. $\{uv\colon u \in \mathcal{U}(\mathcal{H})\}$ is a homogeneous space of $\mathcal{U}(\mathcal{H})$. In [4] the initial value problem was solved without restrictions on the range of v using different techniques. Recall that in the paper [7] the structure of homogeneous space of $\{uv\colon u \in \mathcal{U}_c(\mathcal{H})\}$ was studied. The tangent space at uv is $\{xv\colon x \in \mathcal{K}(\mathcal{H})_{ah}\}$. The quotient metric takes the form

$$\|xv\|_v = \inf\{\|x+a\|\colon av = 0,\ a \in \mathcal{K}(\mathcal{H})_{ah}\}.$$

Note that $xv = 0$ if and only if $xvv^* = 0$. As we mention in the proof of Proposition 3.1 Davis et al. in [8] proved that the operator $x : vv^*(\mathcal{H}) \longrightarrow \mathcal{H}$ has a compact extension z satisfying $\|zv\| = \|z\|$. Hence the existence of minimal liftings is guaranteed in this case. Then the initial value problem of $\{uv\colon u \in \mathcal{U}_c(\mathcal{H})\}$ can be solved with the same techniques that we use for $\mathcal{S}t_c(v)$ and without restrictions on the range of v.

3.2. Some special tangent directions

Throughout this section no assumption on the rank of v is required. We shall give particular curves in $\mathcal{S}t_c(v)$ that remain of minimal length along their paths. We need some facts about the orbit $\mathcal{O}_p$ of a projection p by the natural action of $\mathcal{U}_c(\mathcal{H})$ on the set of projections, i.e.

$$\mathcal{O}_p = \{upu^*\colon u \in \mathcal{U}_c(\mathcal{H})\}.$$

The tangent space at $p_0 \in \mathcal{O}_p$ is given by

$$(T\mathcal{O}_p)_{p_0} = \{xp_0 - p_0x\colon x \in \mathcal{K}(\mathcal{H})_{ah}\}.$$

It is a real analytic submanifold of $p + \mathcal{K}(\mathcal{H})$ and a homogeneous space of $\mathcal{U}_c(\mathcal{H})$. The Lie algebra of the isotropy group at $p_0 \in \mathcal{O}_p$ is

$$\mathcal{G}_{p_0} = \{x \in \mathcal{K}(\mathcal{H})_{ah}\colon xp_0 = p_0x\}.$$

One can define a quotient metric $\|.\|_{p_0}$ using the Banach quotient norm of $\mathcal{K}(\mathcal{H})_{ah}/\mathcal{G}_{p_0}$,

$$\|xp_0 - p_0x\|_{p_0} = \inf\{\|x+a\|\colon a \in \mathcal{G}_{p_0}\}.$$

In this homogeneous space, the quotient metric can be computed. Given a projection p a 2×2 operator matrix x is co-diagonal if $pxp = (1-p)x(1-p) = 0$. It is a well-know fact that a co-diagonal matrix with respect to a projection has minimal operator norm. Thus the quotient norm equals the operator norm of each tangent vector, i.e. $\|xp_0 - p_0x\|_{p_0} = \|xp_0 - p_0x\|$. Then one measures the length of a piecewise smooth curve γ in $\mathcal{O}_p$ by

$$L(\gamma) = \int_0^1 \|\dot\gamma(t)\|\, dt.$$

It was proved in the article [16] that the curves $\delta(t) = e^{tx}p_0e^{-tx}$, $\|x\| \leqslant \pi/2$ and x co-diagonal with respect to p_0 are geodesics of minimal length joining their endpoints in the unitary orbit of a projection in an arbitrary C^*-algebra. Since $\mathcal{O}_{p_0}$ is contained in the unitary orbit of p_0 in $\mathcal{B}(\mathcal{H})$, the curves $\delta(t) = e^{tx}p_0e^{-tx}$, $\|x\| \leqslant \pi/2$, x compact and x co-diagonal with respect to p_0 have minimal length in $\mathcal{O}_{p_0}$.

Given fixed projections p, q, we can consider the product manifold $\mathcal{O}_p \times \mathcal{O}_q$. It is a homogeneous space of the group $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ and a real analytic submanifold of $(p,q) + \mathcal{K}(\mathcal{H}) \times \mathcal{K}(\mathcal{H})$. We endow it with the product metric (or quotient metric) given by

$$\|(xp_0 - p_0x, q_0y - yq_0)\| = \max\{\|xp_0 - p_0x\|, \|yq_0 - q_0y\|\},$$

where $p_0 \in \mathcal{O}_p$, $q_0 \in \mathcal{O}_q$ and $x, y \in \mathcal{K}(\mathcal{H})_{ah}$. Since it will be clear by the context we shall use the same notation $L(\gamma)$ for the length of a curve γ in $\mathcal{O}_p \times \mathcal{O}_q$. The following result is now apparent.

Lemma 3.7. *Let $x, y \in \mathcal{K}(\mathcal{H})_{ah}$ such that $\|(x,y)\| \leqslant \pi/2$. Suppose that x is co-diagonal with respect to $p_0 \in \mathcal{O}_p$ and y is co-diagonal with respect to $q_0 \in \mathcal{O}_q$. Then $\delta(t) = (e^{tx}p_0e^{-tx}, e^{ty}q_0e^{-ty})$ has minimal length among all piecewise smooth curves in $\mathcal{O}_p \times \mathcal{O}_q$ joining the same endpoints.*

We denote $p = vv^*$ and $q = v^*v$ as in Section 3.1. Consider the following map

$$\varphi : \mathcal{S}t_c(v) \longrightarrow \mathcal{O}_p \times \mathcal{O}_q, \qquad \varphi(uvw^*) = (upu^*, wqw^*).$$

It is easy to check that φ is well defined and smooth. The differential of φ at $v_0 \in \mathcal{S}t_c(v)$ is given by

$$(d\varphi)_{v_0} : \big(T\mathcal{S}t_c(v)\big)_{v_0} \longrightarrow (T\mathcal{O})_{p_0} \times (T\mathcal{O})_{q_0}, \qquad (d\varphi)_{v_0}(xv_0 - v_0y) = (xp_0 - p_0x, yq_0 - q_0y).$$

In the next lemma we prove that this map reduces lengths when one endows $\mathcal{S}t_{\mathfrak{J}}(v)$ with the quotient metric and $\mathcal{O}_p \times \mathcal{O}_q$ with product metric given by the spectral norm.

Lemma 3.8. *Let $v_0 \in \mathcal{S}t_c(v)$ and $x, y \in \mathcal{K}(\mathcal{H})_{ah}$. Then*

$$\big\|(d\varphi)_{v_0}(xv_0 - v_0y)\big\| \leqslant \|xv_0 - v_0y\|_{v_0}.$$

In particular, if γ is a curve in $\mathcal{S}t_c(v)$, then $L(\varphi\gamma) \leqslant L(\gamma)$.

Proof. Note that φ is equivariant for the corresponding actions of $\mathcal{U}_c(\mathcal{H}) \times \mathcal{U}_c(\mathcal{H})$ over $\mathcal{S}t_{\mathfrak{J}}(v)$ and $\mathcal{O}_p \times \mathcal{O}_q$. Moreover, both actions are isometric with respect to the metrics. Therefore it suffices to prove our statement for $v_0 = v$, $p_0 = p$ and $q_0 = q$.

Recall that $av = vb$ if and only if $ap = pa$, $bq = qb$ and $qbq = v^*av$. Then, we have

$$\begin{aligned}\big\|(d\varphi)_{v_0}(xv_0 - v_0y)\big\| &= \big\|(xp - px, qy - yq)\big\| \\ &= \inf\big\{\big\|(x+a, y+b)\big\| : ap = pa,\ qb = bq \text{ and } a, b \in \mathcal{K}(\mathcal{H})_{ah}\big\} \\ &\leqslant \inf\big\{\big\|(x+a, y+b)\big\| : av = vb,\ a, b \in \mathcal{K}(\mathcal{H})_{ah}\big\} = \|xv - vy\|_v,\end{aligned}$$

so our result is proved. The assertion about the curves now follows easily. □

Remark 3.9. The above inequality is sharp. If $x, y \in \mathcal{K}(\mathcal{H})_{ah}$ such that x is co-diagonal with respect to p and y is co-diagonal with respect to q, then it is plain that both quotient metrics attain the infimum at $(a, b) = (0, 0)$. Then,

$$\big\|(xp - px, qy - yq)\big\| = \|xv - vy\|_v.$$

In particular, this implies that the curve $\delta(t) = e^{tx}ve^{-ty}$ satisfies $L(\delta) = L(\varphi\delta)$.

Proposition 3.10. *Let $v_0 \in \mathcal{S}t_c(v)$ and $x, y \in \mathcal{K}(\mathcal{H})_{ah}$ such that $\|(x, y)\| \leqslant \pi/2$. Suppose that x is co-diagonal with respect to $p_0 = v_0v_0^*$ and y is co-diagonal with respect to $q_0 = v_0^*v_0$. Then the curve $\delta(t) = e^{tx}v_0e^{-ty}$, $t \in [0, 1]$, has minimal length among all piecewise smooth curves in $\mathcal{S}t_c(v)$ joining the same endpoints.*

Proof. Let γ be a curve in $\mathcal{S}t_c(v)$ joining v_0 and $e^xv_0e^{-y}$. Observe that the curves $\varphi\gamma$ and $\varphi\delta$ join the same points in $\mathcal{O}_p \times \mathcal{O}_q$. Hence by Lemma 3.7 we have $L(\varphi\gamma) \leqslant L(\gamma)$. Then, by Lemma 3.8 and Remark 3.9 we obtain

$$L(\delta) = L(\varphi\delta) \leqslant L(\varphi\gamma) \leqslant L(\gamma),$$

and our statement holds. □

Acknowledgement

The author wishes to thank Professor Esteban Andruchow for many helpful conversations during the preparation of the article.

References

[1] E. Andruchow, E. Chiumiento, G. Larotonda, Homogeneous manifolds from noncommutative measure spaces, J. Math. Anal. Appl. (2009), in press.
[2] E. Andruchow, G. Larotonda, The rectifiable distance in the unitary Fredholm group, Studia Math. (2009), in press.
[3] E. Andruchow, G. Larotonda, L. Recht, Finsler geometry and actions of the p-Schatten unitary groups, Trans. Amer. Math. Soc. 362 (2010) 319–344.
[4] E. Andruchow, L. Recht, A. Varela, Metric geodesics of isometries in a Hilbert space and the extension problem, Proc. Amer. Math. Soc. 135 (2007) 2527–2537.
[5] J.E. Avron, R. Seiler, B. Simon, The index of a pair of projections, J. Funct. Anal. 120 (1) (1994) 220–237.
[6] D. Beltiţă, Smooth Homogeneous Structures in Operator Theory, Monographs and Surveys in Pure and Applied Mathematics, vol. 137, Chapman and Hall/CRC, 2006.
[7] E. Chiumiento, Geometry of $\mathfrak{J}$-Stiefel manifolds, Proc. Amer. Math. Soc. 138 (2010) 341–353.
[8] C. Davis, W.M. Kahan, H.F. Weinberger, Norm preserving dilations and their applications to optimal error bounds, SIAM J. Numer. Anal. 19 (1982) 445–469.

[9] C. Durán, L. Mata-Lorenzo, L. Recht, Metric geometry in homogeneous spaces of the unitary group of a C^*-algebra. Part I: Minimal curves, Adv. Math. 184 (2) (2004) 342–366.
[10] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Non-Self-Adjoint Operators, Amer. Math. Soc., Providence, RI, 1960.
[11] P.R. Halmos, J.E. Mc Laughlin, Partial isometries, Pacific J. Math. 13 (1963) 585–596.
[12] M.G. Krein, The theory of self-adjoint extensions of semibounded hermitian transformations and its applications, Mat. Sb. 20 (1947) 365–404 (in Russian).
[13] L. Mata-Lorenzo, L. Recht, Infinite-dimensional homogeneous reductive spaces, Acta Cient. Venezolana 43 (1992) 76–90.
[14] M. Mbekhta, S. Stratila, Homotopy classes of partial isometries in von Neumann algebras, Acta Sci. Math. (Szeged) 68 (2002) 271–277.
[15] R.R. Phelps, Convex Functions, Monotone Operators and Differentiability, Lectures Notes in Math., vol. 1364, Springer-Verlag, Berlin, 1989.
[16] H. Porta, L. Recht, Minimality of geodesics in Grassmann manifolds, Proc. Amer. Math. Soc. 100 (3) (1987) 164–166.
[17] M. Takesaki, Theory of Operator Algebras III, Springer-Verlag, 1979.
[18] H. Upmeier, Symmetric Banach Manifolds and Jordan C^*-Algebras, North-Holland Math. Stud., vol. 104, North-Holland, Amsterdam, 1985, Notas de Matemática, vol. 96.