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GEOMETRY OF $\mathfrak{I}$-STIEFEL MANIFOLDS

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ABSTRACT. Let $\mathfrak{I}$ be a separable Banach ideal in the space of bounded operators acting in a Hilbert space $\mathcal{H}$ and $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ the Banach-Lie group of unitary operators which are perturbations of the identity by elements in $\mathfrak{I}$. In this paper we study the geometry of the unitary orbits

$$\{UV : U \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}\}$$

and

$$\{UVW^* : U, W \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}\},$$

where V is a partial isometry. We give a spatial characterization of these orbits. It turns out that both are included in $V + \mathfrak{I}$, and while the first one consists of partial isometries with the same kernel of V, the second is given by partial isometries such that their initial projections and V^*V have null index as a pair of projections. We prove that they are smooth submanifolds of the affine Banach space $V + \mathfrak{I}$ and homogeneous reductive spaces of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ and $\mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ respectively. Then we endow these orbits with two equivalent Finsler metrics, one provided by the ambient norm of the ideal and the other given by the Banach quotient norm of the Lie algebra of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ (or $\mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$) by the Lie algebra of the isotropy group of the natural actions. We show that they are complete metric spaces with the geodesic distance of these metrics.

1. INTRODUCTION

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators acting in $\mathcal{H}$. It will cause no confusion to denote by $\| \, . \, \|$ the spectral norm and the norm of $\mathcal{H}$. By a Banach ideal we mean a two-sided ideal $\mathfrak{I}$ of $\mathcal{B}(\mathcal{H})$ equipped with a norm $\| \, . \, \|_{\mathfrak{I}}$ satisfying $\|T\| \leq \|T\|_{\mathfrak{I}} = \|T^*\|_{\mathfrak{I}}$ and $\|ATB\|_{\mathfrak{I}} \leq \|A\|\|T\|_{\mathfrak{I}}\|B\|$ whenever $A, B \in \mathcal{B}(\mathcal{H})$. In the sequel, $\mathfrak{I}$ stands for a separable Banach ideal.

Denote by $\mathcal{U}(\mathcal{H})$ the group of unitary operators in $\mathcal{H}$ and by $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ the group of unitaries which are perturbations of the identity by an operator in $\mathfrak{I}$, i.e.

$$\mathcal{U}(\mathcal{H})_{\mathfrak{I}} = \{ U \in \mathcal{U}(\mathcal{H}) : U - I \in \mathfrak{I} \}.$$

It is a real Banach-Lie group with the topology defined by the metric $(U_1, U_2) \mapsto \|U_1 - U_2\|_{\mathfrak{I}}$ (see [5]). The Lie algebra of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ is given by

$$\mathfrak{I}_{ah} = \{ A \in \mathfrak{I} : A^* = -A \}.$$

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Let us recall the definition of the (orthogonal) Stiefel manifold $\mathcal{S}t(n,k)$ in $\mathbb{C}^n$,

$$\mathcal{S}t(n,k) = \{ \text{ orthonormal } k\text{-tuples of vectors in } \mathbb{C}^n \} \quad (k \le n).$$

Any element $(v_1, \dots, v_k) \in \mathcal{S}t(n,k)$ identifies with the partial isometry that maps the first k elements of the standard basis of $\mathbb{C}^n$ to the elements of the k-tuple. We want to extend this notion to an infinite dimensional separable Hilbert space $\mathcal{H}$, where the partial isometries can have infinite dimensional range and corange, but only taking partial isometries which are compatible with a fixed partial isometry V and the ideal $\mathfrak{J}$. This leads us to study the following orbit:

$$\mathcal{S}t(V)_{\mathfrak{J}} := \{ UV \,:\, U \in \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \},$$

which we call the *$\mathfrak{J}$-Stiefel manifold associated to V*. Clearly, this is an orbit of the left action of $\mathcal{U}(\mathcal{H})_{\mathfrak{J}}$ on the set of partial isometries $\mathcal{I}$ given by $\mathcal{U}(\mathcal{H})_{\mathfrak{J}} \times \mathcal{I} \to \mathcal{I}$, $(U,V) \mapsto UV$. Moreover, if one wants to move the initial space of V too, then it is natural to consider

$$\mathcal{G}\mathcal{S}t(V)_{\mathfrak{J}} := \{ UVW^* \,:\, U, W \in \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \},$$

which we call the *generalized $\mathfrak{J}$-Stiefel manifold associated to V*. The left action of $\mathcal{U}(\mathcal{H})_{\mathfrak{J}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{J}}$ on $\mathcal{I}$ is given by $\mathcal{U}(\mathcal{H})_{\mathfrak{J}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \times \mathcal{I} \to \mathcal{I}$, $(U,W,V) \mapsto UVW^*$. Note that each Stiefel manifold is contained in the affine Banach space $V + \mathfrak{J}$, so there is an obvious topology coming from the metric $(V_1\,,\,V_2) \mapsto \|V_1 - V_2\|_{\mathfrak{J}}$.

There are several papers concerning the geometry and topology of partial isometries endowed with the spectral norm (see, for instance, [2], [3], [9] and [12]) . On the other hand, the article [1] is devoted to studying partial isometries of finite rank with the Hilbert-Schmidt norm. The aim of this work is to understand some aspects of the geometry of $\mathcal{S}t(V)_{\mathfrak{J}}$ and $\mathcal{G}\mathcal{S}t(V)_{\mathfrak{J}}$ with V a partial isometry of eventually infinite dimensional range and corange.

Let us describe the contents of this paper and the main results.

In Section 2, we establish a spatial characterization of the orbits. The first action defined above is transitive on the set of partial isometries contained in $V + \mathfrak{J}$ with initial projection equal to V^*V, while the second action is transitive on the set of partial isometries contained in $V + \mathfrak{J}$ such that its initial projection and V^*V have null index as a pair of projections. This result is closely related to the characterization of the connected components of the restricted Grassmannian (see [6], [7]).

In Section 3 we prove that $\mathcal{G}\mathcal{S}t(V)_{\mathfrak{J}}$ is a submanifold of the affine Banach space $V + \mathfrak{J}$ and the map $\pi_V : \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \to \mathcal{G}\mathcal{S}t(V)_{\mathfrak{J}}$, $\pi_V(U,W) = UVW^*$ is a submersion. Analogous results hold for $\mathcal{S}t(V)_{\mathfrak{J}}$ and the map $\pi_V : \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \to \mathcal{S}t(V)_{\mathfrak{J}}$, $\pi_V(U) = UV$. Moreover, we prove that each Stiefel manifold is a homogeneous reductive space of the corresponding unitary group which acts on it.

We define in Section 4 two equivalent Finsler metrics over the Stiefel manifolds, the ambient Finsler metric induced as a submanifold of $V + \mathfrak{J}$ and the quotient Finsler metric provided by the homogeneous space structure. Since $\mathcal{S}t(V)_{\mathfrak{J}}$ and $\mathcal{G}\mathcal{S}t(V)_{\mathfrak{J}}$ are infinite dimensional manifolds, there are several, in general nonequivalent, notions of completeness [10]. Using the characterization proved in the second section, we give a short proof of the fact that the Stiefel manifolds are complete metric spaces in the metric given by the infimum of lengths of smooth curves.

2. Spatial characterization

In this section we give an alternative description of the Stiefel manifolds. Let us recall the notion of the index of a Fredholm pair of orthogonal projections (see for instance [4], [15]). Let P, Q be orthogonal projections in $\mathcal{H}$ with range $R(P), R(Q)$ respectively. The pair (P, Q) is Fredholm if $QP : R(P) \longrightarrow R(Q)$ is Fredholm. The index of this operator is the index of the pair (P, Q) and is indicated by $j(P, Q)$.

Fix a partial isometry V and a separable Banach ideal $\mathfrak{I}$. Consider the following set of partial isometries:

$$\mathcal{X}_{V\,,\,\mathfrak{I}} := \{\, V_1 \in \mathcal{B}(\mathcal{H}) \,:\, V_1 = V_1V_1^*V_1,\; V - V_1 \in \mathfrak{I},\; \ker(V_1) = \ker(V)\,\}.$$

Note that if V, V_1 are partial isometries such that $V - V_1$ is a compact operator, then $(V^*V, V_1^*V_1)$ and $(VV^*, V_1V_1^*)$ are Fredholm pairs. Then, consider also the following set:

$$\mathcal{Y}_{V\,,\,\mathfrak{I}} := \{\, V_1 \in \mathcal{B}(\mathcal{H}) \,:\, V_1 = V_1V_1^*V_1,\; V - V_1 \in \mathfrak{I},\; j(V^*V, V_1^*V_1) = 0\,\}.$$

We shall prove that $\mathcal{X}_{V\,,\,\mathfrak{I}} = \mathcal{S}t(V)_{\mathfrak{I}}$ and $\mathcal{Y}_{V\,,\,\mathfrak{I}} = \mathcal{GS}t(V)_{\mathfrak{I}}$. These statements depend on two results. The first one was proved by Stratila and Voiculescu in [15] when $\mathfrak{I}$ is the ideal of Hilbert-Schmidt operators. Then, Carey in [7] generalized it to an arbitrary symmetrically normed separable ideal.

Lemma 2.1 (Carey)**.** *Let P, Q be orthogonal projections. Then $P - Q \in \mathfrak{I}$ and $j(P, Q) = 0$ is equivalent to $UPU^* = Q$ for some $U \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$.*

The second result was given in [14].

Lemma 2.2 (Serban-Turcu)**.** *Let $\mathcal{H}$, $\mathcal{K}$ be separable, infinite dimensional Hilbert spaces. Let $\mathcal{H}_0$, $\mathcal{H}_1$ be infinite dimensional subspaces of $\mathcal{H}$ and let P_i be the orthogonal projection onto $\mathcal{H}_i$ $(i = 1, 2)$. The following are equivalent:*

i) *There exist two isometries V_1, V_2 in $\mathcal{B}(\mathcal{K}, \mathcal{H})$ with ranges H_1 and H_2 such that $V_1 - V_2$ is compact.*
ii) *$P_1 - P_2$ is compact and $j(P_1, P_2) = 0$.*

The following statement could be read as a factorization result for isometries in the $\mathfrak{I}$-Stiefel manifold associated to V.

Theorem 2.3. *Let V be a partial isometry and $\mathfrak{I}$ a separable Banach ideal. Then $\mathcal{X}_{V\,,\,\mathfrak{I}} = \mathcal{S}t(V)_{\mathfrak{I}}$.*

Proof. It suffices to prove that the action of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ on $\mathcal{X}_{V\,,\,\mathfrak{I}}$ given by $U \cdot V_1 = UV_1$ is transitive. Notice that if $U \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ and $V_1 \in \mathcal{X}_{V\,,\,\mathfrak{I}}$, then

$$UV_1 - V = (U - I)V_1 + (V_1 - V) \in \mathfrak{I}.$$

Clearly $\ker(UV_1) = \ker(V_1) = \ker(V)$. Hence $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ acts on $\mathcal{X}_{V\,,\,\mathfrak{I}}$.

To prove transitivity, take $V_1 \in \mathcal{X}_{V\,,\,\mathfrak{I}}$. Assume first that $\dim R(V) = \infty$. Since we have $R(V^*) = \ker(V)^\perp = \ker(V_1)^\perp$, then

$$U_1 := V_1V^* : R(V) \longrightarrow R(V_1)$$

defines a surjective isometry. Therefore $\dim R(V_1) = \infty$, and $V, V_1 \in \mathcal{B}(\ker(V)^\perp, \mathcal{H})$ are isometries such that $V - V_1$ is compact. Then Lemma 2.2 applies to obtain $j(VV^*, V_1V_1^*) = 0$. The projections onto the orthogonal of the respective ranges satisfy $(I - VV^*) - (I - V_1V_1^*) \in \mathfrak{I}$ and $0 = j(VV^*, V_1V_1^*) = -j(I - VV^*, I - V_1V_1^*)$.

Then there exists $U_2 \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ such that $U_2(I - VV^*)U_2^* = I - V_1V_1^*$ by Lemma 2.1. Notice that U_2 maps $R(V)^\perp$ onto $R(V_1)^\perp$; thus the restriction

$$U_2 : R(V)^\perp \longrightarrow R(V_1)^\perp$$

is a surjective isometry. Then set $U := U_1 \oplus U_2$, which is a unitary in $\mathcal{H}$ such that $UV = V_1$. Moreover, this unitary satisfies $(U - I)VV^* = V_1V^* - VV^* \in \mathfrak{I}$ and $(U - I)(I - VV^*) = (U_2 - I)(I - VV^*) \in \mathfrak{I}$; thus we conclude that $U \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$.

Assume now that $\dim R(V) < \infty$. As before we define an isometry from $R(V)$ onto $R(V_1)$, so we obtain $\dim R(V_1) < \infty$. Therefore the orthogonal complements of the ranges satisfy $\dim R(V)^\perp \cap R(V_1)^\perp = \dim(R(V) + R(V_1))^\perp = \infty$. Let $\{ f_j : j \in \mathbb{N} \}$ be an orthonormal basis of $R(V)^\perp \cap R(V_1)^\perp$ and $\{ e_j : j \in \mathbb{N} \}$ an orthonormal basis of $\mathcal{H}$ such that the first n vectors form a basis of $\ker(V)^\perp$. Set

$$\tilde{V}e_j = \begin{cases} Ve_j\,, & 1 \le j \le n, \\ f_{j-n}\,, & j > n, \end{cases} \qquad \tilde{V}_1 e_j = \begin{cases} V_1e_j\,, & 1 \le j \le n, \\ f_{j-n}\,, & j > n. \end{cases}$$

So $\tilde{V}, \tilde{V}_1$ are isometries such that $\tilde{V}_1 - \tilde{V}$ is compact. Then by the first part of our proof we obtain $\tilde{V}_1 = U\tilde{V}$ for some $U \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$. Finally, it is clear that $V_1 = UV$. ☐

The following factorization result gives an idea of how much the initial space of an isometry in $\mathcal{GS}t(V)_{\mathfrak{I}}$ can change and still lie in the orbit.

Theorem 2.4. *Let V be a partial isometry and $\mathfrak{I}$ a separable Banach ideal. Then $\mathcal{Y}_{V\,,\mathfrak{I}} = \mathcal{GS}t(V)_{\mathfrak{I}}$.*

Proof. It suffices to prove that the action of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ on $\mathcal{Y}_{V\,,\mathfrak{I}}$ given by $(U, W)\cdot V_1 = UV_1W^*$ is transitive. Note that it is indeed an action of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ on $\mathcal{Y}_{V\,,\mathfrak{I}}$. Let $V_1 \in \mathcal{Y}_{V\,,\mathfrak{I}}$, $U, W \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$. Then

$$UV_1W^* - V = (U - I)V_1W^* + V_1(W^* - I) + (V_1 - V) \in \mathfrak{I}.$$

Since $(UV_1W^*)^*(UV_1W^*) = W(V_1^*V_1)W^*$ and applying Lemma 2.1, we get $j(V_1^*V_1, (UV_1W^*)^*(UV_1W^*)) = 0$. We use the following formula proved in [4], which is valid because $V^*V - V_1^*V_1$ is compact:

$$j(V^*V, (UV_1W^*)^*(UV_1W^*)) = j(V^*V, V_1^*V_1) + j(V_1^*V_1, (UV_1W^*)^*(UV_1W^*)) = 0.$$

Thus $UV_1W^* \in \mathcal{Y}_{V\,,\mathfrak{I}}$.

In order to establish transitivity, note that if $V_1 \in \mathcal{Y}_{V\,,\mathfrak{I}}$, then $V^*V - V_1^*V_1 \in \mathfrak{I}$ and $j(V^*V, V_1^*V_1) = 0$. Again Lemma 2.1 yields

$$W(V^*V)W^* = V_1^*V_1$$

for some $W \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$. In particular, this unitary maps isometrically $\ker(V)$ (resp. $\ker(V)^\perp$) onto $\ker(V_1)$ (resp. $\ker(V_1)^\perp$). Define

$$T : R(V) \longrightarrow R(V_1), \quad Te = V_1WV^*e.$$

This gives a surjective isometry. Notice that if $e \in \ker(V)^\perp$,

$$TVe = V_1WV^*Ve = V_1We.$$

If $e \in \ker(V)$,

$$TVe = 0 = V_1We.$$

So we obtain $TV = V_1W$.

The task is now to change T for a unitary in $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$. Observe that

$$TVV^* - VV^* \in \mathfrak{I}. \tag{2.1}$$

This follows because $V_1 - V \in \mathfrak{I}$ implies $V_1V^* - VV^* \in \mathfrak{I}$. Therefore,

$$TVV^* - VV^* = V_1WV^* - VV^* = V_1(W-I)V^* + (V_1V^* - VV^*) \in \mathfrak{I}.$$

On the other hand, note that $\ker(V) = \ker(V_1W)$ and by (2.1) we have

$$V - V_1W = V - TV \in \mathfrak{I}.$$

Thus, Theorem 2.3 holds, and there exists $U \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ with $V_1W = UV$, i.e. $V_1 = UVW^*$. □

Remark 2.5. One implication of the preceding result says that if V, V_1 are partial isometries satisfying $V - V_1 \in \mathfrak{I}$ and $j(V^*V, V_1^*V_1) = 0$, then $V_1 = UVW^*$ for some $U, W \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$. In particular, this gives $V_1V_1^* = U(VV^*)U^*$. Hence by Lemma 2.1, we obtain $j(VV^*, V_1V_1^*) = 0$. Therefore two partial isometries with difference in $\mathfrak{I}$ and null index of their initial projections also have null index of their final projections. A similar statement holds for the final projections in place of the initial projections by taking the adjoint of the partial isometries.

This is not true for arbitrary partial isometries. For instance take $V = I$ and V_1 the unilateral shift operator. Then $j(V^*V, V_1^*V_1) = 0$, but $j(VV^*, V_1V_1^*) = 1$.

3. Submanifold and homogeneous reductive structures

In this section we prove that $\mathcal{GS}t(V)_{\mathfrak{I}}$ is a real analytic submanifold of $V + \mathfrak{I}$ and a homogeneous reductive space of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$. Analogous results hold for $\mathcal{S}t(V)_{\mathfrak{I}}$ and the group $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$, where the proofs follow easily.

First we prove that the action of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ on $\mathcal{GS}t(V)_{\mathfrak{I}}$ admits continuous local cross sections.

Lemma 3.1. *The map*

$$\pi_V : \mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}} \longrightarrow \mathcal{GS}t(V)_{\mathfrak{I}} \subset V + \mathfrak{I}, \qquad \pi_V((W,U)) = UVW^*$$

has continuous local cross sections. In particular, it is a locally trivial fiber bundle.

Proof. Let $V_1 \in \mathcal{GS}t(V)_{\mathfrak{I}}$ such that $\|V_1 - V\|_{\mathfrak{I}} < 1$. The idea to find unitaries that depend continuously on V_1 is adapted from [3]. Recall that $\|\,.\,\|$ denotes the spectral norm. Set $P = VV^*$, $P_1 = V_1V_1^*$. We have

$$\begin{aligned}\|P - PP_1\| &= \|VV^* - VV^*V_1V_1^*\| \le \|V^*(I - V_1V_1^*)\| \\ &= \|(V^* - V_1^*)(I - V_1V_1^*)\| \le \|V^* - V_1^*\| \le \|V_1 - V\|_{\mathfrak{I}} < 1.\end{aligned}$$

Then

$$\|P - PP_1P\| \le \|P - PP_1\| < 1.$$

Therefore we obtain that PP_1P is invertible on $R(P)$. Taking the inverse on $R(P)$, put $S = P_1(PP_1P)^{-1/2} = P_1|P_1P|^{-1}$. Note the following:

$$S^*S = (PP_1P)^{-1/2}P_1(PP_1P)^{-1/2} = (PP_1P)^{-1/2}(PP_1P)(PP_1P)^{-1/2} = P.$$

The next step is to prove that $SS^* = P_1$. We first check that $P_1P = S|P_1P|$ is actually the polar decomposition, proving the following two conditions:

i) $S|P_1P| = P_1|P_1P|^{-1}|P_1P| = P_1P$.

ii) Clearly $R(P) = R(PP_1P) \subseteq R(PP_1) \subseteq R(P)$, i.e. $R(P) = R(PP_1)$. Thus

$$\ker(S) = \ker(P) = R(P)^\perp = R(PP_1)^\perp = \ker(P_1P).$$

Since S is the partial isometry given by the polar decomposition, its final space coincides with $R(P_1P) = R(P_1)$ (this equality can be proved as in ii), changing the roles of P_1 and P). Therefore, $SS^* = P_1$.

By the same argument as above,

$$\|(I-P) - (I-P)(I-P_1)\| = \|P - PP_1\| < 1,$$

so there exists a partial isometry $S' : \ker(P) \longrightarrow \ker(P_1)$ implementing the equivalence between $I-P$ and $I-P_1$. Let us define $T = S+S'$, which satisfies $T \in \mathcal{U}(\mathcal{H})$ and $TVV^*T^* = V_1V_1^*$.

Analogously, we construct a $W \in \mathcal{U}(\mathcal{H})$ satisfying $WV^*VW^* = V_1^*V_1$.

Notice that the partial isometries TVW^* and V_1 have the same initial and final spaces. Then, taking $R = V_1(TVW^*)^* + I - V_1V_1^*$, we clearly have $R \in \mathcal{U}(\mathcal{H})$. Moreover,

$$RTVW^* = V_1(TVW^*)^*(TVW^*) + (1 - V_1V_1^*)TVW^* = V_1V_1^*V_1 = V_1\,.$$

Finally take $U = RT \in \mathcal{U}(\mathcal{H})$, which satisfies $UVW^* = V_1$.

Claim. $U, W \in \mathcal{U}(\mathcal{H})_{\mathfrak{J}}$.

For this purpose, recall that $V_1 - V \in \mathfrak{J}$. Then $P_1 - P \in \mathfrak{J}$. Therefore,

$$|P_1P|^2 - P = PP_1P - P = P(P_1 - P)P \in \mathfrak{J}. \tag{3.1}$$

Since $|P_1P|$ and P commute, we can write

$$|P_1P|^2 - P = (|P_1P| + P)(|P_1P| - P).$$

Moreover, $|P_1P| + P$ is invertible in $R(P)$. Then by equation (3.1) we obtain

$$|P_1P| - P = (|P_1P| + P)^{-1}(|P_1P|^2 - P) \in \mathfrak{J}.$$

In particular, this implies that

$$|P_1P|^{-1} - P = |P_1P|^{-1}(P - |P_1P|) \in \mathfrak{J}.$$

Therefore,

$$S - P = P_1|P_1P|^{-1} - P = (P_1 - P)|P_1P|^{-1} + P(|P_1P|^{-1} - P) \in \mathfrak{J}.$$

Analogously we can show that $S' - (I - P) \in \mathfrak{J}$. Then,

$$T - I = (S - P) + (S' - (I - P)) \in \mathfrak{J}.$$

By the same argument we have $W \in \mathcal{U}(\mathcal{H})_{\mathfrak{J}}$. Since $R = V_1(TVW^*)^* + I - V_1V_1^*$, we obtain

$$\begin{aligned} R - I &= V_1(TVW^*)^* - V_1V_1^* \\ &= V_1(W - I)V^*T^* + V_1V^*(T^* - I) + V_1(V^* - V_1^*) \in \mathfrak{J}. \end{aligned}$$

Hence $U = RT \in \mathcal{U}(\mathcal{H})_{\mathfrak{J}}$.

Claim. The map

$$\{V_1 \in \mathcal{GS}t(V)_{\mathfrak{J}} : \|V_1 - V\|_{\mathfrak{J}} < 1\} \subseteq V + \mathfrak{J} \longrightarrow \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{J}}, V_1 \mapsto (U(V_1), W(V_1))$$

is continuous.

We will write this map as a composition of continuous maps. First, consider the following map:

$$\mathcal{GS}t(V)_{\mathfrak{J}} \longrightarrow P + P\mathfrak{J}P, \quad V_1 \mapsto V_1V_1^*P.$$

It is clearly continuous. Since $\mathbb{C}P + P\mathfrak{J}P$ is a $*$-Banach algebra, multiplication and taking inverses are continuous. Then the map

$$P + P\mathfrak{J}P \longrightarrow P + P\mathfrak{J}P, \quad V_1V_1^*P \mapsto V_1V_1^*P|V_1V_1^*P|^{-1}$$

is continuous. Therefore, if we define

$$S : \mathcal{GS}t(V)_{\mathfrak{J}} \longrightarrow P + P\mathfrak{J}P, \quad S(V_1) = V_1V_1^*P|V_1V_1^*P|^{-1},$$

it is a continuous map. Analogously, we can prove that

$$S' : \mathcal{GS}t(V)_{\mathfrak{J}} \longrightarrow I - P + (I-P)\mathfrak{J}(I-P),$$
$$S'(V_1) = (I - V_1V_1^*)(I-P)|(I - V_1V_1^*)(I-P)|^{-1}$$

defines a continuous map. Adding these maps, we obtain that

$$T : \mathcal{GS}t(V)_{\mathfrak{J}} \longrightarrow \mathcal{U}(\mathcal{H})_{\mathfrak{J}}, \quad T(V_1) = S(V_1) + S'(V_1)$$

is a continuous map. On the other hand, since W is constructed like T, we have that $V_1 \mapsto W(V_1)$ is continuous. Then, the following map, which is multiplication and taking adjoint,

$$R : \mathcal{GS}t(V)_{\mathfrak{J}} \longrightarrow \mathcal{U}(\mathcal{H})_{\mathfrak{J}}, \quad R(V_1) = V_1(T(V_1)VW(V_1)^*)^* + I - VV^*,$$

is continuous. So we conclude that $V_1 \mapsto (R(V_1)T(V_1), W(V_1))$ is continuous. $\square$

The same result can be proved for $\mathcal{S}t(V)_{\mathfrak{J}}$.

Corollary 3.2. *The map*

$$\pi_V : \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \longrightarrow \mathcal{S}t(V)_{\mathfrak{J}} \subset V + \mathfrak{J}, \quad \pi_V(U) = UV$$

has continuous local cross sections. In particular, it is a locally trivial fiber bundle.

We need the following consequence of the implicit function theorem in Banach spaces, which is contained in the appendix of [13].

Lemma 3.3. *Let G be a Banach-Lie group acting smoothly on a Banach space X. For a fixed $x_0 \in X$, denote by $\pi_{x_0} : G \longrightarrow X$ the smooth map $\pi_{x_0}(g) = g \cdot x_0$. Suppose that*

1. *π_{x_0} is an open mapping, when regarded as a map from G onto the orbit $\{ g \cdot x_0 : g \in G \}$ of x_0 (with the relative topology of X).*
2. *The differential $(d\pi_{x_0})_1 : (TG)_1 \longrightarrow X$ splits: its kernel and range are closed complemented subspaces.*

Then the orbit $\{ g \cdot x_0 : g \in G \}$ is a smooth submanifold of X, and the map $\pi_{x_0} : G \longrightarrow \{ g \cdot x_0 : g \in G \}$ is a smooth submersion.

Note that the isotropy group at $V_1 \in \mathcal{GS}t(V)_{\mathfrak{J}}$ of the action of $\mathcal{U}(\mathcal{H})_{\mathfrak{J}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{J}}$ on $\mathcal{GS}t(V)_{\mathfrak{J}}$ is given by

$$G_{V_1} = \{ (U, W) \in \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{J}} : UV_1 = V_1W \}.$$

The Lie algebra is

$$\mathcal{G}_{V_1} = \{ (X, Y) \in \mathfrak{J}_{ah} \times \mathfrak{J}_{ah} : XV_1 = V_1Y \}.$$

Recall that a reductive structure for $\mathcal{GS}t(V)_{\mathfrak{J}}$ is a smooth distribution of horizontal spaces $\{ \mathcal{H}_{V_1} : V_1 \in \mathcal{GS}t(V)_{\mathfrak{J}} \}$, which are supplements for the Lie algebra of the isotropy groups: $\mathcal{H}_{V_1} \oplus \mathcal{G}_{V_1} = \mathfrak{I}_{ah} \times \mathfrak{I}_{ah}$. Each $\mathcal{H}_{V_1}$ has to be invariant under the inner action of G_{V_1} (see [11]). Now we can state the main theorem of this section.

Theorem 3.4. *Let $V \in \mathcal{B}(\mathcal{H})$ be a partial isometry. Then $\mathcal{GSt}(V)_{\mathfrak{I}}$ is a real analytic submanifold of $V + \mathfrak{I}$ and the map*

$$\pi_V : \mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}} \longrightarrow \mathcal{GSt}(V)_{\mathfrak{I}}$$

is a real analytic submersion. Moreover, $\mathcal{GSt}(V)_{\mathfrak{I}}$ is a homogeneous reductive space of the group $\mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$.

Proof. We only have to apply Lemma 3.3 with $G = \mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$, $X = V + \mathfrak{I}$ and $x_0 = V$. Notice that π_V is open by Lemma 3.1. The differential map of π_V at I is given by

$$\delta_V := (d\pi_V)_I : \mathfrak{I}_{ah} \times \mathfrak{I}_{ah} \longrightarrow \mathfrak{I}, \quad \delta_V(X,Y) = XV - VY.$$

The kernel of this map is the Lie algebra of the isotropy at V, which can be expressed as

(3.2)

$$\mathcal{G}_V = \{\, (\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}_{VV^*}, \begin{pmatrix} V^*X_{11}V & 0 \\ 0 & Y_{22} \end{pmatrix}_{V^*V}) : X_{11},\, X_{22},\, Y_{11} \in \mathfrak{I}_{ah} \,\}.$$

Here the subscripts VV^* and V^*V indicate that the matrices are regarded with respect to this projections. A closed complement for $\mathcal{G}_V$ is

(3.3)

$$\mathcal{H}_V = \{(\begin{pmatrix} X_{11} & X_{12} \\ -X_{12}^* & 0 \end{pmatrix}_{VV^*}, \begin{pmatrix} 0 & Y_{12} \\ -Y_{12}^* & 0 \end{pmatrix}_{V^*V}) : X_{11} \in \mathfrak{I}_{ah},\, X_{12},\, Y_{12} \in \mathfrak{I} \}.$$

The argument in [1] to show that the range is closed does not depend on the dimension of the range of V. We repeat it here for the convenience of the reader. Consider the real linear map,

$$\mathcal{K}_V : \mathfrak{I} \longrightarrow \mathfrak{I} \times \mathfrak{I}, \quad \mathcal{K}_V(A) = (\mathcal{K}_1, \mathcal{K}_2),$$
$$\mathcal{K}_1 = \frac{1}{4}VV^*AV^* - \frac{1}{4}VA^*VV^* + (I - VV^*)AV^* - VA^*(I - VV^*),$$
$$\mathcal{K}_2 = -\frac{1}{4}V^*AV^*V + \frac{1}{4}V^*VA^*V - V^*A(I - V^*V) + (I - V^*V)A^*V.$$

It can be proved that $\delta_V \circ \mathcal{K}_V \circ \delta_V = \delta_V$. Therefore $\delta_V \circ \mathcal{K}_V$ is an idempotent operator on $\mathfrak{I}$ whose range is closed and equal to the range of δ_V. Since the action is real analytic we have that $\mathcal{GSt}(V)_{\mathfrak{I}}$ is a real analytic submanifold of $V + \mathfrak{I}$ and π_V is a real analytic submersion.

Therefore $\mathcal{GSt}(V)_{\mathfrak{I}}$ is a homogeneous space of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}} \times \mathcal{U}(\mathcal{H})_{\mathfrak{I}}$. The reductive structure is given by $\{\, \mathcal{H}_{V_1} : V_1 \in \mathcal{GSt}(V)_{\mathfrak{I}} \,\}$ as in (3.3). It is a straightforward computation using the matrix decomposition above that these supplements satisfy $Ad(U,W)(\mathcal{H}_V) = \mathcal{H}_V$, for all $(U,W) \in \mathcal{G}_V$. □

Now we consider $\mathcal{St}(V)_{\mathfrak{I}}$. Observe that the isotropy group at V_1 of the action of $\mathcal{U}(\mathcal{H})_{\mathfrak{I}}$ on $\mathcal{St}(V)_{\mathfrak{I}}$ is given by

$$G_{V_1} = \{\, U \in \mathcal{U}(\mathcal{H})_{\mathfrak{I}} : UV_1 = V_1 \,\}.$$

The Lie algebra is

$$\mathcal{G}_{V_1} = \{\, X \in \mathfrak{I}_{ah} : XV_1 = 0 \,\}.$$

Corollary 3.5. *Let $V \in \mathcal{B}(\mathcal{H})$ be a partial isometry. Then $\mathcal{S}t(V)_{\mathfrak{J}}$ is a real analytic submanifold of $V + \mathfrak{J}$ and the map*

$$\pi_V : \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \longrightarrow \mathcal{GS}t(V)_{\mathfrak{J}}$$

is a real analytic submersion. Moreover, $\mathcal{S}t(V)_{\mathfrak{J}}$ is a homogeneous space of $\mathcal{U}(\mathcal{H})_{\mathfrak{J}}$.

Proof. Apply Lemma 3.3 with $G = \mathcal{U}(\mathcal{H})_{\mathfrak{J}}$, $X = V + \mathfrak{J}$ and $x_0 = V$. It is a corollary of the proof of Theorem 3.4. $\square$

4. Completeness as metric space

In this section we prove that $\mathcal{GS}t(V)_{\mathfrak{J}}$ and $\mathcal{S}t(V)_{\mathfrak{J}}$ are complete metric spaces with the geodesic distance given by the ambient metric and a quotient metric. First we consider $\mathcal{GS}t(V)_{\mathfrak{J}}$. Recall that since π_{V_1} is a submersion, the tangent space of $\mathcal{GS}t(V)_{\mathfrak{J}}$ at V_1 is

$$(\, T\mathcal{GS}t(V)_{\mathfrak{J}}\,)_{V_1} = \{\, XV_1 - V_1Y \,:\, X, Y \in \mathfrak{J}_{ah}\,\}.$$

We shall describe two metrics which are invariant by the action of $\mathcal{U}(\mathcal{H})_{\mathfrak{J}} \times \mathcal{U}(\mathcal{H})$ on $\mathcal{GS}t(V)_{\mathfrak{J}}$. For $XV_1 - V_1Y \in (\, T\mathcal{GS}t(V)_{\mathfrak{J}}\,)_{V_1}$ we define the ambient Finsler metric by

$$F_a(XV_1 - V_1Y) := \|XV_1 - V_1Y\|_{\mathfrak{J}}.$$

On the other hand, we have a natural Finsler quotient metric. Fix a symmetric norming function Φ in $\mathbb{R}^2$, i.e. a norm which is invariant under permutations, only depends on the absolute values of the coordinates and satisfies $\Phi(1, 0) = 1$. For $XV_1 - V_1Y \in (\, T\mathcal{GS}t(V)_{\mathfrak{J}}\,)_{V_1}$ define the quotient metric by

$$F_q(XV_1 - V_1Y) := \inf\{\, \Phi(\, \|X + A\|_{\mathfrak{J}}\,, \|Y + B\|_{\mathfrak{J}}\,) \,:\, AV_1 = V_1B,\ A, B \in \mathfrak{J}_{ah}\,\}.$$

Indeed, in each tangent space, this is the quotient norm of $(\mathfrak{J}_{ah} \times \mathfrak{J}_{ah})/\mathcal{G}_{V_1}$. Now we show that both metrics are equivalent with bounds that do not depend on the partial isometry V, the ideal $\mathfrak{J}$ or the symmetric norming function Φ.

Proposition 4.1. *Let $XV_1 - V_1Y \in (T\mathcal{GS}t(V)_{\mathfrak{J}})_{V_1}$. Then*

$$\frac{1}{4}\, F_q(XV_1 - V_1Y) \leq F_a(XV_1 - V_1Y) \leq 2\, F_q(XV_1 - V_1Y).$$

Proof. Since both metrics are invariant by the action, we can assume $V = V_1$. We shall use the following elementary inequality (see [8]): For any symmetric norming function Φ and $(x, y) \in \mathbb{R}^2$ we have

$$\max\{\, |x|\,, |y|\,\} \leq \Phi(x, y) \leq |x| + |y|. \tag{4.1}$$

Then, to prove the first inequality, observe that for any $(A, B) \in \mathcal{G}_V$, we obtain

$$F_q(XV_1 - V_1Y) \leq \Phi(\, \|X + A\|_{\mathfrak{J}}\,, \|Y + B\|_{\mathfrak{J}}\,) \leq 2\, \max\{\, \|X + A\|_{\mathfrak{J}}\,, \|Y + B\|_{\mathfrak{J}}\,\}. \tag{4.2}$$

In particular, we can choose

$$A = \begin{pmatrix} \frac{-X_{11} - VY_{11}V^*}{2} & 0 \\ 0 & -X_{22} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{-V^*X_{11}V - Y_{11}}{2} & 0 \\ 0 & -Y_{22} \end{pmatrix},$$

where the matrix decompositions are regarded with respect to the same projections as in (3.2). Therefore,

$$\begin{aligned}
\|X+A\|_{\mathfrak{I}} &= \left\| \begin{pmatrix} \frac{X_{11}-VY_{11}V^*}{2} & X_{12} \\ -X_{12}^* & 0 \end{pmatrix} \right\|_{\mathfrak{I}} \\
&\leq \left\| \begin{pmatrix} \frac{X_{11}-VY_{11}V^*}{2} & 0 \\ -X_{12}^* & 0 \end{pmatrix} \right\|_{\mathfrak{I}} + \left\| \begin{pmatrix} 0 & X_{12} \\ 0 & 0 \end{pmatrix} \right\|_{\mathfrak{I}} \\
&= \left\| \begin{pmatrix} \frac{X_{11}V-VY_{11}}{2} & 0 \\ -X_{12}^*V & 0 \end{pmatrix} \right\|_{\mathfrak{I}} + \left\| \begin{pmatrix} 0 & X_{12}V \\ 0 & 0 \end{pmatrix} \right\|_{\mathfrak{I}} \\
&\leq 2 \left\| \begin{pmatrix} X_{11}V-VY_{11} & 0 \\ -X_{12}^*V & 0 \end{pmatrix} \right\|_{\mathfrak{I}} \leq 2 \left\| \begin{pmatrix} X_{11}V-VY_{11} & VY_{12} \\ -X_{12}^*V & 0 \end{pmatrix} \right\|_{\mathfrak{I}} \\
&= 2\, \|XV-VY\|_{\mathfrak{I}}.
\end{aligned}$$

A similar argument shows that $\|Y+B\|_{\mathfrak{I}} \leq 2\,\|XV-VY\|_{\mathfrak{I}}$. Hence by (4.2) we obtain

$$\frac{1}{4}\, F_q(XV_1 - V_1Y) \leq F_a(XV_1 - V_1Y).$$

In order to prove the other inequality, fix $\epsilon > 0$ and take $(A, B) \in \mathcal{G}_V$ such that

$$\Phi(\|X+A\|_{\mathfrak{I}}\,,\ \|Y+B\|_{\mathfrak{I}}\,) < F_q(XV-VY) + \epsilon.$$

Then by (4.1) we have

$$\begin{aligned}
F_a(XV-VY) &= \|XV-VY\|_{\mathfrak{I}} = \|(X+A)V - V(Y+B)\|_{\mathfrak{I}} \\
&\leq \|X+A\|_{\mathfrak{I}} + \|Y+B\|_{\mathfrak{I}} \leq 2\Phi(\,\|X+A\|_{\mathfrak{I}}\,,\ \|Y+B\|_{\mathfrak{I}}\,) \\
&< 2\,F_q(XV-VY) + 2\epsilon.
\end{aligned}$$

Hence we conclude that

$$F_a(XV-VY) \leq 2\,F_q(XV-VY),$$

and the proposition follows. □

Remark 4.2. When we consider the ideal $\mathcal{B}_2(\mathcal{H})$ of Hilbert-Schmidt operators and that the symmetric norming function is the Euclidean norm in $\mathbb{R}^2$, we have that

$$F_a(XV_1 - V_1Y) = \sqrt{2}\,F_q(XV_1 - V_1Y). \tag{4.3}$$

In this case, the quotient norm can be explicitly computed as follows. Let us define a real bounded projection onto $\mathcal{G}_{V_1}$ by

$$\begin{aligned}
&P_{V_1} : \mathcal{B}_2(\mathcal{H})_{ah} \times \mathcal{B}_2(\mathcal{H})_{ah} \longrightarrow \mathcal{G}_{V_1}\,, \\
&P_{V_1}\big((X,Y)\big) = \left(\begin{pmatrix} \frac{X_{11}+VY_{11}V^*}{2} & 0 \\ 0 & X_{22} \end{pmatrix}, \begin{pmatrix} \frac{V^*X_{11}V+Y_{11}}{2} & 0 \\ 0 & Y_{22} \end{pmatrix}\right).
\end{aligned}$$

It is easy to see that P_{V_1} is the orthogonal projection onto $\mathcal{G}_{V_1}$ if one considers in $\mathcal{B}_2(\mathcal{H}) \times \mathcal{B}_2(\mathcal{H})$ the induced inner product

$$\langle (A_1, B_1), (A_2, B_2) \rangle = Tr(A_1A_2^*) + Tr(B_1B_2^*),$$

where Tr denotes the trace. Then, the expression of the quotient norm reduces to

$$F_q(XV_1 - V_1Y) = \|\,(I - P_{V_1})((X,Y))\,\|_2,$$

where $\|\,.\,\|_2$ is the Hilbert-Schmidt norm in $\mathcal{B}_2(\mathcal{H}) \times \mathcal{B}_2(\mathcal{H})$ given by the inner product above. Now the equality stated in (4.3) is a straightforward computation.

We measure the length of a piecewise smooth curve $\gamma(t)$ in $\mathcal{GS}t(V)_{\mathfrak{J}}$ defined for $0 \le t \le 1$ by

$$L_a(\gamma) = \int_0^1 F_a(\dot{\gamma}(t))\, dt.$$

Therefore, the geodesic distance is given by

$$d_a(V_1, V_2) = \inf\{\, L_a(\gamma) \,:\, \gamma \subset \mathcal{GS}t(V)_{\mathfrak{J}},\ \gamma(0) = V_1,\ \gamma(1) = V_2 \,\},$$

where the curves γ considered are piecewise smooth.

Remark 4.3. Actually, since we are in an infinite dimensional Finsler manifold, we have to prove that $d_a(\,.\,,\,.\,)$ is a metric in $\mathcal{GS}t(V)_{\mathfrak{J}}$. The only nontrivial fact to check is that $d_a(V_1, V_2) = 0$ implies $V_1 = V_2$. Consider a piecewise smooth curve γ in $\mathcal{GS}t(V)_{\mathfrak{J}}$ joining V_1 and V_2. Then

$$\|V_1 - V_2\|_{\mathfrak{J}} \le \int_0^1 F_a(\dot{\gamma}(t))\, dt.$$

Therefore, we obtain

$$\|V_1 - V_2\|_{\mathfrak{J}} \le d_a(V_1, V_2), \tag{4.4}$$

which clearly proves our assertion.

Let $P \in \mathcal{B}(\mathcal{H})$ be an orthogonal projection and $\mathfrak{J}$ any symmetric normed ideal. We call the $\mathfrak{J}$-Grassmannian $Gr(P)_{\mathfrak{J}}$ corresponding to the polarization $\mathcal{H} = R(P) \oplus R(P)^{\perp}$ to the following unitary orbit:

$$Gr(P)_{\mathfrak{J}} = \{\, UPU^* \,:\, U \in \mathcal{U}(\mathcal{H})_{\mathfrak{J}} \,\}.$$

By Lemma 2.1 we have that $Q \in Gr(P)_{\mathfrak{J}}$ if and only if $P - Q \in \mathfrak{J}$ and $j(P, Q) = 0$. We need to show that $Gr(P)_{\mathfrak{J}}$ is closed with the ideal norm.

Lemma 4.4. *Let $P \in \mathcal{B}(\mathcal{H})$ be an orthogonal projection. Let $(P_n)_{n \ge 1}$ be a sequence in $Gr(P)_{\mathfrak{J}}$ such that $\lim_{n\to\infty} \|P_n - P_0\|_{\mathfrak{J}} = 0$. Then $P_0 \in Gr(P)_{\mathfrak{J}}$.*

Proof. It is obvious that P_0 is an orthogonal projection satisfying $P - P_0 \in \mathfrak{J}$. We only have to prove that $j(P_0, P) = 0$. Fix $n \ge 1$ satisfying $\|P_0 - P_n\|_{\mathfrak{J}} < 1$. The fact that $j(P_n, P) = 0$ implies that

$$j(P_0, P) = j(P_0, P_n) + j(P_n, P) = j(P_0, P_n).$$

Since $j(P_0, P_n) \ne 0$, we have $\dim(\ker(P_0) \cap R(P_n)) \ne \dim(\ker(P_n) \cap R(P_0))$. If we suppose that $\dim(\ker(P_0) \cap R(P_n)) > \dim(\ker(P_n) \cap R(P_0))$, then in particular there exists $e \in \ker(P_0) \cap R(P_n)$, $\|e\| = 1$. Therefore, we have a contradiction since

$$1 = \|e\| = \|(P_0 - P_n)e\| \le \|P_0 - P_n\| \le \|P_0 - P_n\|_{\mathfrak{J}} < 1.$$

The same argument follows in case $\dim(\ker(P_0) \cap R(P_n)) < \dim(\ker(P_n) \cap R(P_0))$. $\square$

Using the spatial characterization of $\mathcal{GS}t(V)_{\mathfrak{J}}$ given in Section 2 we have a short proof of the fact that it is a complete metric space.

Theorem 4.5. *$\mathcal{GS}t(V)_{\mathfrak{J}}$ is a complete metric space with the geodesic distance d_a.*

Proof. Let $(V_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{GS}t(V)_\mathfrak{I}$ for the metric d_a. By equation (4.4), $(V_n)_{n\geq 1}$ is also Cauchy for the norm $\|\,.\|_\mathfrak{I}$. Since $\mathfrak{I}$ is a Banach space, there exists $V_0 \in V + \mathfrak{I}$ such that $\|V_n - V_0\|_\mathfrak{I} \to 0$. Using Lemma 4.4 we obtain that $V_0 \in \mathcal{GS}t(V)_\mathfrak{I}$.

In Lemma 3.1 we prove that $\pi_{V_0} : \mathcal{U}(\mathcal{H})_\mathfrak{I} \times \mathcal{U}(\mathcal{H})_\mathfrak{I} \longrightarrow \mathcal{GS}t(V)_\mathfrak{I}$ has continuous local cross sections. Hence for $n \geq 1$ large enough there exists $U_n, W_n \in \mathcal{U}(\mathcal{H})_\mathfrak{I}$ satisfying $V_n = U_n V_0 W_n^*$, $\|U_n - I\|_\mathfrak{I} \to 0$ and $\|W_n - I\|_\mathfrak{I} \to 0$. Since $\mathcal{U}(\mathcal{H})_\mathfrak{I}$ is a Banach-Lie group, the exponential map is a local diffeomorphism. Then there exists $X_n, Y_n \in \mathfrak{I}_{ah}$ such that $V_n = e^{X_n} V_0 e^{-Y_n}$, $\|X_n\|_\mathfrak{I} \to 0$ and $\|Y_n\|_\mathfrak{I} \to 0$. Taking the curves $\gamma_n(t) = e^{tX_n} V_0 e^{-tY_n}$, we conclude that

$$d_a(V_n, V_0) \leq L_a(\gamma_n) \leq \|X_n\|_\mathfrak{I} + \|Y_n\|_\mathfrak{I} \to 0,$$

and the theorem follows. □

In $\mathcal{S}t(V)_\mathfrak{I}$ we can prove the same result. Since the map $\pi_{V_1} : \mathcal{U}(\mathcal{H})_\mathfrak{I} \longrightarrow \mathcal{S}t(V)_\mathfrak{I}$ is a submersion, the tangent space of $\mathcal{S}t(V)_\mathfrak{I}$ at V_1 is given by

$$(\, T\mathcal{S}t(V)_\mathfrak{I}\,)_{V_1} = \{\, XV_1 \,:\, X \in \mathfrak{I}_{ah}\,\}.$$

The quotient metric in $\mathcal{S}t(V)_\mathfrak{I}$ reduces to the following expression:

$$\|XV_1\|_{V_1} = \inf\{\, \|X + A\|_\mathfrak{I} \,:\, AV_1 = 0,\ A \in \mathfrak{I}_{ah}\,\}.$$

Corollary 4.6. *$\mathcal{S}t(V)_\mathfrak{I}$ is a complete metric space with the geodesic distance d_a.*

Proof. It suffices to show that $\mathcal{S}t(V)_\mathfrak{I}$ is d_a-closed in $\mathcal{GS}t(V)_\mathfrak{I}$. Moreover, we will prove that $\mathcal{S}t(V)_\mathfrak{I}$ is $\|\,.\|_\mathfrak{I}$-closed in $\mathcal{GS}t(V)_\mathfrak{I}$. Let $(V_n)_{n\geq 1}$ a sequence in $\mathcal{S}t(V)_\mathfrak{I}$ satisfying $\|V_n - V_0\|_\mathfrak{I} \to 0$, where $V_0 \in \mathcal{GS}t(V)_\mathfrak{I}$. It suffices to demonstrate that $\ker(V_0) = \ker(V)$. Let $e \in \ker(V_0)$, $\|e\| = 1$. For all $n \geq 1$,

$$\|V_n e\| = \|(V_n - V_0)e\| \leq \|V_n - V_0\| \leq \|V_n - V_0\|_\mathfrak{I} \to 0.$$

Since $V_n \in \mathcal{S}t(V)_\mathfrak{I}$, there exists $U_n \in \mathcal{U}(\mathcal{H})_\mathfrak{I}$ such that $V_n = U_n V$. Then,

$$\|Ve\| = \|U_n^* V_n e\| = \|V_n e\|,$$

which implies that

$$\|Ve\| = \lim_{n\to\infty} \|V_n e\| = 0.$$

In order to prove the other inclusion, let $e \in \ker(V) = \ker(V_n)$, $\|e\| = 1$. Observe that

$$\|V_0 e\| \leq \|(V_0 - V_n)e\| \leq \|V_0 - V_n\| \leq \|V_0 - V_n\|_\mathfrak{I} \to 0,$$

so we have $e \in \ker(V_0)$, and the proof is complete. □

Note that we can also define a geodesic distance d_q induced by the Finsler quotient metric F_q. The next result follows immediately from Proposition 4.1.

Corollary 4.7. *The metric spaces $\mathcal{S}t(V)_\mathfrak{I}$ and $\mathcal{GS}t(V)_\mathfrak{I}$ are complete with the geodesic distance d_q.*

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