



## The complexity of clique graph recognition<sup>☆</sup>

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### ARTICLE INFO

#### Article history:

Received 19 March 2007

Received in revised form 9 December 2008

Accepted 17 January 2009

Communicated by O. Watanabe

#### Keywords:

Clique graphs

NP-complete problems

Helly property

Intersection graphs

### ABSTRACT

A *complete set* of a graph  $G$  is a subset of vertices inducing a complete subgraph. A *clique* is a maximal complete set. Denote by  $\mathcal{C}(G)$  the *clique family* of  $G$ . The *clique graph* of  $G$ , denoted by  $K(G)$ , is the intersection graph of  $\mathcal{C}(G)$ . Say that  $G$  is a *clique graph* if there exists a graph  $H$  such that  $G = K(H)$ . The clique graph recognition problem asks whether a given graph is a clique graph. A sufficient condition was given by Hamelink in 1968, and a characterization was proposed by Roberts and Spencer in 1971. However, the time complexity of the problem of recognizing clique graphs is a long-standing open question. We prove that the clique graph recognition problem is NP-complete.

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## 1. Introduction

Consider finite, simple and undirected graphs.  $V$  and  $E$  denote the vertex set and the edge set of the graph  $G$ , respectively. A *complete set* of  $G$  is a subset of  $V$  inducing a complete subgraph. A *clique* is a maximal complete set. The *clique family* of  $G$  is denoted by  $\mathcal{C}(G)$ . The *clique graph* of  $G$  is the intersection graph of  $\mathcal{C}(G)$ .

The *clique operator*,  $K$ , assigns to each graph  $G$  its clique graph which is denoted by  $K(G)$ . On the other hand, say that  $G$  is a *clique graph* if  $G$  belongs to the image of the clique operator, i.e. if there exists a graph  $H$  such that  $G = K(H)$ .

Clique operator and its image were widely studied. First articles focused on recognizing clique graphs [20,36]. In [4,13], graphs for which the clique graph changes whenever a vertex is removed are considered. Graphs fixed under the operator  $K$  or fixed under the iterated clique operator,  $K^n$ , for some positive integer  $n$ ; and the behavior under these operators of parameters such as the number of vertices or diameter were studied in [5,8,9,12,26,30] and more recently in [7,14,21–23, 29]. For several classes of graphs, the image of the class under the clique operator was characterized [10,18,19,24,34,37]; and, in some cases, also the inverse image of the class [16,28,35]. Results of the previous bibliography can be found in the survey [39]. Clique graphs have been much studied as intersection graphs and are included in several books [11,25,33].

In this paper we are concerned with the time complexity of the problem of recognizing clique graphs, this is the time complexity of the following decision problem.

<sup>☆</sup> An extended abstract published in: Proceedings of Graph-theoretic concepts in computer science, WG 2006, Lecture Notes in Comput. Sci., vol. 4271, Springer, 2006, pp. 269–277. This research was partially supported by FAPERJ–Cientistas do Nosso Estado project, by PROCIENCIA project, and by CNPq Project PROSUL–Proc. no. 490333/04–4.

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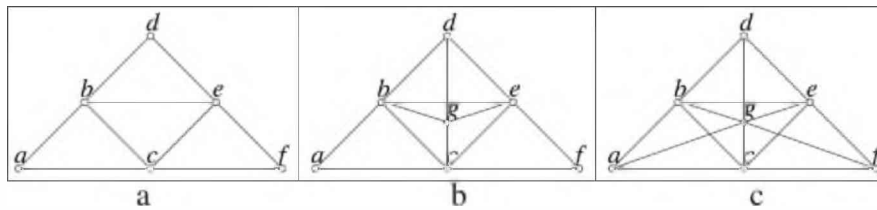


Fig. 1. (a) Non-clique graph thus non-clique-Helly graph; (b) non-clique-Helly graph but clique graph; (c) clique-Helly graph thus clique graph.

CLIQUE GRAPH

INSTANCE: A graph  $G = (V, E)$ .

QUESTION: Is there a graph  $H$  such that  $G = K(H)$ ?

In spite of the characterizations of clique graphs given in [36] and more recently in [1], the time complexity of CLIQUE GRAPH is a long-standing open question [11,32,36,39].

Our main theorem proves that CLIQUE GRAPH is NP-complete by a reduction from a specially chosen version of the 3-satisfiability problem. In Section 2, definitions and basic concepts about clique graphs are presented. Besides, we detail a proof that CLIQUE GRAPH is in NP, and we state the selected 3SAT<sub>3</sub> version of satisfiability problem.

In Section 3, we describe the construction of instance  $G_I$  of CLIQUE GRAPH from instance  $I = (U, C)$  of 3SAT<sub>3</sub>; and analyze some of its properties. In Section 4, we state and prove the main theorem by showing that the constructed graph  $G_I$  is a clique graph if and only if  $C$  is satisfiable. In Section 5, we have our concluding remarks.

The extended abstract [3] recently published contains the description of the special graph  $G_I$  constructed from the satisfiability instance  $I = (U, C)$  but omits most of the proof. The present paper presents the full required proof – a difficult and long case analysis – and highlights the properties of the constructed graph  $G_I$  for the full understanding of the complexity of the recognition problem and the subsequent study of the problem for special classes of graphs.

2. Definitions and basic concepts

Given a set family  $\mathcal{F} = (F_i)_{i=1}^n$ , the sets  $F_i$  are called *members* of the family.  $F \in \mathcal{F}$  means that  $F$  is a member of  $\mathcal{F}$ . The family is *pairwise intersecting* if the intersection of any two members is not the empty set. The *intersection* or *total intersection* of  $\mathcal{F}$  is the set  $\bigcap \mathcal{F} = \bigcap_{i=1}^n F_i$ . The family  $\mathcal{F}$  has the *Helly property*, if any pairwise intersecting subfamily has nonempty total intersection.

The edge with end vertices  $u$  and  $v$  is represented by  $uv$ . We say that the complete set  $C$  covers the edge  $uv$  when  $u$  and  $v$  belong to  $C$ . A *complete set edge cover* of a graph  $G$  is a family of complete sets of  $G$  covering all edges of  $G$ .

The following theorem is a well-known characterization of Clique Graphs.

**Theorem 1** (Roberts and Spencer [36]).  *$G$  is a clique graph if and only if there exists a complete set edge cover of  $G$  satisfying the Helly property.*

Notice that for any graph  $G$  the clique family  $\mathcal{C}(G)$  is a complete set edge cover of  $G$ , but, in general, this family does not satisfy the Helly property. Graphs such that  $\mathcal{C}(G)$  satisfies the Helly property are called *clique-Helly* graphs. It follows from Theorem 1 that every clique-Helly graph is a clique graph. The reciprocal implication is not true: there exist clique graphs which are not clique-Helly graphs. We have depicted in Fig. 1 three examples: (a) a non-clique graph (no complete set edge cover satisfies the Helly property [36]); (b) a clique graph that is not a clique-Helly graph (the clique family does not satisfy the Helly property, but the complete set edge cover  $\{a, b, c\}, \{c, e, f\}, \{b, d, g\}, \{d, e, g\}, \{b, c, e, g\}$  does); and (c) a clique graph that is a clique-Helly graph (the clique family has the Helly property). Examples given in Fig. 1 also show that being a clique graph or being a clique-Helly graph are not hereditary properties.

In [38], clique-Helly graphs are characterized and a polynomial-time algorithm for their recognition is presented. Next lemma extends that result and leads to a polynomial-time algorithm to check if a given complete set edge cover of a graph satisfies the Helly property; this lemma is used to show that CLIQUE GRAPH is in NP and in the proof of Theorem 8.

A *triangle* is a complete set with exactly 3 vertices. The set of triangles of  $G$  is denoted by  $T(G)$ . Let  $\mathcal{F}$  be a complete set edge cover of  $G$  and  $T$  a triangle, and denote by  $\mathcal{F}_T$  the subfamily of  $\mathcal{F}$  formed by all the members containing at least two vertices of  $T$ .

**Lemma 2** (Alcón and Gutierrez [2]). *Let  $\mathcal{F}$  be a complete set edge cover of  $G$ . The following conditions are equivalent:*

- (i)  $\mathcal{F}$  has the Helly property.
- (ii) For every  $T \in T(G)$ , the subfamily  $\mathcal{F}_T$  has the Helly property.
- (iii) For every  $T \in T(G)$ , the subfamily  $\mathcal{F}_T$  has nonempty intersection, this means  $\bigcap \mathcal{F}_T \neq \emptyset$ .

As noted by Roberts and Spencer [36], Theorem 1 yields a polynomial certificate of  $G$  being a clique graph. First, for the polynomial size of the edge cover certificate, note that if  $\mathcal{F}$  has the Helly property, then every subfamily  $\mathcal{F}'$  of  $\mathcal{F}$  has the Helly property as well. In addition, we prove that if  $G$  admits a complete set edge cover  $\mathcal{F}$ , then  $G$  admits a complete set edge cover  $\mathcal{F}'$  of size at most  $|E|$  which is our considered certificate: just sequentially scan the edges of  $E$ , select for  $\mathcal{F}'$

one complete set of  $\mathcal{F}$  covering the first edge, and for each edge  $e$  not yet covered by  $\mathcal{F}'$ , select for  $\mathcal{F}'$  one complete set of  $\mathcal{F}$  covering  $e$ . Clearly this procedure labels each selected set with a corresponding scanned edge of  $E$ , yielding a subfamily  $\mathcal{F}'$  of size at most  $|E|$ . Notice, in addition, that no member of  $\mathcal{F}'$  is contained in another one. Second, for the polynomial verification of the certificate, a result of Berge [6] says that a family of sets has the Helly property, if and only if for any triple of elements, the subfamily of sets containing at least two out of these three elements has nonempty intersection. Actually, by Lemma 2, it is enough to consider the triples of vertices  $a, b, c$  of  $G$  defining a triangle  $T$ . We consider the members of  $\mathcal{F}'_T$  and check for every vertex  $v$  of  $V$  if  $v$  belongs to  $\bigcap \mathcal{F}'_T$ . This produces an  $O(|V|^4|E|)$  algorithm that checks if a complete set edge cover  $\mathcal{F}'$  of size  $O(|E|)$  is Helly. Thus CLIQUE GRAPH belongs to NP.

A consequence of the above analysis is that a graph admits a complete set edge cover with the Helly property if and only if the graph admits a complete set edge cover with the Helly property such that no member is contained in another; such cover is called an RS-family of the graph. Thus Theorem 1 is equivalent to the following simpler statement:

$G$  is a clique graph if and only if  $G$  admits an RS-family.

The following two facts are stated and proved by Roberts and Spencer [36] and as we explain after each statement, they are explicitly used in our NP-completeness proof.

**Fact 3** (Lemma 1 of [36]). *Let  $\mathcal{F}$  be an RS-family of a graph  $G$ . Then  $\mathcal{F}$  contains a complete set of size 2 if and only if this complete set is a clique of  $G$ .*

Our constructed instance of CLIQUE GRAPH is a graph where every edge is in a triangle, which means that no complete set of size 2 is a clique. In our proof we apply the definition of RS-family and Fact 3 to our constructed graph to know that any possible RS-family does not contain complete sets of size 1 or 2.

**Fact 4** (Proof of Theorem 3 of [36]). *If a triangle  $T$  is a clique of  $G$ , then  $T$  is a member of every complete set edge cover of  $G$  that satisfies the Helly property.*

Our constructed instance of CLIQUE GRAPH has many auxiliary vertices of degree 2. Each vertex of degree 2 is contained in exactly one corresponding auxiliary triangle  $T$ , which implies that this auxiliary triangle  $T$  is a clique. In our proof we use Fact 4 to know that every such auxiliary triangle  $T$  of our constructed graph is a member of any possible RS-family.

We show that CLIQUE GRAPH is NP-complete by a reduction from the following version of the 3-satisfiability problem with at most 3 occurrences per variable. Let  $U = \{u_i, 1 \leq i \leq n\}$  be a set of boolean variables. A literal is either a variable  $u_i$  or its complement  $\bar{u}_i$ . A clause over  $U$  is a set of literals of  $L$ . Let  $C = \{c_j, 1 \leq j \leq m\}$  be a collection of clauses over  $U$ . We say that variable  $u_i$  occurs in clause  $c_j$  (and then in  $C$ ) if  $u_i$  or  $\bar{u}_i \in c_j$ . We say that variable  $u_i$  occurs in clause  $c_j$  as literal  $u_i$  (or that literal  $u_i$  occurs in  $c_j$ ) if  $u_i \in c_j$ , and as literal  $\bar{u}_i$  (or that literal  $\bar{u}_i$  occurs in  $c_j$ ) if  $\bar{u}_i \in c_j$ .

3SAT<sub>3</sub>

INSTANCE:  $I = (U, C)$ , where  $U = \{u_i, 1 \leq i \leq n\}$  is a set of boolean variables, and  $C = \{c_j, 1 \leq j \leq m\}$  a set of clauses over  $U$  such that each clause has two or three variables, each variable occurs at most three times in  $C$ .

QUESTION: Is there a truth assignment for  $U$  such that each clause in  $C$  has at least one true literal?

It is a known result that 3SAT<sub>3</sub> is an NP-complete problem [15,27]. Note that the instances of the chosen version of 3SAT must satisfy a linear dependency between parameters  $n$  and  $m$ : every 3SAT<sub>3</sub> instance satisfies  $2m \leq 3n$ .

In order to reduce 3SAT<sub>3</sub> to CLIQUE GRAPH we need to construct in polynomial time a particular instance  $G_I$  of CLIQUE GRAPH from a generic instance  $I = (U, C)$  of 3SAT<sub>3</sub>, in such a way that the constructed graph  $G_I$  is a clique graph if and only if  $C$  is satisfiable.

### 3. Construction of $G_I$ from $I = (U, C)$

Let  $I = (U, C)$  be any instance of 3SAT<sub>3</sub>. We assume with no loss of generality that each variable occurs two or three times in  $C$ , and no variable occurs twice in the same clause. In addition, if variable  $u_i$  occurs twice in  $C$ , then we assume it is once as literal  $u_i$  and once as literal  $\bar{u}_i$ ; and if variable  $u_i$  occurs three times in  $C$ , then we assume it is once as literal  $u_i$  and twice as literal  $\bar{u}_i$ .

For each variable  $u_i$ , let  $j_i$  be the subindex of the unique clause where variable  $u_i$  occurs as literal  $u_i$ ; and  $\bar{J}_i = \{j \mid \text{literal } \bar{u}_i \text{ occurs in } c_j\}$ .

For each clause  $c_j$  with  $|c_j| = 3$ , let  $I_j = \{i \mid \text{variable } u_i \text{ occurs in } c_j\}$ ; and for each clause  $c_j$  with  $|c_j| = 2$ , let  $I_j = \{i \mid \text{variable } u_i \text{ occurs in } c_j\} \cup \{n + 1\}$ . Notice that in any case  $|I_j| = 3$ . Given  $I_j = \{i_1, i_2, i_3\}$ , with  $i_1 < i_2 < i_3$ , let  $i_1^* = i_2$ ,  $i_2^* = i_3$  and  $i_3^* = i_1$ .

From instance  $I = (U, C)$ , we construct a graph  $G_I = (V, E)$  as follows (we give immediately after an easier alternative modular description of the graph  $G_I$ ):

The vertex set  $V$  is the union:

$$V = \bigcup_{1 \leq i \leq n} \{a_{j_i}^i, c_{j_i}^i, d_{j_i}^i, e_{j_i}^i, f_{j_i}^i, g_{j_i}^i, h_{j_i}^i\} \cup \bigcup_{1 \leq i \leq n} \bigcup_{j \in \bar{J}_i} \{a_j^i, c_j^i, d_j^i, e_j^i, f_j^i, g_j^i, h_j^i, z_j^i, v_j^i, w_j^i\} \\ \cup \bigcup_{1 \leq j \leq m, |c_j|=2} \{a_j^{n+1}, c_j^{n+1}, d_j^{n+1}, e_j^{n+1}, f_j^{n+1}, g_j^{n+1}, h_j^{n+1}\}.$$

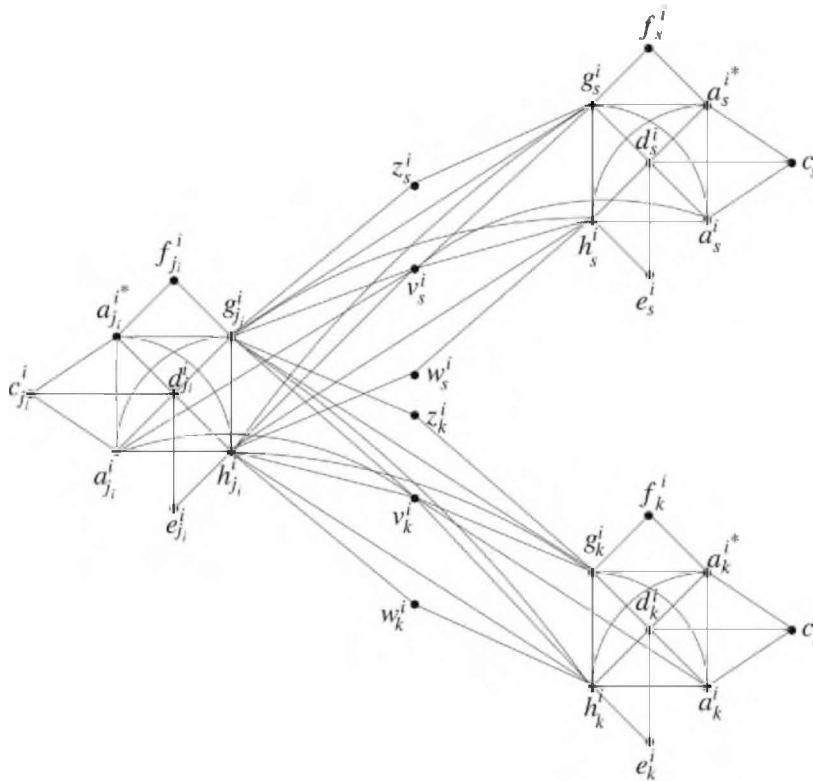


Fig. 2. Truth Setting component  $T_i$  for variable  $u_i$  with  $J_i = \{s, k\}$ .

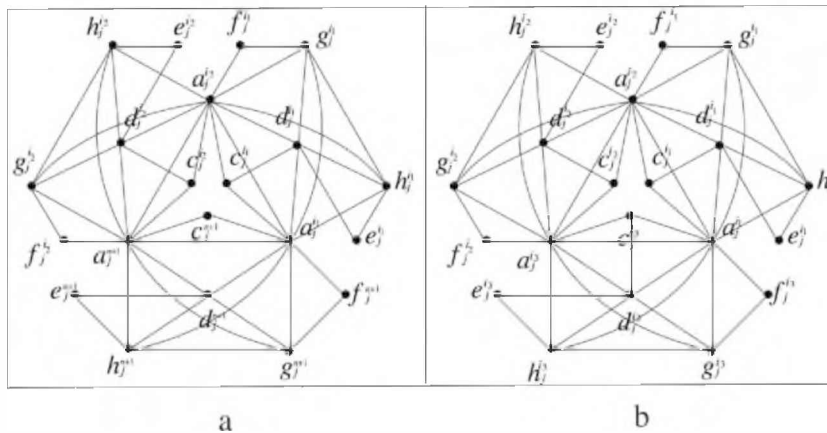


Fig. 3. Satisfaction Testing component  $S_j$  for clause  $c_j$  with: (a) 2 literals corresponding to variables  $u_{i_1}$  and  $u_{i_2}$ ; and (b) 3 literals corresponding to variables  $u_{i_1}$ ,  $u_{i_2}$  and  $u_{i_3}$ . Vertices  $\{a_j^i, d_j^i, g_j^i, h_j^i \mid i \in I_j\}$  induce a complete graph on 12 vertices but for simplicity some edges are not drawn.

Since  $|\bar{J}_i| \leq 2$ ,  $|V|$  is bounded by  $n \times 7 + n \times 2 \times 10 + m \times 7 = 27n + 7m$ . Actually, the linear dependency  $2m \leq 3n$  between parameters  $n$  and  $m$  gives a linear upper bound on  $n$ :  $|V|$  is bounded by  $27n + 7m \leq 27n + \frac{7}{2}n = \frac{75}{2}n$ .

The edge set  $E$  contains:

For each  $j$ ,  $1 \leq j \leq m$ , the edges of the complete graph induced by the vertex set  $K_{12}(j) = \{a_j^i, d_j^i, g_j^i, h_j^i \mid i \in I_j\}$ ; the edges of the sets  $\{c_j^i d_j^i \mid i \in I_j, i \neq n + 1\}$  and  $\{c_j^i a_j^i, c_j^i a_j^{i+1}, e_j^i d_j^i, e_j^i h_j^i, f_j^i g_j^i, f_j^i a_j^{i+1} \mid i \in I_j\}$ .

And for each  $i$ ,  $1 \leq i \leq n$ , for each  $j \in \bar{J}_i$ , the edges of the complete graph induced by the vertex set  $K_5(j, i) = \{h_j^i, g_j^i, v_j^i, h_j^i, g_j^i\}$ ; and the edges of the set  $\{h_j^i w_j^i, w_j^i h_j^i, g_j^i z_j^i, z_j^i g_j^i, a_j^i v_j^i, v_j^i a_j^i\}$ .

Notice that for each variable  $u_i$ , graph  $G_i$  contains an induced subgraph, Truth Setting component  $T_i$ , the graph depicted in Fig. 2; and for each clause  $c_j$ , graph  $G_j$  contains as induced subgraph, Satisfaction Testing component  $S_j$ , the graph depicted either in Fig. 3(a) or (b), depending on the number of variables in clause  $c_j$ . Notice that in that figure some edges have been omitted for simplicity. A subgraph  $T_i$  intersects a subgraph  $S_j$  if and only if variable  $u_i$  occurs in clause  $c_j$ ; and, in that case, the

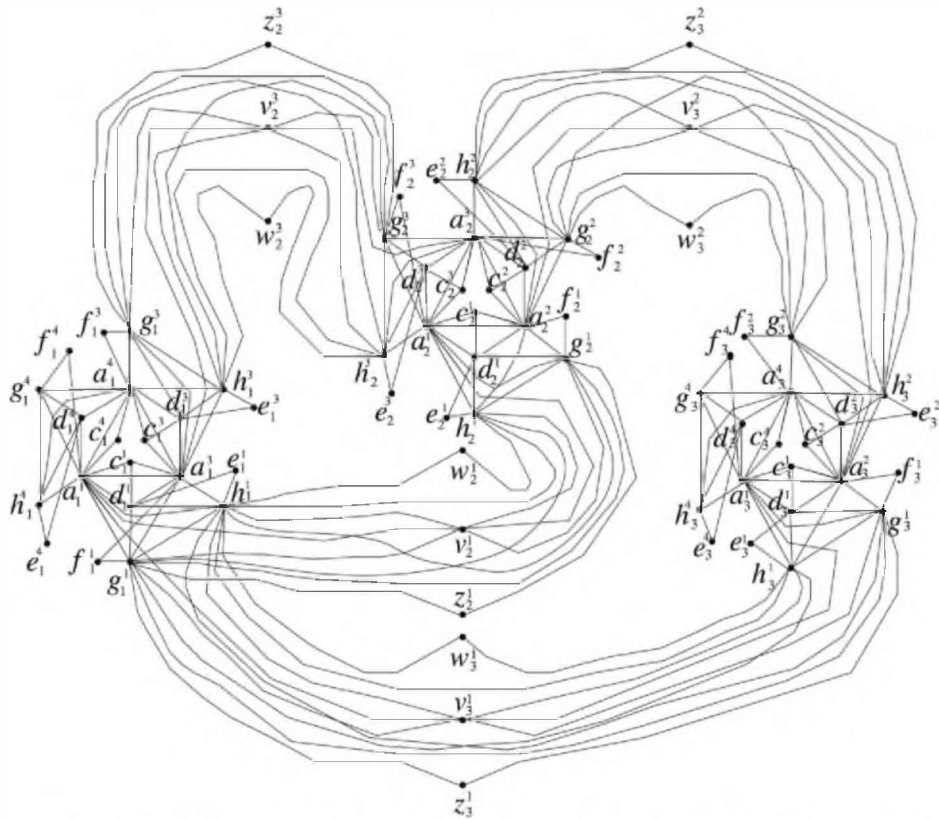


Fig. 4. Graph  $G_I$  obtained from the  $3SAT_7$  instance  $I = (U, C), U = \{u_1, u_2, u_3\}, C = \{\{u_1, u_3\}, \{\bar{u}_1, u_2, \bar{u}_3\}, \{\bar{u}_1, \bar{u}_2\}\}$ .

intersection is the subgraph induced by vertices  $a_j^i, a_j^{i'}, c_j^i, d_j^i, e_j^i, f_j^i, g_j^i$ , and  $h_j^i$ . We obtain the whole graph  $G_I$  by superposing the subgraphs  $T_i$  and  $S_j, 1 \leq i \leq n, 1 \leq j \leq m$ .

For the convenience of the reader we offer an example in Fig. 4 of graph  $G_I$  obtained from the instance  $I = (U, C), U = \{u_1, u_2, u_3\}, C = \{\{u_1, u_3\}, \{\bar{u}_1, u_2, \bar{u}_3\}, \{\bar{u}_1, \bar{u}_2\}\}$ .

In Section 3.1, we shall prove some properties about the RS-families of the constructed graph  $G_I$ . Note that in the constructed graph  $G_I$ , every edge is in a triangle, so by Fact 3 we know that every complete set in an RS-family of  $G_I$  has size at least 3. Note the presence in the constructed graph  $G_I$  of auxiliary vertices of degree 2:  $e_j^i, f_j^i, w_j^i, z_j^i$ . The auxiliary triangles containing those auxiliary degree 2 vertices are cliques and by Fact 4 must be present in every RS-family of the constructed graph  $G_I$ . For the convenience of the reader, we list all cliques of the constructed graph  $G_I$ .

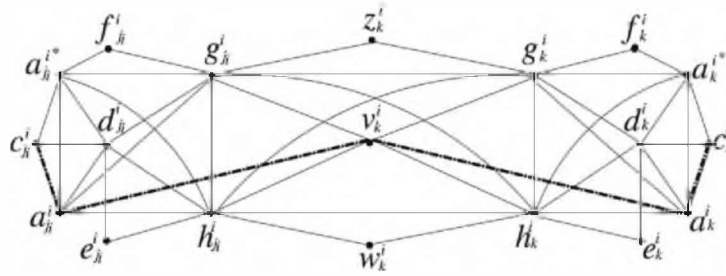
**Fact 5.** The cliques of  $G_I$  can be listed as follows:

- For each  $j, 1 \leq j \leq m$ , the complete set  $K_{12}(j) = \{a_j^i, d_j^i, g_j^i, h_j^i \mid i \in I_j\}$ ;
- For each  $i, 1 \leq i \leq n$ , for each  $j \in \bar{J}_i$ , the complete set  $K_5(j, i) = \{h_{j_1}^i, g_{j_1}^i, v_j^i, h_j^i, g_j^i\}$ ;
- For each  $j, 1 \leq j \leq m, i \in I_j, i \neq n + 1$ , one complete set  $K_4$  containing  $c_j^i: \{a_j^i, c_j^i, d_j^i, a_j^{i'}\}$ ;
- For each  $i, 1 \leq i \leq n$ , for each  $j \in \bar{J}_i$ , two complete sets  $K_4$  containing  $v_j^i: \{a_{j_1}^i, h_{j_1}^i, g_{j_1}^i, v_j^i\}, \{a_j^i, h_j^i, g_j^i, v_j^i\}$ ;
- All triangles with a vertex of degree 2. For each  $j, 1 \leq j \leq m$ , for each  $i \in I_j$ , the triangles  $\{f_j^i, a_j^{i'}, g_j^i\}, \{e_j^i, d_j^i, h_j^i\}$ ; and the triangle  $\{c_j^{n+1}, a_j^{n+1}, a_j^{n+1}\}$  when  $n + 1 \in I_j$ . For each  $i, 1 \leq i \leq n$ , for each  $j \in \bar{J}_i$ , the triangles  $\{g_{j_1}^i, z_j^i, g_j^i\}, \{h_{j_1}^i, w_j^i, h_j^i\}$ .

### 3.1. About RS-families of graph $G_I$

Our main theorem is proved in the next section by showing that the constructed graph  $G_I$  admits an RS-family if and only if there exists a truth assignment for  $U$  that satisfies  $C$ . The following lemmata are used in that proof when given an RS-family of  $G_I$  we exhibit a truth assignment for  $U$  that satisfies  $C$ . The truth value of each variable  $u_i$  will be assigned depending on the member of the RS-family of  $G_I$  covering the edge  $a_{j_1}^i c_{j_1}^i$ .

For the convenience of the reader, we recall our notation. The notation explicitly defines a correspondence between indices of variables and clauses of instance  $I$ , and the vertices of the constructed graph  $G_I$ . For each variable  $u_i$ , let  $j_i$  be the subindex of the unique clause where variable  $u_i$  occurs as literal  $u_i$ ; and  $\bar{J}_i = \{j \mid \text{literal } \bar{u}_i \text{ occurs in } c_j\}$ . For each clause  $c_j$  with



**Fig. 5.** Figure used in the proof of Lemma 6. Any RS-family of  $G_I$  must contain the highlighted triangles. In order to cover highlighted edge  $a_j^i c_j^i$ , exactly one of the triangles  $\{a_j^i, c_j^i, a_j^{i*}\}$  or  $\{a_j^i, c_j^i, d_j^i\}$  must belong to  $\mathcal{F}$ .

$|c_j| = 3$ , let  $I_j = \{i \mid \text{variable } u_i \text{ occurs in } c_j\}$ ; and for each clause  $c_j$  with  $|c_j| = 2$ , let  $I_j = \{i \mid \text{variable } u_i \text{ occurs in } c_j\} \cup \{n+1\}$ . Notice that in any case  $|I_j| = 3$ . Given  $I_j = \{i_1, i_2, i_3\}$ , with  $i_1 < i_2 < i_3$ , let  $i_1^* = i_2, i_2^* = i_3$  and  $i_3^* = i_1$ .

Notice that, in case  $i = n + 1$ , vertex  $c_j^{n+1}$  is of degree 2 contained in precisely one triangle  $\{a_j^{n+1}, c_j^{n+1}, a_j^{(n+1)*}\}$  which is the only clique covering edge  $a_j^{n+1} c_j^{n+1}$ . From Fact 4, triangle  $\{a_j^{n+1}, c_j^{n+1}, a_j^{(n+1)*}\}$  must belong to any RS-family. Lemma 6 considers the possible ways an RS-family may cover edge  $a_j^i c_j^i$ , when  $i \neq n + 1$ .

**Lemma 6 (Two Cover Lemma).** Let  $\mathcal{F}$  be an RS-family of the graph  $G_I$ . For each  $j, 1 \leq j \leq m$ , and for each  $i \in I_j, i \neq n + 1$ , exactly one of the triangles  $\{a_j^i, c_j^i, a_j^{i*}\}, \{a_j^i, c_j^i, d_j^i\}$  belongs to  $\mathcal{F}$ .

**Proof.** Consider any  $j, 1 \leq j \leq m$ , and  $i \in I_j, i \neq n + 1$ . Assume with no loss of generality,  $j = j_i$ , and refer to the left side of Fig. 5. Indeed, to prove the case  $j \in \bar{J}_i$ , refer to the right side of Fig. 5 accordingly.

We have to prove that exactly one of the triangles  $\{a_j^i, c_j^i, a_j^{i*}\}, \{a_j^i, c_j^i, d_j^i\}$  of Fig. 5 belongs to  $\mathcal{F}$ .

Notice that, by Fact 4, triangles  $\{d_j^i, e_j^i, h_j^i\}$  and  $\{a_j^{i*}, f_j^i, g_j^i\}$  highlighted in Fig. 5 belong to  $\mathcal{F}$ , as  $e_j^i$  and  $f_j^i$  are vertices of degree 2 in  $G_I$ .

From Facts 3 and 5, in order to cover the highlighted edge  $a_j^i c_j^i$ , at least one of the three complete sets  $\{a_j^i, c_j^i, a_j^{i*}\}, \{a_j^i, c_j^i, d_j^i\}$  or  $\{a_j^i, c_j^i, a_j^{i*}\}$  must belong to  $\mathcal{F}$ .

We claim that the complete set  $\{a_j^i, c_j^i, a_j^{i*}, d_j^i\} \notin \mathcal{F}$ .

Indeed, from Facts 3 and 5, in order to cover the highlighted edge  $a_j^i v_k^i$ , at least one of the three complete sets  $\{a_j^i, g_j^i, v_k^i, h_j^i\}, \{a_j^i, g_j^i, v_k^i\}$ , or  $\{a_j^i, h_j^i, v_k^i\}$  must belong to  $\mathcal{F}$ . Now, assuming  $\{a_j^i, c_j^i, a_j^{i*}, d_j^i\} \in \mathcal{F}$ , implies that no one of these three complete sets is a member of  $\mathcal{F}$ , as each of the following three subfamilies:

- $\{\{a_j^i, g_j^i, v_k^i, h_j^i\}, \{a_j^i, c_j^i, a_j^{i*}, d_j^i\}, \{a_j^{i*}, f_j^i, g_j^i\}\},$
- $\{\{a_j^i, g_j^i, v_k^i\}, \{a_j^i, c_j^i, a_j^{i*}, d_j^i\}, \{a_j^{i*}, f_j^i, g_j^i\}\},$  and
- $\{\{a_j^i, h_j^i, v_k^i\}, \{a_j^i, c_j^i, a_j^{i*}, d_j^i\}, \{d_j^i, e_j^i, h_j^i\}\}$  violate the Helly property.

It follows that  $\{a_j^i, c_j^i, a_j^{i*}, d_j^i\} \notin \mathcal{F}$ , and that it remains to prove that both triangles  $\{a_j^i, c_j^i, a_j^{i*}\}$  and  $\{a_j^i, c_j^i, d_j^i\}$  are not together in  $\mathcal{F}$ .

Indeed, again by considering the highlighted edge  $a_j^i v_k^i$ , the presence of triangles:  $\{a_j^{i*}, f_j^i, g_j^i\}, \{d_j^i, e_j^i, h_j^i\}, \{a_j^i, c_j^i, a_j^{i*}\},$  and  $\{a_j^i, c_j^i, d_j^i\}$  together in  $\mathcal{F}$  implies that none of the three complete sets that may possibly cover  $a_j^i v_k^i$  is a member of  $\mathcal{F}$ , as each of the following three subfamilies:

- $\{\{a_j^i, g_j^i, v_k^i, h_j^i\}, \{a_j^i, c_j^i, a_j^{i*}\}, \{a_j^{i*}, f_j^i, g_j^i\}\},$
- $\{\{a_j^i, g_j^i, v_k^i\}, \{a_j^i, c_j^i, a_j^{i*}\}, \{a_j^{i*}, f_j^i, g_j^i\}\},$  and
- $\{\{a_j^i, h_j^i, v_k^i\}, \{a_j^i, c_j^i, d_j^i\}, \{d_j^i, e_j^i, h_j^i\}\}$  violate the Helly property.

It follows that exactly one of the triangles  $\{a_j^i, c_j^i, a_j^{i*}\}$  or  $\{a_j^i, c_j^i, d_j^i\}$  belongs to  $\mathcal{F}$ ; and the proof is completed.  $\square$

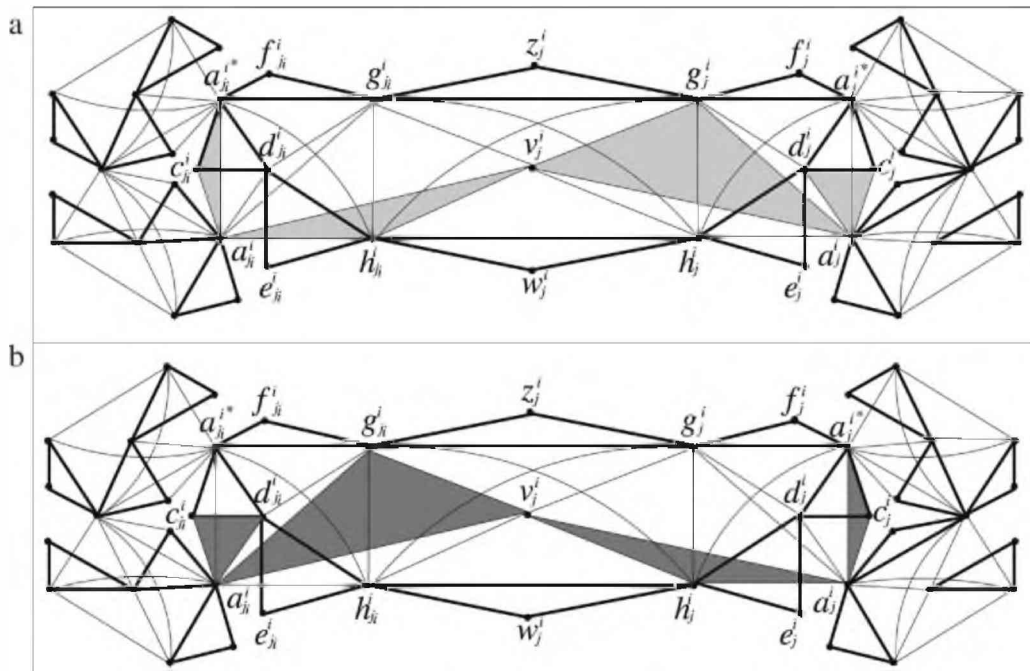
**Lemma 7 (Literal Communication Lemma).** Let  $\mathcal{F}$  be an RS-family of the graph  $G_I$ . For each  $i, 1 \leq i \leq n$ , and for each  $j \in \bar{J}_i$ , if  $\{a_j^i, c_j^i, d_j^i\} \in \mathcal{F}$  then  $\{a_j^i, c_j^i, a_j^{i*}\} \in \mathcal{F}$ , (Fig. 6(a)), and if  $\{a_j^i, c_j^i, a_j^{i*}\} \in \mathcal{F}$  then  $\{a_j^i, c_j^i, d_j^i\} \in \mathcal{F}$ , (Fig. 6(b)).

**Proof.** Please refer to Fig. 6. Consider any  $i, 1 \leq i \leq n$ , and  $j \in \bar{J}_i$ . First assume  $\{a_j^i, c_j^i, d_j^i\} \in \mathcal{F}$  and refer to the right side of Fig. 6(a), where the assumed triangle  $\{a_j^i, c_j^i, d_j^i\}$  is a filled triangle.

Notice that, by Fact 4, triangles  $\{e_j^i, h_j^i, d_j^i\}, \{g_j^i, g_j^i, z_j^i\}, \{e_j^i, h_j^i, d_j^i\}$  highlighted in Fig. 6 belong to  $\mathcal{F}$ , as  $e_j^i, z_j^i$  and  $e_j^i$  are vertices of degree 2 in  $G_I$ .

As in the proof of the previous Lemma, in order to cover edge  $a_j^i v_j^i$ , the complete set  $\{a_j^i, g_j^i, v_j^i\}$  must be in  $\mathcal{F}$ , as each of the following two subfamilies:

- $\{\{a_j^i, h_j^i, g_j^i, v_j^i\}, \{a_j^i, c_j^i, d_j^i\}, \{e_j^i, h_j^i, d_j^i\}\},$  and
- $\{\{a_j^i, h_j^i, v_j^i\}, \{a_j^i, c_j^i, d_j^i\}, \{e_j^i, h_j^i, d_j^i\}\}$  violate the Helly property.



**Fig. 6.** Figure used in the proof of Lemma 7. For any variable  $u_i$  and  $j \in \bar{J}_i$ , any RS-family of  $G_i$  must contain either the filled triangles in (a) or the filled triangles in (b). In any case the bold triangles must belong to the RS-family.

Now, again, in order to cover the edge  $a_j^l v_j^l$ , the complete set  $\{a_j^l, h_j^l, v_j^l\}$  must be in  $\mathcal{F}$ , as each of the following two subfamilies:

- $\{\{a_j^l, g_j^l, v_j^l, h_j^l\}, \{a_j^l, g_j^l, v_j^l\}, \{g_j^l, g_j^l, z_j^l\}\}$ , and
- $\{\{a_j^l, g_j^l, v_j^l\}, \{a_j^l, g_j^l, v_j^l\}, \{g_j^l, g_j^l, z_j^l\}\}$  violate de Helly property.

So far, the assumption of triangle  $\{a_j^l, c_j^l, d_j^l\} \in \mathcal{F}$ , forced that the triangles  $\{a_j^l, g_j^l, v_j^l\}$  and  $\{a_j^l, h_j^l, v_j^l\}$  are both in  $\mathcal{F}$ . Note these three triangles are filled triangles in Fig. 6(a).

Please refer to the left side of Fig. 6(a). By the previous lemma, we know that precisely one of the two triangles  $\{a_j^l, c_j^l, a_j^r\}$  or  $\{a_j^l, c_j^l, d_j^l\}$  belongs to  $\mathcal{F}$ .

Since the assumption of  $\{a_j^l, c_j^l, d_j^l\} \in \mathcal{F}$  implies a subfamily  $\{\{a_j^l, h_j^l, v_j^l\}, \{e_j^l, d_j^l, h_j^l\}, \{a_j^l, c_j^l, d_j^l\}\}$  that violates de Helly property, we conclude that  $\{a_j^l, c_j^l, a_j^r\} \in \mathcal{F}$ . Note, this is the fourth filled triangle in Fig. 6(a).

To get the second implication, an analogous argument refers to Fig. 6(b), starts with the assumption of  $\{a_j^l, c_j^l, a_j^r\} \in \mathcal{F}$ , a filled triangle on the right side of Fig. 6(b), and obtains as a consequence and in turn the other three filled triangles of Fig. 6(b):  $\{a_j^l, h_j^l, v_j^l\}$ ,  $\{a_j^l, g_j^l, v_j^l\}$ , and finally  $\{a_j^l, c_j^l, d_j^l\}$ . □

These two lemmata are the basis of the proof of the main theorem. The first implies that given any variable  $u_i$  and any clause  $c_j$  where  $u_i$  occurs, any RS-family of  $G_i$  is forced to choose exactly one of the triangles  $\{a_j^l, a_j^r, c_j^l\}$ ,  $\{a_j^l, c_j^l, d_j^l\}$  to cover the edge  $a_j^l c_j^l$ . The second lemma implies that if  $c_k$  and  $c_{h_i}$  are clauses where variable  $u_i$  occurs as different literals, then any RS-family of  $G_i$  is forced to choose different types of triangles to cover respectively the edges  $a_k^l c_k^l$  and  $a_{h_i}^l c_{h_i}^l$ . It follows from the structure of the Truth Setting component  $T_i$ , that if  $c_s$  and  $c_k$  are clauses where variable  $u_i$  occurs as the same literal, then any RS-family of  $G_i$  is forced to choose the same type of triangle to cover the edges  $a_s^l c_s^l$  and  $a_k^l c_k^l$ , respectively. The correspondence between the two possible truth assignments of variable  $u_i$  and the two possible triangles used to cover the edge  $a_j^l c_j^l$  is clear.

**4. Main theorem**

**Theorem 8.** CLIQUE GRAPH is NP-complete.

**Proof.** As shown in Section 2, CLIQUE GRAPH belongs to NP.

Given any instance  $I = (U, C)$  of 3SAT<sub>3</sub>, let  $G_I$  be the graph obtained by Section 3 process. We show that  $G_I$  is a clique graph if and only if  $C$  is satisfiable.

First, suppose  $G_I$  is a clique graph and let  $\mathcal{F}$  be an RS-family for  $G_I$ . We exhibit a truth assignment for  $U$  that satisfies  $C$ : For each variable  $u_i \in U$ , set

$$u_i \text{ equal to true if and only if } \{a_{h_i}^l, c_{h_i}^l, d_{h_i}^l\} \in \mathcal{F}. \tag{1}$$

To see that this truth assignment satisfies  $C$  consider a clause  $c_j$ .

Please refer to Fig. 3. The Helly property of  $\mathcal{F}$  implies there exists  $i \in I_j$  such that the triangle  $\{a_j^i, c_j^i, a_j^{i^*}\} \notin \mathcal{F}$ ; notice that  $i \neq n + 1$ , because  $c_j^{n+1}$  is a vertex of degree 2 in  $G_j$ . It follows, by the Two Cover Lemma, that

$$\{a_j^i, c_j^i, d_j^i\} \in \mathcal{F}. \tag{2}$$

Observe that, since  $i \neq n + 1$ , subindex  $i$  corresponds to a variable  $u_i$  which occurs in clause  $c_j$ . There are two possibilities:

$u_i$  occurs as literal  $u_i$  in  $c_j$ : then  $j = j_i$  and condition (2) says  $\{a_{j_i}^i, c_{j_i}^i, d_{j_i}^i\} \in \mathcal{F}$ . Then, by (1), variable  $u_i$  is true; and clause  $c_j$  is satisfied.

$u_i$  occurs as literal  $\bar{u}_i$  in  $c_j$ : then  $j \in \bar{J}_i$ .

By condition (2) and Literal Communication Lemma,  $\{a_{j_i}^i, a_{j_i}^{i^*}, c_{j_i}^i\} \in \mathcal{F}$ ; then, by Two Cover Lemma,  $\{a_{j_i}^i, c_{j_i}^i, d_{j_i}^i\} \notin \mathcal{F}$ ; thus, by (1),  $u_i$  is false; and clause  $c_j$  is satisfied.

Conversely, given a truth assignment of  $U$  that satisfies  $C$ , we exhibit a complete set edge cover  $\mathcal{F}$  of  $G_j$ .

For each  $j$ ,  $1 \leq j \leq m$ , complete set  $K_{12}(j) = \{a_j^i, d_j^i, g_j^i, h_j^i \mid i \in I_j\}$ .

For each  $j$ ,  $1 \leq j \leq m$ , for each  $i \in I_j$ , the triangles  $\{f_j^i, a_j^{i^*}, g_j^i\}$ ,  $\{e_j^i, d_j^i, h_j^i\}$ .

For each  $j$ ,  $1 \leq j \leq m$ , for each  $i \in I_j$ ,  $i \neq n + 1$ ,  $\{c_j^i, a_j^i, d_j^i\}$ ; and for  $i = n + 1$ ,  $\{c_j^{n+1}, a_j^{(n+1)^*}, a_j^{n+1}\}$ .

For each  $i$ ,  $1 \leq i \leq n$ , for each  $j \in \bar{J}_i$ , the complete set  $K_5(j, i) = \{h_{j_i}^i, g_{j_i}^i, v_j^i, h_j^i, g_j^i\}$ .

For each  $i$ ,  $1 \leq i \leq n$ , for each  $j \in \bar{J}_i$ ,  $\{g_{j_i}^i, z_j^i, g_j^i\}$ ,  $\{h_{j_i}^i, w_j^i, h_j^i\}$ .

For each  $i$ ,  $1 \leq i \leq n$ , such that variable  $u_i$  is true,  $\{c_{j_i}^i, d_{j_i}^i, a_{j_i}^i\}$ ; and for each  $j \in \bar{J}_i$ ,  $\{a_{j_i}^i, g_{j_i}^i, v_j^i\}$ ,  $\{v_j^i, h_{j_i}^i, a_j^i\}$ ,  $\{a_j^i, a_{j_i}^{i^*}, c_j^i\}$ .

For each  $i$ ,  $1 \leq i \leq n$ , such that variable  $u_i$  is false,  $\{c_{j_i}^i, a_{j_i}^{i^*}, a_{j_i}^i\}$ ; and for each  $j \in \bar{J}_i$ ,  $\{a_{j_i}^i, h_{j_i}^i, v_j^i\}$ ,  $\{v_j^i, g_{j_i}^i, a_j^i\}$ ,  $\{a_j^i, d_{j_i}^i, c_j^i\}$ .

Please refer to Fig. 6. Notice that the triangles depicted in bold are present in the above defined complete set edge cover regardless of the truth assignment. On the other hand, according to the truth assignment of variable  $u_i$ , in the above defined complete set edge cover, there are the filled triangles depicted in either Fig. 6(a) or (b). Observe that, when  $i \neq n + 1$ , according to the truth assignment of variable  $u_i$ , precisely one of the triangles  $\{c_{j_i}^i, d_{j_i}^i, a_{j_i}^i\}$  or  $\{c_{j_i}^i, a_{j_i}^{i^*}, a_{j_i}^i\}$  is selected in order to cover the edge  $a_{j_i}^i c_{j_i}^i$ . On the other hand, when  $i = n + 1$ , the triangle  $\{c_j^{n+1}, a_j^{(n+1)^*}, a_j^{n+1}\}$  is always selected to cover the edge  $a_j^{n+1} c_j^{n+1}$ , regardless of the truth assignment.

Observe that in particular given a 2-sized clause  $c_j = \{u_{i_1}, u_{i_2}\}$ ,  $i_1 < i_2$ , a satisfiable truth assignment must set  $u_{i_1}$  or  $u_{i_2}$  to true, hence the complete set edge cover  $\mathcal{F}$  defines the forced triangles  $\{c_j^{i_1}, d_j^{i_1}, c_j^{i_1}\}$ ,  $\{a_j^{i_2}, d_j^{i_2}, c_j^{i_2}\}$ , and according to  $u_{i_1}$  or  $u_{i_2}$  be set to true respectively triangles  $\{a_j^{i_1}, d_j^{i_1}, c_j^{i_1}\}$  or  $\{a_j^{i_2}, d_j^{i_2}, c_j^{i_2}\}$  in the corresponding Satisfaction Testing component  $S_j$ .

The proof is concluded by showing that the complete set edge cover  $\mathcal{F}$  of  $G_j$  has the Helly property. By Lemma 2, it is enough to show that for each triangle  $T \in T(G_j)$ ,  $\bigcap \mathcal{F}_T \neq \emptyset$ .

If a triangle  $T$  contains an edge  $e$  for which any complete set of  $\mathcal{F}$  covering  $e$  contains also  $T$ , then  $\bigcap \mathcal{F}_T \neq \emptyset$ . We call such a triangle an *easy triangle*. Note, in particular, that if  $T$  is a triangle of  $G_j$  with a vertex of degree 2, and  $e$  is any of the two edges of  $T$  incident to the vertex of degree 2, then the only member of the defined family  $\mathcal{F}$  covering  $e$  is the triangle  $T$  itself – which clearly contains  $T$ . It follows that the triangles of  $G_j$  containing a vertex of degree 2 are easy triangles.

For the analysis below, please refer to Fig. 6. We classify the triangles of  $G_j$  into types according to either they are, or they are not contained in a  $K_{12}(j)$  or in a  $K_5(j, i)$ .

(1) First we consider the triangles of  $G_j$  which are not contained in a  $K_{12}(j)$  nor in a  $K_5(j, i)$ . These can be classified as follows.

(a) Triangles containing  $c_j^i$ ,  $1 \leq j \leq m$ ,  $i \in I_j$ ,  $i \neq n + 1$ . Note, in this case, vertex  $c_j^i$  is of degree 4, and is contained in precisely 3 triangles of  $G_j$ :

$$\{c_j^i, a_j^i, a_j^{i^*}\}, \{c_j^i, a_j^i, d_j^i\}, \{c_j^i, a_j^{i^*}, d_j^i\}.$$

For any of these three types of triangles  $T$  the members of  $\mathcal{F}_T$  are:

If either  $u_i$  is true and occurs as literal  $u_i$  in  $c_j$ , or  $u_i$  is false and occurs as literal  $\bar{u}_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:

$$K_{12}(j), \{c_j^i, a_j^i, d_j^i\}, \{a_j^i, c_j^i, d_j^i\}. \text{ Thus } d_j^i \in \bigcap \mathcal{F}_T.$$

If either  $u_i$  is true and occurs as literal  $\bar{u}_i$  in  $c_j$ , or  $u_i$  is false and occurs as literal  $u_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:

$$K_{12}(j), \{c_j^i, a_j^{i^*}, d_j^i\}, \{a_j^i, c_j^i, a_j^{i^*}\}. \text{ Thus } a_j^i \in \bigcap \mathcal{F}_T.$$

(b) Triangles containing  $c_j^{n+1}$ ,  $1 \leq j \leq m$ ,  $n + 1 \in I_j$ :

$$\{c_j^{n+1}, a_j^{n+1}, a_j^{(n+1)^*}\}. \text{ All these triangles are easy.}$$

(c) Triangles containing  $e_j^i$  or  $f_j^i$  or  $w_j^i$  or  $z_j^i$ ,  $1 \leq j \leq m$ ,  $i \in I_j$ :

$$\{e_j^i, d_j^i, h_j^i\}, \{f_j^i, a_j^{i^*}, g_j^i\}, \{w_j^i, h_{j_i}^i, h_j^i\}, \{z_j^i, g_{j_i}^i, g_j^i\}. \text{ All these triangles are easy.}$$

(d) Triangles containing  $v_j^i$ ,  $1 \leq i \leq n$ ,  $j \in \bar{J}_i$ ,

$$\{v_j^i, g_{j_i}^i, a_{j_i}^i\}, \{v_j^i, h_{j_i}^i, a_{j_i}^i\}, \{v_j^i, g_j^i, a_j^i\}, \{v_j^i, h_j^i, a_j^i\};$$



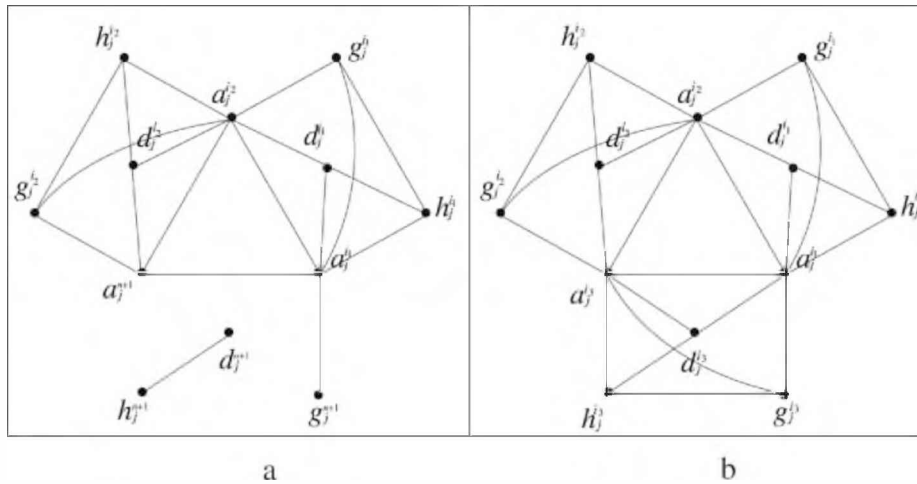


Fig. 7. Triangles of  $K_{12}(j)$  which may be not easy, they contain edges covered not only by the  $K_{12}(j)$  itself but also by other members of  $\mathcal{F}$ .

(i)  $T = \{v_j^i, g_j^i, a_j^i\}$ .

If  $u_i$  is true, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j_i), K_5(j, i), \{a_j^i, g_j^i, v_j^i\}$ , where  $k \in \bar{J}_i$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ . If  $u_i$  is false, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j_i), K_5(j, i), \{a_j^i, h_j^i, v_j^i\}$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ .

(ii)  $T = \{v_j^i, h_j^i, a_j^i\}$ :

If  $u_i$  is true, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j_i), K_5(j, i), \{a_j^i, g_j^i, v_j^i\}$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ . If  $u_i$  is false, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j_i), K_5(j, i), \{a_j^i, h_j^i, v_j^i\}$ , where  $k \in \bar{J}_i$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ .

(iii)  $T = \{v_j^i, g_j^i, a_j^i\}$ :

If  $u_i$  is true, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(j, i), \{v_j^i, h_j^i, a_j^i\}$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ . If  $u_i$  is false, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(j, i), T$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ .

(iv)  $T = \{v_j^i, h_j^i, a_j^i\}$ :

If  $u_i$  is true, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(j, i), T$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ . If  $u_i$  is false, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(j, i), \{v_j^i, g_j^i, a_j^i\}$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ .

(2) Consider the triangles of  $G_I$  contained in a  $K_{12}(j)$  or in a  $K_5(j, i)$ .

(a) Now we study the triangles contained in  $K_{12}(j) = \{a_j^i, d_j^i, g_j^i, h_j^i \mid i \in I_j\}, 1 \leq j \leq m$ . Among these triangles the ones with at least one edge covered only by  $K_{12}(j)$  are easy triangles. The remaining triangles are the triangles whose three edges are edges of  $K_{12}(j)$  covered by a complete set of  $\mathcal{F}$  besides  $K_{12}(j)$  itself.

Notice, by looking at the list of sets of the complete set edge cover  $\mathcal{F}$  or by looking at Fig. 6, that the only edges of  $K_{12}(j)$  that may be covered by a complete set of  $\mathcal{F}$  besides  $K_{12}(j)$  are:

$a_j^i a_j^i, a_j^i d_j^i, a_j^i g_j^i, a_j^i h_j^i, a_j^i g_j^i, a_j^i d_j^i, h_j^i d_j^i, h_j^i g_j^i$ , for  $i \neq n + 1$ ; and  
 $a_j^i a_j^i, a_j^i g_j^i, h_j^i d_j^i$ , for  $i = n + 1$ , which are depicted in Fig. 7.

Thus the triangles of  $K_{12}(j)$  which are not easy triangles can be classified into only five types, as follows (see Fig. 7):

(i)  $T = \{a_j^i, i \in I_j\}$ .

Since  $c_j$  is satisfied there exists  $i \in I_j$  such that either  $u_i$  occurs as literal  $u_i$  in  $c_j$  and variable  $u_i$  is true or  $u_i$  occurs as literal  $\bar{u}_i$  in  $c_j$  and variable  $u_i$  is false. In any case, by the construction of  $\mathcal{F}$ ,  $a_j^i a_j^i$  is covered only by  $K_{12}(j)$ . Since  $K_{12}(j)$  also contains the third vertex of  $T$ , it follows that  $T$  is an easy triangle.

(ii)  $T = \{a_j^i, a_j^i, d_j^i\}, i \in I_j, i \neq n + 1$ .

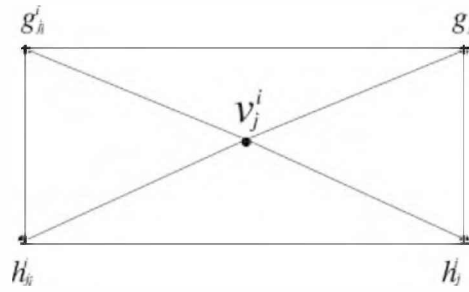
If either  $u_i$  is true and occurs as literal  $u_i$  in  $c_j$ , or  $u_i$  is false and occurs as literal  $\bar{u}_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), \{c_j^i, a_j^i, d_j^i\}, \{a_j^i, c_j^i, d_j^i\}$ . Thus  $d_j^i \in \bigcap \mathcal{F}_T$ .

If either  $u_i$  is true and occurs as literal  $\bar{u}_i$  in  $c_j$ , or  $u_i$  is false and occurs as literal  $u_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), \{c_j^i, a_j^i, d_j^i\}, \{a_j^i, c_j^i, a_j^i\}$ . Thus  $a_j^i \in \bigcap \mathcal{F}_T$ .

(iii)  $T = \{a_j^i, d_j^i, h_j^i\}, i \in I_j, i \neq n + 1$ .

If either  $u_i$  is true and occurs as literal  $u_i$  in  $c_j$ , or  $u_i$  is false and occurs as literal  $\bar{u}_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), \{e_j^i, h_j^i, d_j^i\}, \{a_j^i, c_j^i, d_j^i\}$ . Thus  $d_j^i \in \bigcap \mathcal{F}_T$ .

If  $u_i$  is true and occurs as literal  $\bar{u}_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), \{e_j^i, h_j^i, d_j^i\}, \{a_j^i, h_j^i, v_j^i\}$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ . If  $u_i$  is false and occurs as literal  $u_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), \{e_j^i, h_j^i, d_j^i\}, \{a_j^i, h_j^i, v_k^i\}, k \in \bar{J}_i$ . Please



**Fig. 8.** Triangles of  $K_5(j, i)$  which may be not easy, they contain edges covered not only by the  $K_5(j, i)$  itself but also by other members of  $\mathcal{F}$ .

refer to the proposed complete set edge cover  $\mathcal{F}$ , where  $\{a_j^i, h_j^i, v_k^i\}$  is listed in the 7th item as  $\{a_{j_1}^i, h_{j_1}^i, v_j^i\}$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ .

(iv)  $T = \{a_j^i, a_j^i, g_j^i\}, i \in I_j, i \neq n + 1.$

If  $u_i$  is true and occurs as literal  $u_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), \{a_j^i, f_j^i, g_j^i\}, \{a_j^i, g_j^i, v_k^i\}, k \in \bar{J}_i$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ .

If either  $u_i$  is true and occurs as literal  $\bar{u}_i$  in  $c_j$ , or  $u_i$  is false and occurs as literal  $u_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), \{a_j^i, f_j^i, g_j^i\}, \{a_j^i, a_j^i, c_j^i\}$ . Thus  $a_j^i \in \bigcap \mathcal{F}_T$ .

If  $u_i$  is false and occurs as literal  $\bar{u}_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), \{a_j^i, f_j^i, g_j^i\}, \{a_j^i, g_j^i, v_j^i\}$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ .

(v)  $T = \{a_j^i, h_j^i, g_j^i\}, i \in I_j, i \neq n + 1.$

If  $u_i$  is true and occurs as literal  $u_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(k, i), \{a_k^i, g_k^i, v_k^i\}, k \in \bar{J}_i$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ .

If  $u_i$  is false and occurs as literal  $u_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(k, i), \{a_k^i, h_k^i, v_k^i\}, k \in \bar{J}_i$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ .

If  $u_i$  is true and occurs as literal  $\bar{u}_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(j, i), \{a_j^i, h_j^i, v_j^i\}$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ .

If  $u_i$  is false and occurs as literal  $\bar{u}_i$  in  $c_j$ , the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(j, i), \{a_j^i, g_j^i, v_j^i\}$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ .

(b) Now we consider the triangles of  $G_i$  which are contained in a  $K_5(j, i) = \{h_{j_1}^i, g_{j_1}^i, v_j^i, h_j^i, g_j^i\}$ , for each  $1 \leq i \leq n$  and  $j \in \bar{J}_i$ . Among these triangles the ones with at least one edge covered only by  $K_5(j, i)$  are easy triangles. Notice the edges  $g_{j_1}^i h_{j_1}^i$  and  $h_{j_1}^i g_j^i$  are covered only by  $K_5(j, i)$ , which implies that six triangles formed by one of these two edges and an additional vertex of  $K_5(j, i)$  are all easy. Every edge of the remaining four triangles may be covered by a complete set besides  $K_5(j, i)$ . Notice that such edges are:  $g_{j_1}^i g_j^i, g_{j_1}^i v_j^i, g_{j_1}^i h_{j_1}^i, h_{j_1}^i v_j^i, h_{j_1}^i g_j^i, h_{j_1}^i h_{j_1}^i, v_j^i h_{j_1}^i, v_j^i g_j^i$ , which are depicted in Fig. 8.

Thus we study four types of triangles contained in  $K_5(j, i)$  (see Fig. 8):  $\{g_{j_1}^i, g_j^i, v_j^i\}, \{h_{j_1}^i, h_j^i, v_j^i\}, \{g_{j_1}^i, h_{j_1}^i, v_j^i\}, \{g_j^i, h_j^i, v_j^i\}$ .

(i)  $T = \{g_{j_1}^i, g_j^i, v_j^i\}$ .

If  $u_i$  is true, then the members of  $\mathcal{F}_T$  are:  $K_5(j, i), \{g_{j_1}^i, g_j^i, z_j^i\}, \{a_{j_1}^i, g_{j_1}^i, v_j^i\}$ . Thus  $g_{j_1}^i \in \bigcap \mathcal{F}_T$ . If  $u_i$  is false, then the members of  $\mathcal{F}_T$  are:  $K_5(j, i), \{g_{j_1}^i, g_j^i, z_j^i\}, \{v_j^i, g_j^i, a_j^i\}$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ .

(ii)  $T = \{h_{j_1}^i, h_j^i, v_j^i\}$ .

If  $u_i$  is true, then the members of  $\mathcal{F}_T$  are:  $K_5(j, i), \{h_{j_1}^i, h_j^i, w_j^i\}, \{v_j^i, h_j^i, a_j^i\}$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ . If  $u_i$  is false, then the members of  $\mathcal{F}_T$  are:  $K_5(j, i), \{h_{j_1}^i, h_j^i, w_j^i\}, \{a_{j_1}^i, v_j^i, h_{j_1}^i\}$ . Thus  $h_{j_1}^i \in \bigcap \mathcal{F}_T$ .

(iii)  $T = \{g_{j_1}^i, h_{j_1}^i, v_j^i\}$ .

If  $u_i$  is true, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j), \{a_{j_1}^i, g_{j_1}^i, v_j^i\}, K_5(k, i)$ , where  $k \in \bar{J}_i$ . Thus  $g_{j_1}^i \in \bigcap \mathcal{F}_T$ . If  $u_i$  is false, then the members of  $\mathcal{F}_T$  are  $K_{12}(j), \{a_{j_1}^i, h_{j_1}^i, v_j^i\}, K_5(k, i)$ , where  $k \in \bar{J}_i$ . Thus  $h_{j_1}^i \in \bigcap \mathcal{F}_T$ .

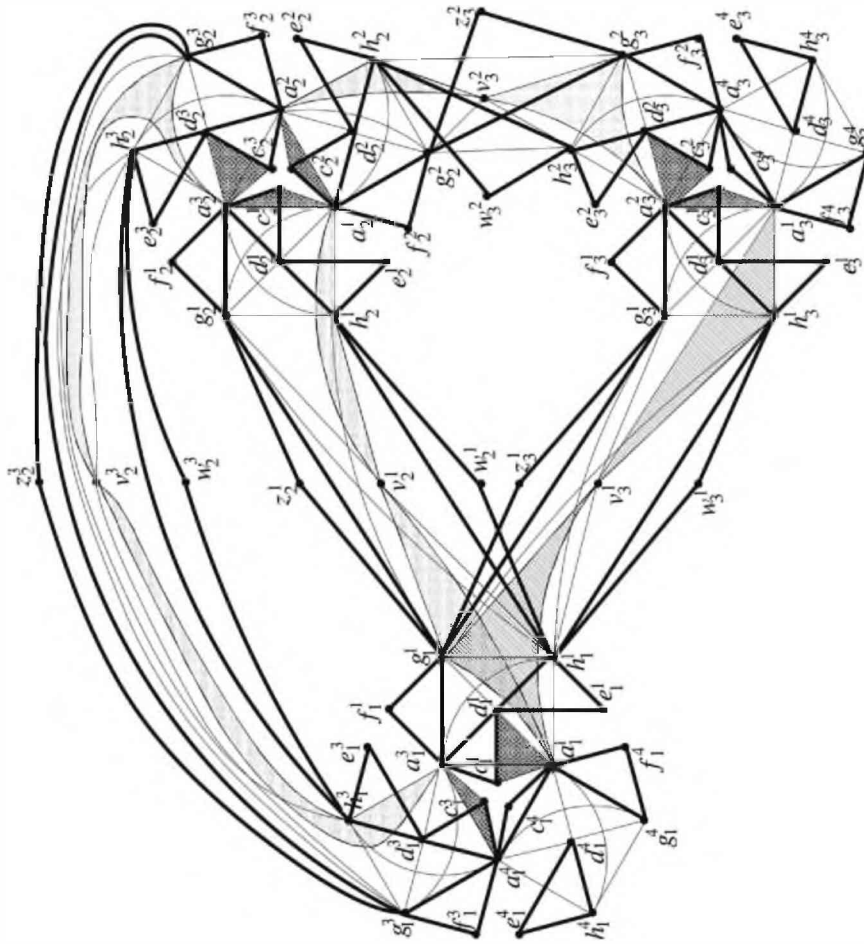
(iv)  $T = \{g_j^i, h_j^i, v_j^i\}$ .

If  $u_i$  is true, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(j, i), \{v_j^i, h_j^i, a_j^i\}$ . Thus  $h_j^i \in \bigcap \mathcal{F}_T$ . If  $u_i$  is false, then the members of  $\mathcal{F}_T$  are:  $K_{12}(j), K_5(j, i), \{v_j^i, g_j^i, a_j^i\}$ . Thus  $g_j^i \in \bigcap \mathcal{F}_T$ .  $\square$

In Fig. 9, we give an example of an RS-cover defined by a satisfying truth assignment, according to the proof of Theorem 8.

### 5. Concluding remarks

We have proved that deciding whether a given graph is a clique graph is an NP-complete problem. From the same proof, it follows that the problem remains NP-complete even for graphs with bounded clique size  $\omega$ , and for bounded maximum degree  $\Delta$  graphs. We say that the complete bipartite graph  $K_{1,t}$  is a  $t$ -claw. A graph  $G$  is  $t$ -claw free if  $G$  does not contain a



**Fig. 9.** RS-cover  $\mathcal{F}$  for graph  $G_l$  of Fig. 4. The RS-cover is defined by the satisfying truth assignment where  $u_1$  is true, and  $u_2$  and  $u_3$  are false. Bold edges highlight forced triangles present in every RS-cover of  $G_l$ . Filled connected regions depict triangles of  $\mathcal{F}$  which depend on the truth assignment for  $l = (U, C)$ . Complete sets  $K_5(j, i)$  and  $K_{12}(j)$  belong to the RS-cover  $\mathcal{F}$  but are not depicted in order to make simpler the drawing.

$t$ -claw as an induced subgraph. The instance of CLIQUE GRAPH used to prove NP-completeness has clique size  $\omega = 12$ , has maximum degree  $\Delta = 14$ , and it is 7-claw free. However the problem is polynomial when restricted to graphs with clique size  $\omega < 4$ , and also when restricted to graphs with maximum degree  $\Delta < 5$ . Note that Theorem 3 of the fundamental paper by Roberts and Spencer [36] says: a  $K_4$ -free graph is a clique graph if and only if it is clique-Helly. A graph with  $\Delta < 5$  is a clique graph if and only if it is hereditary clique-Helly [17,31].  $G$  is a 1-claw free graph if and only if each connected component of  $G$  is  $K_1$ , and  $G$  is a 2-claw free graph if and only if each connected component of  $G$  is a complete graph. This suggests the search of the maximum values for the clique size  $3 \leq \omega \leq 11$  and for the maximum degree  $4 \leq \Delta \leq 13$  for which the problem is polynomial, and additionally the maximum value of  $2 \leq t \leq 6$  such that the problem is polynomial for  $t$ -claw free graphs.

The only reference for the problem of recognizing clique graphs restricted to greater bounded maximum clique size graphs is the class of Planar graphs [2]. In that paper, a non-bounded degree subclass of planar clique graphs, larger than clique-Helly planar graphs, and admitting cliques of size 4, is characterized. In addition, a polynomial-time algorithm for the recognition of that subclass of planar clique graphs is given.

Several subclasses of clique graphs have been studied for which polynomial-time recognition is known. In particular, for several classes of graphs the corresponding class of clique graphs is known [39]. Note that it is known that the clique graph of a Chordal graph is a Dually chordal graph, but the complexity of deciding whether a Chordal graph is a clique graph is not known.

The NP-completeness of CLIQUE GRAPH suggests the study of the problem restricted to classes of graphs not properly contained in the class of clique graphs. We leave as open problems the recognition of Planar clique graphs, and of Chordal clique graphs.

**Acknowledgments**

We are grateful to the referees for their careful reading and the many suggestions which helped to improve the paper.

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