Cent. Eur. J. Phys. • 7(3) • 2009 • 624-629
DOI: 10.2478/s11534-009-0080-2

VERSITA

Central European Journal of **Physics**

Statistical, noise-related non-classicality's indicator

Research Article

Flavia Pennini[1,2], Angelo Plastino[1*], Gustavo Ferri[3]

1 *Instituto de Física La Plata–CCT-CONICET. Fac. de Ciencias Exactas, Universidad Nacional de La Plata, C.C. 727, 1900 La Plata, Argentina*

2 *Departamento de Física, Universidad Católica del Norte, Av. Angamos 0610, Antofagasta, Chile*

3 *Facultad de Ciencias Exactas, Universidad Nacional de La Pampa, Peru y Uruguay, Santa Rosa, La Pampa, Argentina*

Received 22 January 2009; accepted 23 April 2009

Abstract: Finding signs of the classical-quantum border is a very important task of perennial interest. We show, using semiclassical arguments, that the frontier between the classical and quantum domains can be characterized by recourse to idiosyncratic features of a delimiter parameter associated with the concepts of i) noise) ii) Husimi distribution functions, iii) Wherl's entropy, and iv) escort distributions.

PACS (2008): 03.65.Sq, 03.67.-a, 05.30.-d, 42.50.Lc

Keywords: Husimi distribution • quantum statistical mechanics • semiclassical methods

1. Introduction

Finding signs of the classical-quantum border is a very important task of perennial interest. Coherent states can be viewed as the analogues of points in phase-space [1]. For an arbitrary quantum state described by the density operator ρ one can ask a natural question: to what an extent it is nonclassical in the sense that its properties diverge from those of coherent states? Kenfack and Zyczkowski [2] ask in this respect whether there is any parameter that may legitimately reflect ρ's degree of non-classicality and consider to that effect the negativity of the Wigner function. In this note we intend to provide a different kind of answer within the semiclassical statistics' realm [3], and with relation to quantum optics' techniques and information theory treatments. Our considerations are related to "noise", that plays a significant informative role with regards to the particle-wave duality [4]. Electromagnetic fluctuations are different if the energy is carried by waves or by particles. The magnitude of energy fluctuations scales linearly with the mean energy for classical waves, while it does so with the square root of the mean energy for classical particles. Since a photon is neither a classical wave nor a classical particle, for it the linear and square-root contributions must coexist. The square-root (particle) contribution dominates at optical frequencies, the linear (wave) contribution taking over at radio-frequencies [5–7]. The diagnostic-power of photon-noise was extended further in the 60's, as it was discovered that fluctuations discriminate between the radiation from a laser and that from a black body. For the former the wave contribution to the fluctuations is null, while it is merely small for a black body. Measurements of noise are now a common technique in quantum optics and Glauber's quantum theory of photon

*E-mail: plastino@fisica.unlp.edu.ar

statistics is textbook material. Thus, coherent states become of central importance in quantum optics [5, 6], being the states of a harmonic oscillator system which mimic in the best possible way the classical motion of a particle in a quadratic potential [1, 8, 9]. Much of the thrust of quantum optics' techniques lies indeed in their ability to exploit classical analogues and most particularly, comparisons with classical noise theory, that allow reduction of purely harmonic systems to non-operator ones, *via* phase space methods [1, 8], where the essentially quantal nature of the problem is transcribed in terms of the interpretation of apparently classical variables, with coherent states [9] playing the starring role. Here that role will be again invoked, within the strictures of semiclassical techniques, in order to provide an answer to the query posed in the first paragraph above. It will be shown that non-classicality can be visualized in terms the idiosyncratic features of a semiclassical delimiter parameter associated to the concepts of i) Husimi distributions, ii) Wherl's entropy, and iii) escort distributions.

2. Preliminaries

2.1. Wehrl entropy and Husimi distributions

The paradigmatic semiclassical concept we appeal to is that of Wehrls's entropy W, a useful measure of localization in phase-space [11] that is built up using coherent states [1, 10, 11]. The pertinent definition reads

$$W = -\int \frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar}\,\mu(x,p)\,\ln\mu(x,p), \tag{1}$$

where $\mu(x,p) = \langle z|\rho|z\rangle$ is a "semi-classical" phase-space distribution function associated to the density matrix ρ of the system [1, 9]. Coherent states are eigenstates of the annihilation operator a, *i.e.*, satisfies $a|z\rangle = z|z\rangle$.
The distribution $\mu(x,p)$ is normalized in the fashion

$$\int (\mathrm{d}x\,\mathrm{d}p/2\pi\hbar)\,\mu(x,p) = 1, \tag{2}$$

and it is often referred to as the Husimi distribution [12]. The last two equations clearly indicate that the Wehrl entropy is simply the "classical entropy" (1) of a Wigner-distribution. Indeed, $\mu(x,p)$ is a Wigner-distribution D_W smeared over an $\hbar$ sized region of phase space [10]. The smearing renders $\mu(x,p)$ a positive function, even if D_W does not have such a character. The semi-classical Husimi probability distribution refers to a special type of probability: that for simultaneous but approximate location of position and momentum in phase space [10]. The uncertainty principle manifests itself through the inequality

$$1 \leq W, \tag{3}$$

which was first conjectured by Wehrl [11] and later proved by Lieb [13].
The usual treatment of equilibrium in statistical mechanics makes use of the Gibbs's canonical distribution, whose associated, "thermal" density matrix is given by

$$\rho = Z^{-1}\mathrm{e}^{-\beta H}, \tag{4}$$

with $Z = \mathrm{Tr}(\mathrm{e}^{-\beta H})$ the partition function, $\beta = 1/k_B T$ the inverse temperature T, and k_B the Boltzmann constant. In order to conveniently write down an expression for W consider an arbitrary Hamiltonian H of eigen-energies E_n and eigenstates $|n\rangle$ (n stands for a collection of all the pertinent quantum numbers required to label the states). One can always write [10]

$$\mu(x,p) = \frac{1}{Z}\sum_n \mathrm{e}^{-\beta E_n}\,|\langle z|n\rangle|^2. \tag{5}$$

A useful route to W starts then with Eq. (5) and continues with Eq. (1). In the special case of the harmonic oscillator the coherent states are of the form [1]

$$|z\rangle = \mathrm{e}^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle, \tag{6}$$

where $|n\rangle$ are a complete orthonormal set of eigenstates and whose spectrum of energy is $E_n = (n+1/2)\hbar\omega$, $n = 0, 1, \ldots$ In this situation we have the useful analytic expressions obtained in Ref. [10]

$$\mu(z) = (1-\mathrm{e}^{-\beta\hbar\omega})\,\mathrm{e}^{-(1-\mathrm{e}^{-\beta\hbar\omega})|z|^2}, \tag{7}$$

$$W = 1 - \ln(1-\mathrm{e}^{-\beta\hbar\omega}). \tag{8}$$

When $T \to 0$, the entropy takes its minimum value $W = 1$, expressing purely quantum fluctuations. On the other hand when $T \to \infty$, the entropy tends to the value $-\ln(\beta\hbar\omega)$ which expresses purely thermal fluctuations.

2.2. An indicator of noise: the Mandel parameter

A convenient noise-indicator of a non-classical field is the so-called Mandel parameter which is defined by [7]

$$Q = \frac{(\Delta N)^2}{\langle \hat{N}\rangle} - 1 \equiv F - 1, \tag{9}$$

which is closely related to the normalized variance (also called the quantum Fano factor F [14]) $F \equiv \sigma = (\Delta N)^2/\langle\hat{N}\rangle$ of the photon distribution. For $F < 1$ ($Q \leq 0$), emitted light is referred to as sub-Poissonian since it has photo-count noise smaller than that of coherent (ideal laser) light with the same intensity ($F = 1$; $Q = 0$), whereas for $F > 1$, ($Q > 0$) the light is called super-Poissonian, exhibiting photo-count noise higher than the coherent-light noise. Of course, one wishes to minimize the Fano factor. For a coherent state (a pure quantum state) the Mandel parameter vanishes, *i.e.*, $Q = 0$ and $F = 1$. A field in a coherent state is considered to be the closest possible quantum-state to a classical field, since it saturates the Heisenberg uncertainty relation and has the same uncertainty in each quadrature component. *The question we will try to answer here is: how close to $Q = 0$ (or $F = 1$) can we get semiclassically?* The answer should help to define the boundary between a classical and a quantum field. If so, it would be clear that both Q and F function as indicators on non-classicality. Indeed, for a thermal state one has $Q > 0$ and $F > 1$, corresponding to a photon distribution broader than the Poissonian. For $Q < 0$, ($F < 1$) the photon distribution becomes narrower than that of a Poisson-PDF and the associated state is non-classical. The most elementary examples of non-classical states are number states. Since they are eigenstates of the photon number operator $\hat{N}$ the fluctuations in $\hat{N}$ vanish and the Mandel parameter reads $Q = -1$ ($F = 0$) [15]. We will below establish a semiclassical link with these ideas.

3. Present considerations: semiclassical $Q-$evaluation

Taking into account that the number operator is connected with the harmonic oscillator Hamiltonian $\hat{H}$ *via* $\hat{N} = \hat{H}/\hbar\omega - 1/2$, we can rewrite the HO-Mandel parameter in this fashion

$$Q = F - 1 = \frac{(\Delta\hat{H})^2}{\hbar\omega\langle\hat{H}\rangle - \hbar^2\omega^2/2} - 1, \tag{10}$$

where we have used that $\hat{H} = \hbar\omega|z|^2$ [3]. Our main protagonist from now on is a semiclassical version Q^{sc} of Mandel's parameter evaluated with Husimi's distribution, *i.e.*,

$$F^{sc} - 1 = Q^{sc} = \frac{(\Delta_\mu N)^2}{\langle\hat{N}\rangle_\mu} - 1, \tag{11}$$

where $\langle\ldots\rangle_\mu$ denotes the semiclassical mean value of any general observable and the subindex μ indicates that we have taken the Husimi distribution (7) as the weight function. It is then easy to see that Q^{sc} reads

$$Q^{sc} + 1 = F^{sc} = \frac{2}{(1 - e^{-\beta\hbar\omega})(2 - (1 - e^{-\beta\hbar\omega}))} \geq 2, \tag{12}$$

and it becomes of the essence to remark that the semiclassical approach impedes us to reach the $Q = 0-$value (for an explanation see note below Eq. (22)). Additional tools, that we turn now to elaborate, will hopefully shed additional light on this crucial (as will be seen) issue.

4. Escort-Fano factor

Given a probability distribution (PD) $f(x)$, there exists an infinite family of associated PDs $f_q(x)$ given by

$$f_q(x) = \frac{f^q(x)}{\int f^q(x)\,dx}; \qquad (q \in \mathcal{R}), \tag{13}$$

that have proved to be quite useful in the investigation of nonlinear dynamical systems, as they often are better able to discern some of the system's features than the original distribution [16]. Things can indeed be improved in the above described scenario by recourse to this concept of escort distribution (ED), introducing it in conjunction with semiclassical Husimi distributions. Thereby one might try to gather "improved" *semiclassical information* from escort Husimi distributions ($q-$HDs) $\gamma_q(x,p)$:

$$\gamma_q(x,p) = \mu(x,p)^q / \left(\int \frac{d^2z}{\pi}\,\mu(x,p)^q\right), \tag{14}$$

where $d^2z/\pi = dxdp/2\pi\hbar$ and whose HO-analytic form can be obtained from Ref. [17], *i.e.*,

$$\gamma_q(z) = q(1 - e^{-\beta\hbar\omega})\exp\left[-q(1 - e^{-\beta\hbar\omega})|z|^2\right]. \tag{15}$$

We compute now the expectation values involved in Eq. (10) with γ_q as a the weight function and find for the relevant Hamiltonian-moments

$$\langle H\rangle_{\gamma_q} = \int \frac{d^2z}{\pi}\gamma_q(z)\hbar\omega|z|^2 = \frac{\hbar\omega}{q(1 - e^{-\beta\hbar\omega})}, \tag{16}$$

$$\langle H^2\rangle_{\gamma_q} = \int \frac{d^2z}{\pi}\gamma_q(z)\hbar^2\omega^2|z|^4 = \frac{2\hbar^2\omega^2}{q^2(1 - e^{-\beta\hbar\omega})^2}, \tag{17}$$

and thus,

$$(\Delta H)^2_{\gamma_q} = \frac{\hbar^2\omega^2}{q^2\,[1 - \exp(-\beta\hbar\omega)]^2}, \tag{18}$$

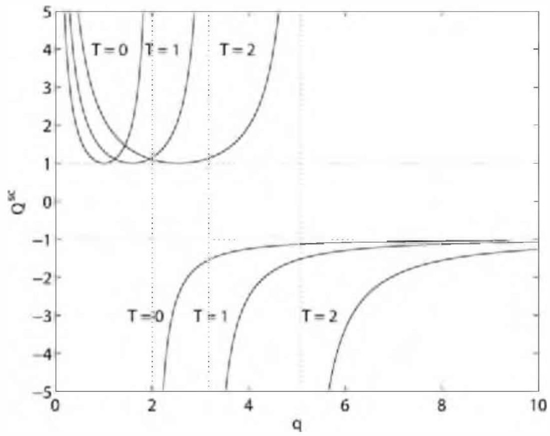

Figure 1. Mandel parameter Q^{sc} evaluated semiclassically by recourse to escort distributions of order q at different temperatures T (given in $\hbar\omega-$units).

so that we finally obtain an "escort"-expression for the Fano factor Mandel parameter (Fano factor):

$$Q_q^{sc} + 1 = F_q^{sc} = \frac{2}{q\,(1-e^{-\beta\hbar\omega})\,(2-q(1-e^{-\beta\hbar\omega}))}. \quad (19)$$

We note that when q tends to unity we have $Q_1^{sc} \equiv Q^{sc}$. The additional degree of freedom acquired *via* q allows for the desired negative values of the Mandel parameter, as depicted in Fig. 1.

In order to interpret these results, additional considerations are in order. First of all let us look at the escort-Wehrl entropy built up using the distributions γ_q, which has the form (found in [18])

$$W_q = W - \ln q, \quad (20)$$

and thus forbids negative $q-$values. Eq. (20), together with the HO-analytic expression (8), entail that, by requiring that the information measure W_q obey both Lieb's bound and positivity (namely, $1 \geq W_q \geq 0$), one must restrict the escort-degree $q-$range to $1 \leq q \leq e \simeq 2.7182818$. Still more sophisticated considerations may further circumscribe the above domain, however. To this effect we appeal now the concept of participation ratio $\mathcal{R}$ of a density operator ρ (giving the number of pure states that enter ρ [19, 20]):

$$\mathcal{R} = 1/Tr(\rho^2); \qquad [1 \leq \mathcal{R} \leq \infty]. \quad (21)$$

We will now concoct a semiclassical "equivalent-notion" by performing an analogous calculation using the $q-$escort Husimi distribution γ_q of the harmonic oscillator. This would yield

$$\mathcal{R}_q^{HO} = \frac{1}{\int \frac{d^2z}{\pi}\,\gamma_q(z)^2} = \frac{2}{q(1-e^{-\beta\hbar\omega})}. \quad (22)$$

Note that $\mathcal{R}_{q=1}^{HO}(T=0) = 2$. Our density operator (4) contains a minimum of two pure states in this "best possible" scenario, which impedes us to semiclassically reaching $Q = 0$ in Eq. (12). Now, invoking $\mathcal{R} \geq 1$ immediately entails, at zero temperature, $q \leq 2$. For higher temperatures the allowed $q-$purview shifts "rightwards" and exceeds the value two. At $T = 0$ a refined region $\mathcal{F}$ of permissible values for q then ensues, namely, $\mathcal{F} = [1 \leq q \leq 2]$, which is crucial to our present discussion, as a glance to Fig. 1 will confirm. As stated, when T grows, $\mathcal{F}$ expands rightwards.

Fig. 1 shows that the realm of negative (and thus quantum) values of the Mandel parameter Q can indeed be attained semiclassically by recourse to the concept of escort distributions of order $2 \leq q \leq e$. However, the physical (quantum) region $-1 \leq Q \leq 1$ remains strictly inaccessible to our modified semiclassical treatment (and thus the quantum-classical border begins at $Q^{sc} = 1$). The $Q^{sc} < -1$ values of Fig. 1 are un-physical since they imply negative fluctuations, which are nonsensical [*Cf.* Eq. (9)]. Note that we do get $Q^{sc} = -1$ at $q = \infty$ (for all temperatures T), but this is un-physical as well, since the accompanying escort-Husimi distribution would be a delta in phase space, violating the uncertainty principle. We proceed now to tackle the same issue *via* a different approach, in order to make sure that our results are not just a Husimi artifact.

5. Escort thermal Wigner distribution

It is well known that the Wigner distribution for a general density matrix is given by [22]

$$f_W(x,p) = \int_{-\infty}^{\infty} ds\, e^{ips/\hbar} \left\langle x - \frac{s}{2}\right|\rho\left|x + \frac{s}{2}\right\rangle, \quad (23)$$

which is normalized according to $\int (d^2z/\pi) f_W(x,p) = 1$. The pertinent analytic expression for the harmonic-oscillator thermal density is thus [22]

$$f_W(x,p) = 2\tanh(\beta\hbar\omega/2)\, e^{-2\tanh(\frac{\beta\hbar\omega}{2})|z|^2}, \quad (24)$$

where $|z| = x^2/4\sigma_x^2 + p^2/4\sigma_p^2$, with $\sigma_x = \hbar/2m\omega$ and $\sigma_p = \hbar m\omega/2$. We define now the escort-thermal Wigner

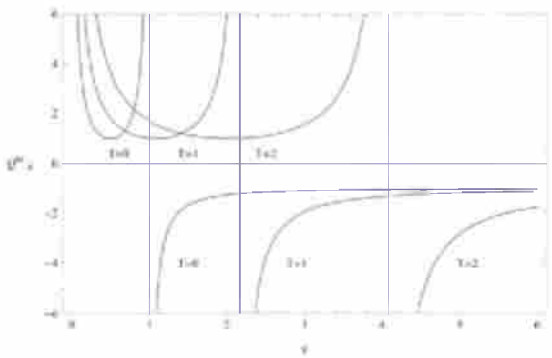

Figure 2. Mandel parameter Q_W^{sc} evaluated semiclassically by recourse to escort distributions of order q at different temperatures T (given in $\hbar\omega-$units).

distribution as follows:

$$\zeta_q(x,p) = f_W(x,p)^q \Big/ \left(\int \frac{d^2 z}{\pi} f_W(x,p)^q \right), \tag{25}$$

so that integrating over phase space we are led to

$$\zeta_q(x,p) = 2q \tanh(\beta\hbar\omega/2) e^{-2q \tanh(\beta\hbar\omega/2)|z|^2}. \tag{26}$$

The Mandel parameter evaluated with ζ acquires then the form

$$Q_W^{SC} = \frac{1}{2q \tanh(\beta\hbar\omega/2)\,(1 - q\tanh(\beta\hbar\omega/2))} - 1. \tag{27}$$

In Fig. 2 we plot Q_W^{SC} as a function of q. We see that we reobtain the behavior illustrated in Fig. 1, albeit with a different $q-$dependence.

6. Conclusions

What has effectively been gained with our escort generalization? Well, to be in a position to ascertain that, when the escort degree q adopts certain specific values, rather strange things happen, which vividly illustrate non-classicality (our goal in this communication). Clearly, such idiosyncracy seems to signal *having reached the classical-quantum border at* $Q^{sc} = 1$. First, take note of what happens at $q = 2$; $T = 0$, when $q-$negativity first becomes possible. Note that the ensuing semiclassical escort-Husimi distribution for e $\geq q \geq 2$ cannot be associated à la (22) to a quantal distribution function derived from a density operator, since its participation ratio would in that case be smaller than unity, a limit value only reached by pure states. This is of no great relevance for the semiclassical treatment, which is not a quantum one by definition, but does point out to an incompatibility between the quantum regime and escort distributions of degree > 2. The concomitant transition is by no means a gentle one, as (remember that $T = 0$), Q^{sc} jumps from plus to minus infinite at $q = 2$. These considerations hold also at finite temperatures, by replacing $q = 2$ by $q = 2/[1 - \exp(-\beta\hbar\omega)]$. Second, we attain the quantal $Q^{sf} = -1$ at the "strange" value $q = \infty$, where the escort distributions turns into a Dirac's delta in phase-space [21]. If we replace the escort-Husimi distribution with the thermal-Wigner one the same qualitative behavior is observed, although the associated $q-$values differ somewhat from the Husimi ones.

Thus, if we want our semiclassically evaluated noise-estimator Q to take values associated to the quantal regime, we encounter the strange behaviors just described. One may dare thus to formulate a conjecture in this respect. Strange behaviors of semiclassical quantities may well be indicators on non-classicality. Although we cannot enter the quantum regime *via* a semiclassical treatment, we have ascertained that ours does "sense" the existence of such quantal regime, which is our main conclusion. Moreover, we can somehow "visualize" non-classicality in, paradoxically, classical terms: it entails having simultaneously *zero-fluctuations* in the particle-number together with *finite ones* in phase-space location, which is not possible classically (because of the Dirac's delta at $q = \infty$). Summing up, we have found in this work a semiclassical indicator of non-classicality, the degree q of the escort distribution of a Husimi distribution. When q attains the value 2 we leave the classical region. Such a scenario ($0 \leq q \leq 2$) is characterized by semiclassical probability distributions (PDs) to which one can not in principle associate a quantum probability distribution (for instance, of the $|\psi|^2-$sort) because for such PDs the participation ratio would be smaller than unity. We also seem to reach the purely quantum instance of zero particle-number fluctuations at $q = \infty$, at the price of violating the uncertainty principle. The escort-semiclassical approach has thus been seen to exhibit idiosyncratic features that help "visualizing" the classical-quantum border.

Acknowledgements

F. Pennini would like to thank partial financial support by FONDECYT, grant 1080487.

References

[1] R. J. Glauber, Phys. Rev. 131, 2766 (1963)

[2] A. Kenfack, K. Zyczkowski, J. Opt. B-Quantum S. O. 6, 396 (2004)

[3] F. Pennini, A. Plastino, Phys. Rev. E 69, 057101 (2004)

[4] C. Beenakker, C. Schonenberger, Phys. Today 56, 37 (2003)

[5] D. F. Walls, G. J. Milburn, Quantum optics (Springer, New York, 1994)

[6] P. Meystre, M. Sargent III, Elements of quantum optics (Springer, New York, 1990)

[7] L. Mandel, E. Wolf, Optical coherence and quantum optics (Cambridge University Press, New York, 1995)

[8] E. C. G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963)

[9] J. R. Klauder, B. S. Skagerstam, Coherent states (World Scientific, Singapore, 1985)

[10] A. Anderson, J. J. Halliwell, Phys. Rev. D 48, 2753 (1993)

[11] A. Wehrl, Rep. Math. Phys. 16, 353 (1979)

[12] K. Husimi, Proceedings of the Physics and Mathematics Society of Japan 22, 264 (1940)

[13] E. H. Lieb, Commun. Math. Phys. 62, 35 (1978)

[14] J. Bajer, A. Miranowicz, J. Opt. B-Quantum S. O. 2, L10 (2000)

[15] F. Haug, M. Freyberger, K. Vogel, W. P. Schleich, Quantum Opt. 5, 65 (1993)

[16] C. Beck, F. Schlogl, Thermodynamics of chaotic systems (Cambridge University Press, New York, 1993)

[17] F. Pennini, A. Plastino, G. L. Ferri, Physica A 383, 782 (2007)

[18] F. Pennini, A. Plastino, G. L. Ferri, F. Olivares, Phys. Lett. A 372, 4870 (2008)

[19] J. Batle, A.R. Plastino, M. Casas, A. Plastino, J. Phys. A-Math. Gen. 35, 10311 (2002)

[20] W. J. Munro, D. F. V. James, A. G. White, P. G. Kwiat, Phys. Rev. A 64, 0303202 (2003)

[21] F. Pennini, A. Plastino, Phys. Lett. A 326, 20 (2004)

[22] D. J. Tannor, Introduction to Quantum Mechanics: Time-Dependent Perspective (University Science Books, USA, 2007)