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Combinatorial functional and differential equations applied to differential posets☆

Matías Menni

Conicet – Lifia, Universidad Nacional de La Plata, 50 y 115 s/n (1900) La Plata, Argentina

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Abstract

We give combinatorial proofs of the primary results developed by Stanley for deriving enumerative properties of differential posets. In order to do this we extend the theory of combinatorial differential equations developed by Leroux and Viennot.

Keywords: Differential posets; Y-graphs; Joyal species; Combinatorial differential equations

1. Introduction

The class of posets known as Y-*graphs* or *differential posets* was discovered independently by Fomin [4,6] and by Stanley [18]. Intuitively, the definition of this class of graphs captures the essential structural properties of Young's lattice (i.e. partitions ordered by inclusion of Young diagrams) which allow a correspondence between Hasse walks in the poset and certain permutations.

Although both authors seem to have been motivated by similar enumerative applications, their strategies are quite different. Fomin shows that for Y-graphs it is possible to define variants of the Schensted algorithm which realize explicit bijections between walks and permutations. On the other hand: "In [18] Stanley was able to derive many enumerative results involving walks or chains in a differential poset by constructing an algebra of operators on the poset. The (formal) solution of certain partial differential equations involving these operators yielded generating functions counting such walks. Stanley's results are powerful but entirely algebraic. Fomin's approach gives bijective proofs of some of Stanley's results" (see the introduction to Chapter 2 of [16]).

Indeed, Fomin's theory of *growths* as described in [6] or [16] provides a very satisfactory bijective account of the enumerative applications which motivated the definition of Y-graphs or differential posets. But despite the success of Fomin's combinatorial approach, there is still interest in the 'entirely algebraic' one (see e.g. Fomin's account in [5], Sloss' thesis [17] or the work on *down-up algebras* as defined in [2]). The main reason is, I believe, the following: the algebraic approach has a striking intuitive appeal which, on the one hand helps at the time of deriving new results, while on the other is not 'explained' by the highly algorithmic approach in [6].

The aim of the present work is to give a combinatorial interpretation of some key ideas in [18] which, so far, seem to have only linear algebraic formalizations. Moreover, we expect to do so in such a way that the intuitive presentation

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E-mail address: matias.menni@gmail.com.

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remains mainly unchanged. Our proposal is to use category theory in the way pioneered by Joyal in [8]. In fact, we will re-interpret Stanley's ideas using a simple generalization of the theory of combinatorial differential equations developed by Leroux and Viennot in [11]. We will review some of the main material in these two references but it seems convenient that the reader be familiar with them.

1.1. Outline

We now outline the structure of the paper. Section 2 is devoted to a review of the parts of [18] that are relevant to the present paper. In Section 3 we review Joyal's *catégorie des espèces linéares* and generalize the monoidal category studied by Leroux and Viennot in [11]. In Section 4 we describe the general theory of functional and differential equations meant to be applied to our generalization of the theory by Leroux and Viennot.

At the core of Stanley's theory there are two linear operators D and U satisfying the equation $DU = r + UD$. The analogous concepts in our context are introduced in Section 5 through the notion of Weyl category. The main results of the paper are stated and proved in Section 6. Two examples of applications are described in Section 7.

In order to read the paper some familiarity with category theory is required (see [12]). We will freely use elementary results about monoidal categories but, although we will not make a strong emphasis on them, we will also rely on more sophisticated results. In particular, we assume that the reader is familiar with [7]. For categories $\mathscr{C}$ and $\mathscr{D}$ we denote the category of functors $\mathscr{C} \to \mathscr{D}$ and natural transformations between them by $[\mathscr{C}, \mathscr{D}]$.

We also assume that the reader has experience with basic combinatorial manipulation of power series. Moreover, although we reproduce all the concepts involved, the reader may find it convenient to have an acquaintance with Joyal's theory of species [8], Leroux and Viennot's [11] and Stanley's work on differential posets [18].

2. Review of Stanley's main result on differential posets

We now briefly discuss some of the main definitions of [18] in order to ease the comparison between Stanley's approach and the one used in our paper. For $x < y$ in a poset P we say that y *covers* x in P if $x \leqslant p \leqslant y$ implies that $x = p$ or $p = y$. As in [18] we denote the set of elements that cover x by x^+. Analogously, the set of elements covered by x is denoted by x^-.

Definition 2.1. Let r be a positive integer. A poset P is called *r-differential* if it satisfies the following three conditions:

(D1) P is locally finite, graded and has a least element that we denote by $\perp$.
(D2) If $x \neq y$ in P and there are exactly k elements of P which are covered by both x and y, then there are exactly k elements of P which cover both x and y.
(D3) If x covers exactly k elements of P, then x is covered by exactly $k + r$ elements.

The main example of a 1-differential poset is given by the set of partitions ordered by inclusion of Young diagrams (see Corollary 1.4 in [18]). Proposition 5.1 in [18] states that if P and Q are r and s-differential, respectively then $P \times Q$ is $(r + s)$-differential. Also in Section 5 of [18] a class of examples called *Fibonacci* differential posets is introduced.

Let K_0 be a field of characteristic 0, let K be the quotient field of the ring of formal Laurent series with coefficients in K_0 and let $\widehat{K}P$ denote the K-vector space of arbitrary linear combinations $\sum_{x\in P} c_x x$ with $c_x \in K$. Assuming that for all $x \in P$, x^+ and x^- are finite Stanley defines operators $U, D : \widehat{K}P \to \widehat{K}P$ by $Ux = \sum_{y\in x^+} y$ and $Dx = \sum_{y\in x^-} y$. It is clear from the definition that a sequence of U's and D's can be thought of as the instructions on how to perform a walk up and down the poset P. It is also clear that the coefficient c_y in the result of applying such a sequence to x will enumerate the number of ways to get from x to y using the instructions given by the sequence of operators. So that enumerative properties of Hasse walks inside P can be deduced from studying such operators. All this without assuming much on P, but if P is differential then the operators U and D are related in such a way that a completely different intuition is available. Theorem 2.2 in [18] states that P is r-differential if and only if $DU - UD = rI$ and using this characterization it is proved in Corollary 2.4 of [18] that if P is r-differential and $f(U) \in K[\![U]\!]$ then $Df(U) = rf'(U) + f(U)D$ where f' denotes formal derivative of the power series $f(U)$. This result allows one to think of D as the derivative $\partial/\partial U$. One of the main observations in [18] is that then it is possible to reduce enumerative

problems to solving partial differential equations. In order to explicitly do this Stanley introduces exponential power series with coefficients in the ring $\mathrm{End}(\widehat{K}P)$ of endomorphisms of $\widehat{K}P$ and defines in the usual way the formal integral and derivative operators. With this machinery in place the fundamental Theorem 2.5 is proved. We now reproduce the first item of this result and its proof so that the reader can appreciate now how Stanley uses differential equations and later compare with our proof using a simple generalization of the theory combinatorial differential equations of Leroux and Viennot.

Theorem 2.2 (*Stanley: Theorem 2.5(a) in [18]*). *Let P be an r-differential poset. Let $f(U), h(U) \in K_0[\![U]\!]$ and $c \in K_0$. Then*

$$\mathrm{e}^{(f(U)+cD)t}h(U) = \mathrm{e}^{\int_0^t f(U+crs)\,\mathrm{d}s}h(U+crt)\mathrm{e}^{cDt}.$$

Proof. Let $H(t) = \mathrm{e}^{(f(U)+cD)t}h(U) = \sum_{n\geqslant 0}(f(U)+cD)^n h(U)\frac{t^n}{n!}$. Then the operator $H(t)$ is uniquely determined by the conditions $\frac{\partial}{\partial t}H = (f(U)+cD)H$ and $H0 = h(U)$. Hence we need to verify only that the right-hand side of the equation in the statement satisfies the above differential equation. But there is a formal computation, taking care that U and D do not commute. Specifically, writing $L(t)$ for $\mathrm{e}^{\int_0^t f(U+crs)\,\mathrm{d}s}h(U+crt)\mathrm{e}^{cDt}$, we have (using that $Df(U) = rf'(U) + f(U)D$),

$$\begin{aligned}
(f(U)+cD)L(t) &= \left[f(U) + cr\int_0^t \frac{\partial}{\partial U}f(U+crs)\,\mathrm{d}s + cr\frac{h'(U+crt)}{h(U+crt)}\right]L(t) + cL(t)D \\
&= \left[f(U) + (f(U+crt)-f(U)) + cr\frac{h'(U+crt)}{h(U+crt)}\right]L(t) + cL(t)D \\
&\quad \text{since } cr\frac{\partial}{\partial U}f(U+crs) = \frac{\partial}{\partial s}f(U+crs) \\
&= \left[f(U+crt) + cr\frac{h'(U+crt)}{h(U+crt)}\right]L(t) + cL(t)D.
\end{aligned}$$

On the other hand,

$$\frac{\partial}{\partial_t}L(t) = \left[f(U+crt) + cr\frac{h'(U+crt)}{h(U+crt)}\right]L(t) + cL(t)D,$$

and $L(0) = h(U)$ (by inspection). So $L(t)$ satisfies the required differential equation. □

We find that this result and its proof show a remarkable insight into the relation between linear algebra and combinatorics. We also believe that the bijective accounts of the enumerative corollaries of Theorem 2.2 do not provide a combinatorial interpretation of the theorem itself. In this paper we provide such an explanation.

3. Linear species, Joyal, Leroux and Viennot

In [8], Joyal proposed to use certain monoidal categories as combinatorial analogues of rings of power series. These categories allow to perform many algebraic calculations without collapsing the combinatorial information that is usually lost when working with identities among power series.

Joyal's theory was developed in several directions but of particular interest to us is the combinatorial theory of differential equations developed by Leroux and Viennot in [11]. In this theory the main objects of study are exponential power series with sets as coefficients.

At an early stage of the development of the present work, we wanted a combinatorial theory of differential equations that could interpret Stanley's statement of Theorem 2.2. Perhaps naively we attempted to apply Leroux and Viennot's theory. But we quickly realized that it is not powerful enough. Intuitively, the coefficients considered in this theory are 'too simple'. More explicitly, there are no objects in **Set** that behave like D and U. This observation led us to develop a theory analogous to that by Leroux and Viennot but with coefficients in categories more general than **Set** and to build categories of coefficients with objects which behave like D and U. Such categories will replace $\mathrm{End}(\widehat{K}P)$. (Recall that Stanley uses exponential power series with coefficients in this ring.)

In this section we recall some of the work by Joyal, Leroux and Viennot and suitably generalize it for our present purposes. The main elementary notion is the following.

Definition 3.1. A *category of coefficients* is a (not necessarily symmetric) monoidal category $\mathbf{k} = (\mathbf{k}_0, \circ, I)$ such that the underlying category $\mathbf{k}_0$ has finite coproducts and such that for any object C in $\mathbf{k}_0$, both $C \circ (\_)$ and $(\_) \circ C$ preserve finite coproducts.

Initial objects are denoted by 0, binary coproduct is denoted by $+$ and we will usually confuse $\mathbf{k}$ with the underlying category $\mathbf{k}_0$. We speak of a *symmetric* category of coefficients if $(\mathbf{k}_0, \circ, I)$ is symmetric monoidal.

The fact that we are dealing with coproducts implies that if $C + C'$ is initial then so are C and C'. So, in spite of the notation, a category of coefficients will never be a field, neither a ring, and so we are forcing ourselves to never use 'negative numbers' in our calculations with coefficients. We see the assumption on the existence of coproducts as ensuring some degree of 'combinatorialness' in the nature of the coefficients.

In many of the cases we will be interested in, the objects of a category of coefficients will be able to be 'differentiated'. In order to capture this property we recall the notion of a Leibniz functor as introduced in Definition 1.8 in [14]. (For the notion of *strength* see, for example, [9].)

Definition 3.2. If $\mathbf{k} = (\mathbf{k}_0, \circ, I)$ is a category of coefficients then a functor $\partial : \mathbf{k}_0 \to \mathbf{k}_0$ is called a *Leibniz functor* if the unique map $! : 0 \to \partial I$ is an isomorphism and ∂ is equipped with strengths $\alpha : \partial F \circ G \to \partial(F \circ G)$ and $\beta : F \circ \partial G \to \partial(F \circ G)$ such that $[\alpha, \beta] : (\partial F \circ G) + (F \circ \partial G) \to \partial(F \circ G)$ is an iso.

Below we will usually use the isomorphism $\partial(F \circ G) \cong (\partial F \circ G) + (F \circ \partial G)$ leaving the transformations α and β implicit.

While it is conceptually useful to have elementary definitions, we will need to assume, at certain key points (e.g. in the construction of free algebras), some non-elementary (co)completeness conditions. Mainly, that the underlying categories are monoidally cocomplete as defined in [7] and recalled below.

Definition 3.3. A monoidal category $(\mathbf{k}, \circ, \delta)$ is called *monoidally cocomplete* if it is cocomplete and moreover, the functors $(\_) \circ C, C \circ (\_) : \mathbf{k} \to \mathbf{k}$ preserve colimits for each C in $\mathbf{k}$.

It is clear that every monoidally cocomplete category is a category of coefficients.

3.1. Joyal's catégorie des espèces linéares

Let $\mathbb{L}$ be the (essentially small) category of finite linear orders and monotone bijections between them. This category is equivalent to the discrete category determined by the set of natural numbers, but it is sometimes convenient to use the whole of $\mathbb{L}$. For each $k \in \mathbb{N}$ denote the total order $\{0 < 1 < \cdots < k-1\}$ by $[k]$. The category $\mathbb{L}$ can be equipped with a (non-symmetric) monoidal structure $(\mathbb{L}, \oplus, \emptyset)$ where $\oplus$ is determined by the condition $[k] \oplus [k'] = [k + k']$.

The category $[\mathbb{L}, \mathbf{Set}]$ is complete and cocomplete and from general considerations about completeness, Day's well-known *convolution* construction (see [7]) produces, out of the monoidal structure $(\mathbb{L}, \oplus, \emptyset)$, a new monoidal structure $([\mathbb{L}, \mathbf{Set}], *, q^0)$ on the category of functors from $\mathbb{L}$ to $\mathbf{Set}$. (The notation for the unit of this monoidal structure will become clear in the rest of the section.) The resulting symmetric monoidal category was called *catégorie des espèces linéares* in [8]. We will denote it by $\mathscr{J}$.

In order to discuss the combinatorial intuition of $\mathscr{J}$ we first recall Joyal's more explicit description of the tensor $F * G$ of two objects F and G in $[\mathbb{L}, \mathbf{Set}]$. (Readers unfamiliar with Day's convolution construction can use the explicit description in Proposition 3.4 as the definition of $\mathscr{J}$.) A *cut* of a linear order l is a partition $l = l_0 l_1$ of l such that l_0 is an initial segment of l and l_1 is a terminal segment of l. (We stress that $(\emptyset, \emptyset)$ is the unique cut of the empty linear order $\emptyset$.)

Proposition 3.4. *For any F, G in $[\mathbb{L}, \mathbf{Set}]$,*

$$(F * G)l \cong \sum_{(l_0, l_1)} Fl_0 \times Gl_1$$

where (l_0, l_1) ranges over the cuts of l. The unit $q^0 : \mathbb{L} \to \mathbf{Set}$ can be described by $q^0\emptyset = 1$ and $q^0 l = 0$ for any non-empty linear order l. Moreover, the monoidal category $\mathscr{J} = ([\mathbb{L}, \mathbf{Set}], *, q^0)$ is actually a symmetric category of coefficients.

Proof. The first part follows from the explicit definition of Day's convolution and from the calculation of colimits in the presheaf category $[\mathbb{L}, \mathbf{Set}]$. The rest of the proof is straightforward calculation. See also [8]. □

The symmetric category of coefficients $\mathscr{J}$ was described by Joyal in Section 4 of [8] as a variant of the theory of species that would play, with respect to ordinary power series, the same role that the original species play in relation with exponential power series. Indeed, Joyal proposes to think of an object F in $[\mathbb{L}, \mathbf{Set}]$ as the series $\sum_{k\geqslant 0} F[k]q^k$ with sets (of 'unlabeled structures') as coefficients. Let us explain this notation in some detail. Notice that every functor $F : \mathbb{L} \to \mathbf{Set}$ induces a sequence $\{F[k]\}_{k\in\mathbb{N}}$ of sets. (We sometimes say that $F[k]$ is the value of F *at stage* $[k]$.) On the other hand, for every sequence $\{F_k\}_{k\in\mathbb{N}}$ of sets there exists, up to iso, a unique functor $F : \mathbb{L} \to \mathbf{Set}$ such that $F[k] = F_k$. The reason to write this information as an ordinary power series is the same as the reason why we write the counting information of certain combinatorial objects in the same way. One of the most interesting aspects of $\mathscr{J}$ is that $*$ behaves as the product of such series. The explicit description of $*$ given in Proposition 3.4 is already a precise formulation of the above statement. But some examples will make things more clear and will also allow us to introduce notation that we will need later on.

The linear species denoted by q^n is defined by the condition that $q^n l$ is 1 (the terminal in **Set**) if l has length n and 0 otherwise. This is consistent with the notation for the unit q^0 and a simple calculation shows that $q^n * q^m = q^{m+n}$. Notice that when defining q^k in this way the notation in terms of ordinary power series acquires an objective meaning since then we do have that $F \cong \sum_{k\geqslant 0} F[k]q^k$ in $\mathscr{J}$.

The object determined by $\sum_{n\geqslant 0} q^n$ will be denoted by $\frac{1}{1-q}$. The object $\sum_{n\geqslant 0} q^{2n}$ will be denoted by $\frac{1}{1-q^2}$. When convenient we will not write $*$ so, for example $q * \frac{1}{1-q^2}$ will be written as $\frac{q}{1-q^2}$. As coproducts are calculated pointwise it should be clear what, for example, $\frac{1}{1-q} + \frac{q}{1-q^2}$ denotes.

Given a ranked poset $\rho : P \to \mathbb{N}$, we denote its associated *rank-generating species* $\sum_{n\geqslant 0}(\rho^{-1}n)q^n$ by $F(P, q)$. So, for example, if P is Young's lattice then $F(P, q) = \prod_{i\geqslant 1} \frac{1}{1-q^i}$. (See p. 929 in [18].)

The *derivative* of a linear species F is defined by $(\partial_q F)l = \sum_{(l_0,l_1)} F(l_0 1 l_1)$ where the sum ranges over the cuts of l and $l_0 1 l_1$ is the obvious linear order obtained by inserting an element between l_0 and l_1. In this way, $\partial_q q^0 = 0$ and $\partial_q q^{n+1} = (n+1)q^n$. It is not difficult to show that $\partial_q : [\mathbb{L}, \mathbf{Set}] \to [\mathbb{L}, \mathbf{Set}]$ is a Leibniz functor (recall Definition 3.2) and that it preserves coproducts. The subindex q in ∂_q is not meant to be a parameter of any kind. It is there just to remind us that the functor is Leibniz w.r.t the monoidal structure $\mathscr{J}$. That is, when we think of the objects of $[\mathbb{L}, \mathbf{Set}]$ as ordinary power series.

It is clear that in the explicit description recalled in Proposition 3.4 one can replace **Set** by an arbitrary category of coefficients $\mathbf{k}$. We will not need this generalization so we refrain from working it out in detail. But we will generalize Leroux and Viennot's theory in a way analogous to the one just suggested. We do this in the next section and it will be essential for the rest of the paper.

3.2. *A generalization of Leroux and Viennot's monoidal structure on* $[\mathbb{L}, \mathbf{Set}]$

Let $(\mathbf{k}, \circ, I)$ be a category of coefficients and consider the category $[\mathbb{L}, \mathbf{k}]$ of functors from $\mathbb{L}$ to $\mathbf{k}$. For F, G in $[\mathbb{L}, \mathbf{k}]$ we define their tensor product as follows

$$(F \cdot G)l = \sum_{l_1 \uplus l_2 = l} Fl_1 \circ Gl_2,$$

where $l_1 \uplus l_2 = l$ denotes a pair (l_1, l_2) where l_1 and l_2 are subsets of l, with their linear order induced from that of l, such that $l_1 \cap l_2 = \emptyset$ and $l_1 \cup l_2 = l$, just as in the original case treated by Leroux and Viennot. The functor $\mathbb{L} \to \mathbf{k}$ that assigns I to the empty linear order and 0 to every other linear order is denoted by t^0. (A word on notation, when tensoring with a constant object $C = Ct^0$ we may sometimes write CF instead of $C \cdot F$.) The monoidal category $([\mathbb{L}, \mathbf{k}], \cdot, t^0)$ will be denoted by $\mathbf{LV}(\mathbf{k})$.

Proposition 3.5. *If* $\mathbf{k}$ *is a category of coefficients then* $\mathbf{LV}(\mathbf{k})$ *is also a category of coefficients. If the former is symmetric then so is the latter.*

Proof. Straightforward. See also [11]. □

For an arbitrary category of coefficients $\mathbf{k}$, and F and G in $\mathbf{LV}(\mathbf{k})$, $F \cdot G$ at stage $[k]$ gives $\sum_{i=0}^{k} \binom{k}{i} (F[i] \circ G[k-i])$. That is, the tensor $\cdot$ behaves as product of exponential power series. So it is fair to think of an object F in $\mathbf{LV}(\mathbf{k})$ as an exponential power series $\sum_{k \geqslant 0} F[k] \frac{t^k}{k!}$ with coefficients in $\mathbf{k}$. The letter t is a notational device analogous to the letter q in the case of $\mathscr{J}$ (recall Section 3.1). The reader should profit from all the advantage of the notation but never forget that we are working with combinatorial objects.

The case treated originally in [11] arises as $\mathbf{LV}(\mathbf{Set}, \times, 1)$. In this case, notice that if $\sharp(\_)$ denotes number of elements of its argument then a simple calculation shows that $\sharp((F \cdot G)[k])$ gives $\sum_{i=0}^{k} \binom{k}{i} \sharp(F[i]) \sharp(G[k-i])$.

Let us look at some examples. The object t in $\mathbf{LV}(\mathbf{k})$ is defined by the condition that t is 1 (the terminal object in $\mathbf{Set}$) at stage [1] and is empty everywhere else. A simple calculation shows that $t \cdot t = 2! \frac{t^2}{2!}$. That is: $t \cdot t$ is the initial object at every stage different from [2] and at stage [2] it has 2! copies of the unit object I in $\mathbf{k}$.

By Proposition 3.4, the monoidal category $\mathscr{J}$ is a symmetric category of coefficients. So we can consider, by Proposition 3.5, the symmetric category of coefficients $\mathbf{LV}(\mathscr{J})$. An interesting example of an object in this category is the object $\sum_{k \geqslant 0} k! (\frac{1}{1-q})^k \frac{t^k}{k!}$ which we should denote by $\frac{1}{1-\frac{1}{1-q}t}$.

For another example let $f_{\lambda/\mu}$ be the number of skew tableaux of shape λ/μ the equality below

$$\sum_k \left(\sum_n \left(\sum f_{\lambda/\mu} \right) q^n \right) \frac{t^k}{k!} = \exp\left(\frac{1}{1-q} t + \frac{1}{1-q^2} \frac{t^2}{2!} \right) \prod_i \frac{1}{1-q^i}$$

(with the unlabeled sum ranging over $\lambda/\mu \vdash k$ and $\mu \vdash n$) is the pale reflection of an isomorphism in $\mathbf{LV}(\mathscr{J})$.

The category of coefficients $\mathbf{LV}(\mathscr{J})$ will play a prominent role in the combinatorial results that we are aiming at. At this point we still have not developed the necessary tools to derive them but we hope that the couple of examples above give at least a hint of why it is useful to consider categories of the form $\mathbf{LV}(\mathbf{k})$ for $\mathbf{k}$ different from $\mathbf{Set}$.

Before we carry on explaining the differential structure of $\mathbf{LV}(\mathbf{k})$ let us highlight the following.

Remark 3.6. In contrast with the case of $\mathscr{J}$, the monoidal structure defined by Leroux and Viennot does not arise as the convolution product associated to a monoidal structure in $\mathbb{L}$. In order to explain why, we recall (see [7]) that, for any monoidal structure in $\mathbb{L}$, the convolution product associated to it makes the Yoneda embedding $\mathbb{L}^{\mathrm{op}} \to [\mathbb{L}, \mathbf{Set}]$ into a monoidal functor. But the image of this functor is not closed under the Leroux–Viennot tensor. (Take for example $t \cdot t$.)

The *derivative* functor $\partial_t : [\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ is defined by $(\partial_t F)l = F(\perp l)$ where $\perp l$ is the linear order obtained by adding a new first element to the linear order l. It is easy to show that the functor ∂_t is a Leibniz functor and that it preserves coproducts. The subindex t in ∂_t is not meant to be a parameter of any kind. It is there just as a notational reminder that the functor will behave as a derivative operation when interacting with the monoidal category $\mathbf{LV}(\mathbf{k})$. This is to distinguish it from the derivative operator in the context of $\mathscr{J}$ (which we have denoted by ∂_q). As functors $[\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$, ∂_t and ∂_q are different, but both are to be thought of as differential operators (with respect to different monoidal structures).

Perhaps the key feature of the categories of the form $\mathbf{LV}(\mathbf{k})$ is that they support an *integral* operation. The functor $\int : [\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ is defined by $(\int F)\emptyset = 0$ and $(\int F)(\perp l) = Fl$. The functor $\int$ also preserves coproducts and, almost trivially, satisfies $\partial_t (\int F) = F$.

One of the main results in [11] shows that when interpreted in $\mathbf{LV}(\mathbf{Set}, \times, 1)$, certain systems of differential equations have solutions determined up to iso and that these can be described in terms of very explicit combinatorial structures (indeed, suitably enriched trees). We will see in Section 4.3 how the functor $\int$ takes part in the construction of solutions for differential equations and we will generalize Leroux and Viennot's result.

One further piece of structure that the categories $\mathbf{LV}(\mathbf{k})$ enjoy is a kind of exponential operator. Indeed, the functor $\mathbf{E} : [\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ is defined by

$$(\mathbf{E}F)l = \sum_{\pi} (Fp_1) \circ \cdots \circ (Fp_k),$$

where π ranges over the partitions of the set underlying l and $p_1 < p_2 < \cdots < p_k$ are the components of π each component with the total order inherited from l and ordered among them according to their least element. (In the trivial case this should be understood as saying that $(\mathbf{E}F)\emptyset = t^0$.)

(A word on notation. We will sometimes write $\prod$ in order to refer to a finite indexed tensor, not necessarily cartesian products. So, for example, we can write $(\mathbf{E}F)l = \sum_\pi \prod_{p\in\pi} Fp$. The underlying monoidal structure will be clear from the context.)

If $(\mathbf{k}, \circ, I)$ is symmetric then there is a natural iso $\mathbf{E}(F+G) \cong \mathbf{E}F \cdot \mathbf{E}G$. But there is no such iso in general. On the other hand, there is a natural iso $\partial_t(\mathbf{E}F) \cong (\partial_t F)\cdot(\mathbf{E}F)$ even if $\mathbf{k}$ is not symmetric.

(It seems relevant to mention that in Section 5 of [11], expressions of the form e^Γ with Γ being certain operators acting on functors $\mathbb{L}^n \to \mathbf{Set}$ are considered. This clearly involves an idea similar to the ones in this section; but we have not explored the precise relation.)

For F in $\mathbf{LV}(\mathbf{k})$ define $\frac{1}{1-F}$ at stage l by $\sum_\pi (Fp_1)\circ\cdots\circ(Fp_k)$ where sum ranges over the *ordered* partions $\pi = \{p_1 < p_2 < \cdots < p_k\}$ and, as in the case of $\mathbf{E}$, each component considered with the order inherited from l.

The object $\frac{1}{1-t}$ built using the definition above is explicitly described by $\sum_{k\geq 0} k! \frac{t^k}{k!}$. This one is a good example of a little 'danger' of the notation in terms of power series. The reader should resist the temptation of canceling the $k!$'s. The object $\frac{1}{1-t}$ has, at stage $[l]$, $l!$ elements. So do not confuse $\frac{1}{1-t}$ with the object $\frac{1}{1-q}$ introduced in Section 3.1. As functors $\mathbb{L} \to \mathbf{Set}$ they are different. But when they interact with other objects using the monoidal structures $\mathbf{LV}(\mathbf{Set}, \times, 1)$ and $\mathscr{J}$, respectively, then they behave in some ways as the function $x \mapsto \frac{1}{1-x}$. Hence the notation.

3.3. *Pushing coefficients forward along functors*

The present short section introduces a couple of simple results that will be used throughout the rest of the paper. Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ be categories and let $\psi : \mathscr{D} \to \mathscr{E}$ be a functor. Then there is a functor $\psi_* : [\mathscr{C}, \mathscr{D}] \to [\mathscr{C}, \mathscr{E}]$ that assigns to each $F : \mathscr{C} \to \mathscr{D}$ the composite functor $\psi F : \mathscr{C} \to \mathscr{E}$.

Lemma 3.7. *Let* $\mathbf{k}$ *and* $\mathbf{k}'$ *be categories of coefficients and let* $\psi : \mathbf{k}_0 \to \mathbf{k}'_0$ *be a functor then*

(1) *there is a natural iso* $\partial_t \psi_* \cong \psi_* \partial_t$;
(2) *if* ψ *preserves initial object then there is a natural iso* $\psi_* \int \cong \int \psi_*$;
(3) *if* ψ *is a monoidal and preserves finite coproducts then* ψ_* *is also monoidal* $\mathbf{LV}(\mathbf{k}) \to \mathbf{LV}(\mathbf{k}')$. *If the categories of coefficients and* ψ *are symmetric then so is* ψ_*. *Moreover, there is a natural isomorphism* $\psi_* \mathbf{E} \cong \mathbf{E}\psi_*$.

Proof. Straightforward, but we prove preservation of $\mathbf{E}$ as an example. Using that ψ is monoidal and that it preserves coproducts we have, for each n and π ranging over the partitions of n, that

$$\psi \sum_\pi \prod_{p\in\pi} Fp \cong \sum_\pi \prod_{p\in\pi} \psi(Fp)$$

so that $\psi((\mathbf{E}F)n) = (\mathbf{E}(\psi_* F))n$. □

We will need also a variant Lemma 3.7 taking care of the case when ψ is $\partial_q : \mathscr{J} \to \mathscr{J}$ which does preserve coproducts but which is not monoidal. To avoid any possible confusion let us stress that $\partial_{q*} = (\partial_q)_*$.

Lemma 3.8. *For any* F *in* $\mathbf{LV}(\mathscr{J})$, $\partial_{q*}(\mathbf{E}F) \cong (\partial_{q*}F)\cdot\mathbf{E}F$.

Proof. We let $F = \sum_k f_k \frac{t^k}{k!}$ and calculate:

$$(\partial_{q*}(\mathbf{E}F))n = \sum_\pi \partial_q \prod_{p\in\pi} f_p = \sum_\pi \sum_{p\in\pi} (\partial_q f_p) * \prod_{t\in\pi/p} f_t$$

$$= \sum_a \left(\partial_q f_a * \sum_\sigma \prod_{s\in\sigma} f_s \right) = ((\partial_{q*}F)\cdot\mathbf{E}F)n,$$

where π ranges over the partitions of n, a over subsets of n and σ over the partitions of n/a in all cases with the induced order. □

3.4. The combinatorial meaning of $h(q) \mapsto h(q + crt)$

The equality $\mathrm{e}^{(f(U)+cD)t}h(U) = \mathrm{e}^{\int_0^t f(U+crs)\,\mathrm{d}s}h(U + crt)\mathrm{e}^{cDt}$ stated in Stanley's theorem involves ordinary power series $f(U)$ and $h(U)$ in $K_0[\![U]\!]$. On the left hand side, $h(U)$ is used as constant exponential power series but on the right, the exponential series $h(U + crt)$ is not constant. Moreover, it is clear from the proof of Theorem 2.2 that the behavior of $h(U + crt)$ plays an important role.

Although we still have not explained how we will interpret U, there is an important part of the assignment $h(U) \mapsto h(U + crt)$ that we can explain combinatorially at this point. First we need a somewhat abstract construction.

Let $\mathbf{k} = (\mathbf{k}_0, \circ, I)$ be a category of coefficients and let $\psi : \mathbf{k}_0 \to \mathbf{k}_0$. For any fixed non-negative integers c, r, define the functor $\psi_{+crt} : \mathbf{k}_0 \to [\mathbb{L}, \mathbf{k}_0]$ as follows:

$$(\psi_{+crt}h)[k] = (cr)^k\psi^k h,$$

for any h in $\mathbf{k}$. (Here, ψ^k denotes the composition of ψ with itself k times. If $k = 0$ then $\psi^k = id$.) We now state two simple properties of ψ_{+crt} that will be useful in Section 5.4. The first one is a 'corecursive' description of ψ_{+crt}.

Lemma 3.9. *For any f in* $\mathbf{k}$, $\psi_{+crt}f = ft^0 + cr\,f(\psi_{+crt}f)$.

Proof. Straightforward. □

The second property concerns the behavior of ψ_{+crt} followed by ∂_t.

Lemma 3.10. *For any h in* $\mathbf{k}$, $\partial_t(\psi_{+crt}h) = cr\psi_{+crt}(\psi h)$.

Proof. Calculate: $(\partial_t(\psi_{+crt}h))[k] = (cr)^{k+1}\psi^{k+1}h = cr(cr)^k\psi^k(\psi h) = (cr\psi_{+crt}(\psi h))[k]$ □

This is meant to be applied to the Leibniz functor $\partial_q : [\mathbb{L}, \mathbf{Set}] \to [\mathbb{L}, \mathbf{Set}]$ in order to obtain a functor $\partial_{q+crt} : [\mathbb{L}, \mathbf{Set}] \to [\mathbb{L}, [\mathbb{L}, \mathbf{Set}]]$ which, although is not monoidal, we choose to think of it as a functor $\mathscr{J} \to \mathbf{LV}(\mathscr{J})$ in the sense that it takes a combinatorial ordinary power series and produces an exponential one (with coefficients in $\mathscr{J}$). So that $\partial_{q+crt}h = \sum_{k\geqslant 0}(cr)^k\partial_q^k h\frac{t^k}{k!}$ for any h in $\mathscr{J}$.

For example,

$$\partial_{q+crt}\frac{1}{1-q} = \sum_{k\geqslant 0} c^k r^k \left(\sum_{n\geqslant 0}\frac{(n+k)!}{n!}q^n\right)\frac{t^k}{k!}.$$

We see the functor ∂_{q+crt} as providing a combinatorial interpretation of the assignment $h(q) \mapsto h(q + crt)$. In Section 5.4 it will be used to explain the meaning of the assignment $h(U) \mapsto h(U + crt)$ needed to understand Stanley's result.

4. Combinatorial functional and differential equations

Let us discuss in some more detail how we deal with differential and functional equations. A standard way to interpret different types of equations and their solutions in general categories is through the use of algebras for endofunctors. Let us recall the main definitions. If $F : \mathscr{C} \to \mathscr{C}$ is an endofunctor then an *F-algebra* is a pair (X, x) where X is an object of $\mathscr{C}$ and $x : FX \to X$ is a map in $\mathscr{C}$. Given F-algebras (X, x) and (Y, y) a *morphism of algebras* $g : (X, x) \to (Y, y)$ is a map $g : X \to Y$ such that $gx = y(Fg)$. Algebras and their morphisms can be organized into a category Alg_F. The intuition is that an equation gives rise to an endofunctor F and that a solution to the equation is an F-algebra that is a *fixed point* of F, that is, an F-algebra (X, x) such that x is an isomorphism. The result known as *Lambek's lemma* states

that the initial object of Alg_F is a fixed point for F and so it is reasonable to think of this algebra (when it exists) as the 'least' fixed point for F.

Initial F-algebras need not always exist but they do under fairly general hypotheses as the following well known result shows.

Lemma 4.1. *If $\mathscr{C}$ is cocomplete and $F : \mathscr{C} \to \mathscr{C}$ preserves directed colimits then the category Alg_F has an initial object.*

Proof. The underlying object of the initial algebra is the colimit of the ω-chain $0 \to F0 \to \cdots \to F^i0 \to \cdots$. We denote the colimit by μF. Using preservation of directed colimits it is easy to obtain a map $F\mu F \to \mu F$. The universal property of μF implies that the F-algebra just obtained is initial. □

For concrete cases of Lemma 4.1 see [1] which deals with *finitary* endofunctors on **Set**, the 'existence' part of the *implicit species theorem* of Section 5.2 of [8] and the construction of solutions of differential equations in [11]. Now, in the last two cases there is another important phenomenon going on which is important to abstract.

Definition 4.2. A functor $F : \mathscr{C} \to \mathscr{C}$ is *special* if Alg_F has an initial object and moreover for every F-algebra A, A is a fixed point of F if and only if A is initial.

This captures the *unicité des solutions* of the implicit species theorem in [8] and the last part of Theorem 3.1 in [11]. It is this property that allows the following kind of argument: to prove that two objects are isomorphic, just prove that the two objects are fixed points for a special functor. Notice that this is the argument displayed (at the level of equations) in Theorem 2.2. Our strategy to give a combinatorial interpretation of Stanley's theory is then to prove that there are categories of combinatorial objects on which Stanley's differential and functional equations induce special functors in the sense of Definition 4.2. In this way Stanley's statements remain mainly unchanged. The proofs will have to be modified because the statements will be referring to objects and morphisms in categories with coproducts. But we hope that the reader will agree that the spirit of Stanley's proofs is also preserved.

Leroux and Viennot note in page 213 of [11] that the canonical solutions for the systems of differential equations "remain at certain 'recursive' level. This is the price to pay for a general method that works for any system of differential equations". Naturally, our generalization will have to pay the same price.

Remark 4.3. The definition of special functor has an existence and a uniqueness part. So it is fair to ask if it is worth focusing on the more general notion of a functor such that every fixed point is initial in the category of algebras. We believe that the generalization is not useful. The reason is that, if the functor has an initial algebra then the functor is special. If it does not have an initial algebra, then the condition *fixed-point implies initial* means that there are no fixed points. So there is not much use for it.

Essentially, the *Théorème des espèces implicites* in [8] says that certain functors $F : \mathbf{F}^r \to \mathbf{F}^r$ are special. The main auxiliary notion in the proof of this theorem is that of *contact*.

Definition 4.4. For $f : A \to B$ in $[\mathbb{L}, \mathbf{k}]$ and $F : [\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ define

(1) f is a *contact at n* if $f_n : A[n] \to B[n]$ is an iso.
(2) f is a *contact of order n* if it is a contact at m for every $m \leqslant n$.
(3) F *preserves contacts* if for every ψ contact at n, $F\psi$ is also a contact at n.
(4) F *raises contacts* if for every ψ contact at n , $F\psi$ is a contact at $n + 1$.

For example, if $F : [\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ preserves small coproducts, then F preserves contacts. The integral $\int : [\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ raises contacts.

In order to state the following result more clearly, let us define a functor $F : [\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ to be *constant at* $\emptyset$ *(with coefficient K)* if there exists a K in $\mathbf{k}$ such that for every A in $[\mathbb{L}, \mathbf{k}]$, $(FA)\emptyset = K$ and for every map $f : A \to B$ in $[\mathbb{L}, \mathbf{k}]$, $(Ff)_\emptyset = id_K : K \to K$.

Proposition 4.5. *Let $F : [\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ be such that Alg_F has an initial object. If F is constant at $\emptyset$ and raises contacts then F is special.*

Proof. For the purpose of the proof assume that F is constant at $\emptyset$ *with coefficient* K. Let $\alpha : FA \to A$ be the initial F-algebra and let $\beta : FB \to B$ is an isomorphism. We need to show that the unique map $u : (A, \alpha) \to (B, \beta)$ is an iso. We do this by showing that u is a contact at n for every n, by induction. At stage $\emptyset$ we have the following diagram

$$\begin{array}{ccc} K = (FA)\emptyset & \xrightarrow{\alpha_\emptyset} & A\emptyset \\ {\scriptstyle \mathrm{id}=(Fu)_\emptyset}\downarrow & & \downarrow{\scriptstyle u_\emptyset} \\ K = (FB)\emptyset & \xrightarrow[\beta_\emptyset]{} & B\emptyset \end{array}$$

in other words, $u_\emptyset\, \alpha_\emptyset = \beta_\emptyset$. As both $\alpha_\emptyset$ and $\beta_\emptyset$ are isos, so is $u_\emptyset$.

At stage $n+1$ we have $u_{n+1}\, \alpha_{n+1} = \beta_{n+1}\, (Fu)_{n+1}$. By inductive hypothesis, u_n is an iso and as F raises contacts, $(Fu)_{n+1}$ is an iso. As α and β are isos, u_{n+1} is an iso. □

It is important to notice that the proof of Proposition 4.5 actually constructs an explicit isomorphism. At the time of applying the proposition, the resulting explicit iso may not be at all transparent. But we have the certainty that with some patience and attention we will be able to extract a recursive program out of the proof above.

4.1. Examples of functional equations in $\mathscr{J}$

We can think of an endofunctor $F : \mathscr{C} \to \mathscr{C}$ as determining a functional equation $y = Fy$. Fixed points can be thought of as solutions and the initial algebra as the 'minimal' solution. If F is special, there is essentially one solution. Let us look at some examples in the monoidal category $\mathscr{J} = ([\mathbb{L}, \mathbf{Set}], *, q^0)$ described in Section 3.1.

Example 4.6. As the functor $q * (\_) : [\mathbb{L}, \mathbf{Set}] \to [\mathbb{L}, \mathbf{Set}]$ preserves all colimits, the functor $H_1 = q^0 + q * (\_)$ preserves directed colimits. It is easy to check that it raises contacts so the functor H_1 is special. It is also easy to calculate the initial algebra μH_1 of H_1. Indeed, $\mu H_1 = \frac{1}{1-q}$. □

The following variant of the example above will be also useful.

Example 4.7. The functor $H_2 = q^0 + q^2 * (\_)$ is special and its initial algebra μH_2 can be described as $\frac{1}{1-q^2}$.

Example 4.8. Consider the functor $H_{1,l} : [\mathbb{L}, \mathbf{Set}] \to [\mathbb{L}, \mathbf{Set}]$, for a fixed linear order $l > 0$, defined by

$$H_{1,l}X = \left(\sum_{l_0 \subset l} \left(\frac{q}{1-q}\right)^{l_0}\right) + q^l * X$$

for any X in $\mathscr{J}$. Since $q^l * (\_)$ preserves all colimits and $H_{1,l}$ is obtained by adding a constant, it follows that the latter functor preserves all directed colimits. So the functor must have an initial algebra. As $q^l * (\_)$ raises contacts, so does $H_{1,l}$ and hence $H_{1,l}$ is special. Calculating the colimit of the canonical ω-chain associated with $H_{1,l}$ may not be as easy as in the case of Example 4.6. But the functor is special so in order to get a picture of what the initial algebra looks like we just need to find a fixed point for it. We claim that the object $(\frac{1}{1-q})^l$ of *weak l-compositions* can be given the structure of a fixed point. Indeed there is an obvious isomorphism $H_{1,l}(\frac{1}{1-q})^l \to (\frac{1}{1-q})^l$ as below

$$\left(\sum_{l_0 \subset l} \left(\frac{q}{1-q}\right)^{l_0}\right) + \left(\frac{q}{1-q}\right)^l \to \left(\frac{1}{1-q}\right)^l,$$

which reflects the fact that a partitions of length l can be split in those that have all components non-empty (i.e. $(\frac{q}{1-q})^l$) and those that have some components empty.

Example 4.8 shows how to obtain $(\frac{1}{1-q})^l$ as the solution to a non-trivial functional equation. For different variations involving r-differential posets we will need also the following variant. The proofs that the functor involved is special and that the initial algebra is what it should be are non-problematic variants of Example 4.8.

Example 4.9. For a fixed linear order $l > 0$, non-negative integer r and fixed K in $\mathscr{J}$ consider the functor $\overline{H} : [\mathbb{L}, \mathbf{Set}] \to [\mathbb{L}, \mathbf{Set}]$ defined by

$$\overline{H}X = l!r^l \left(\sum_{l_0 \subset l} \left(\frac{q}{1-q} \right)^{l_0} \right) * K + q^l * X$$

for any X in $\mathscr{J}$. The functor $\overline{H}$ is special for the same reasons that $H_{1,l}$ is (see Example 4.8). We claim that $l!r^l(\frac{1}{1-q})^l * K$ is a fixed point for $\overline{H}$. In fact, this follows easily using that tensoring with $l!r^l K$ distributes over coproducts and using the fixed point structure described in Example 4.8.

4.2. The functor **Q** *and further functional equations*

Let us look at another example. This time, in the category $\mathbf{LV}(\mathscr{J})$ determined by $\mathscr{J}$ and Proposition 3.5. First we define the functor $\mathbf{Q} : [\mathbb{L}, \mathscr{J}] \to [\mathbb{L}, \mathscr{J}]$ as follows

$$\mathbf{Q}\left(\sum_{k \geqslant 0} p_k \frac{t^k}{k!} \right) = \sum_{k \geqslant 0} q^k * p_k \frac{t^k}{k!},$$

where p_k in $\mathscr{J}$ for every k.

Lemma 4.10. *The functor* **Q** *is cocontinuous, monoidal and preserves exponentials.*

Proof. Cocompleteness follows because colimits are calculated pointwise in $[\mathbb{L}, \mathscr{J}]$ and because tensoring with q^k is cocontinuous. The rest is straightforward calculation. □

We are interested in the functor $\mathbf{E}t \cdot \mathbf{Q} : [\mathbb{L}, \mathscr{J}] \to [\mathbb{L}, \mathscr{J}]$ but since it preserves initial object then we cannot apply exactly the same argument as in the previous cases. To deal with the present case fix an object K in $\mathscr{J}$ and consider the subcategory $\mathscr{C}$ of $[\mathbb{L}, \mathscr{J}]$ determined by those objects X such that $X\emptyset = K$ and those morphisms $f : X \to Y$ such that $f_\emptyset = id_K$. The object K is initial in $\mathscr{C}$ so the functor $\mathbf{E}t \cdot \mathbf{Q}$ does not preserve initial object when restricted to $\mathscr{C}$. Obviously the inclusion $\mathscr{C} \to [\mathbb{L}, \mathscr{J}]$ does not preserve the initial object either. But it should be clear that the inclusion creates directed colimits. So the colimits (in $\mathscr{C}$) involved in the construction of free algebras are calculated as in $[\mathbb{L}, \mathscr{J}]$. It follows that $\mathbf{E}t \cdot \mathbf{Q} : \mathscr{C} \to \mathscr{C}$ preserves directed colimits. (The idea of building a category with a different initial object is also what accounts for functional equations with an initial condition.)

Proposition 4.11. *The functor* $\mathbf{E}t \cdot \mathbf{Q}$ *is special when restricted to the category* $\mathscr{C}$*. Its initial algebra can be described as* $\mathbf{E}(\frac{1}{1-q}t)K$.

Proof. Denote the functor in the statement by T so that $TX = \mathbf{E}t \cdot \mathbf{Q}X$. The discussion above implies that T has an initial algebra. To describe it let $A_n = \mathbf{E}((\sum_{i=0}^{n} q^i)t)K$ and then, using that $\mathbf{Q}$ preserves exponentials, it follows that $TA_n = A_{n+1}$. Moreover, $TK = (\mathbf{E}t)K = A_0$. So that, at stage l we have the colimit diagram:

$$A_0 l \longrightarrow A_1 l \quad \cdots \longrightarrow A_n l \longrightarrow \cdots \quad \longrightarrow (\mu T)l,$$

which reduces to the following diagram

$$(q^0)^l * K \longrightarrow (q^0 + q)^l * K \longrightarrow \left(\sum_{i=0}^{n} q^i \right)^l * K \longrightarrow \cdots \longrightarrow (\mu T)l$$

in $\mathscr{J}$. A simple argument shows that $(\mu T)l = (\frac{1}{1-q})^l * K$ so we can conclude that $\mu T = \mathbf{E}(\frac{1}{1-q}t)K$.

We must show that T is special. So assume that we have a fixed point $\beta : TB \to B$. We claim that for each l, $Bl = (\frac{1}{1-q})^l * K$. We prove this by induction. The base case is trivial because $B\emptyset = K$. So assume that $l \neq \emptyset$ and then

notice that

$$(\mathbf{E}t \cdot \mathbf{Q}B)l = \sum_{l_0,l_1 \subseteq l} (\mathbf{E}t)l_0 * (\mathbf{Q}B)l_1 = \sum_{l_1 \subseteq l} q^{l_1} * Bl_1$$

so $(\mathbf{E}t \cdot \mathbf{Q}B)l = (\sum_{l_1 \subset l} q^{l_1} * (\frac{1}{1-q})^{l_1} * K) + q^l * (Bl)$ by induction. But then β_l is essentially providing Bl with a fixed point structure for the functor $H_{1,l}$ of Example 4.8 (actually, a variant tensoring with constant K as in Example 4.9). As this functor is special, it must be the case that $Bl \cong (\frac{1}{1-q})^l K$. □

Using the same ideas as in Proposition 4.11 one proves the following result (we leave the details to the reader).

Proposition 4.12. *The functor* $\mathbf{E}(rt + r\frac{t^2}{2}) \cdot \mathbf{Q} : \mathscr{C} \to \mathscr{C}$ *is special and its initial algebra can be described as* $\mathbf{E}(\frac{r}{1-q}t + \frac{r}{1-q^2}\frac{t^2}{2}) \cdot K$.

We need one more example which we discuss in more detail. The category $[\mathbb{L}, \mathbf{k}]$ has another symmetric monoidal structure which can be described by

$$(F \otimes G)l = \sum_{\substack{l_0+l_1=l \\ l_0'+l_1'=l}} Fl_0 \circ Gl_1,$$

where the cardinalities of l_i and l_i' coincide for i equal to 0 or 1. The idea is to think of an object in $[\mathbb{L}, \mathbf{k}]$ as a series $\sum_{k \geqslant 0} a_k \frac{t^k}{k!^2}$ and then

$$\left(\sum_{k \geqslant 0} a_k \frac{t^k}{k!^2}\right) \otimes \left(\sum_{k \geqslant 0} a_k \frac{t^k}{k!^2}\right) = \sum_k \left(\sum_{i=0}^{k} \binom{k}{i}^2 a_i b_{k-i}\right) \frac{t^k}{k!^2}.$$

But we will not need to go deeper into this monoidal structure. We just need to tensor with a constant.

Proposition 4.13. *The functor* $\frac{1}{1-t} \otimes \mathbf{Q}(\_)$ *is special when restricted to* $\mathscr{C}$ *and its initial algebra can be described as* $\frac{1}{1-\frac{1}{1-q}t} \cdot K$.

Proof. This is similar to Proposition 4.11 but we go through some of the details. We denote the functor in the statement by T so that $TX = \frac{1}{1-t} \otimes \mathbf{Q}X$. If we let $A_n = \frac{1}{1-(\sum_{i=0}^{n} q^i)t} K$ then it is easy to prove that $TA_n = A_{n+1}$. Moreover, $TK = \frac{1}{1-t}K = A_0$. In other words, we have that $A_n l = l!(\sum_{i=0}^{n} q^i)^l K$ and so, much as in Proposition 4.11, we can conclude that $(\mu T)l = l!(\frac{1}{1-q})^l K$ and hence that $\mu T = \frac{1}{1-\frac{1}{1-q}t} K$.

We must show that T is special. So assume that we have a fixed point $\beta : TB \to B$. We claim that for each l, $Bl = l!(\frac{1}{1-q})^l K$. We prove this by induction. The base case is trivial because $B\emptyset = K$. So assume that $l \neq \emptyset$ and then notice that

$$\begin{aligned}\left(\frac{1}{1-t} \otimes \mathbf{Q}B\right) l &= \sum_{i=0}^{l} \binom{l}{l-i}^2 (l-i)!q^i * Bi \\ &= \left(\sum_{i=0}^{l-1} \binom{l}{l-i}^2 (l-i)!q^i * Bi\right) + q^l * Bl\end{aligned}$$

so that by induction

$$\begin{aligned}\left(\frac{t}{1-t} \otimes \mathbf{Q}B\right) l &= \left(\sum_{i=0}^{l-1} \binom{l}{l-i}^2 (l-i)!i!q^i * \left(\frac{1}{1-q}\right)^i K\right) + q^l * Bl \\ &= l! \left(\sum_{i=0}^{l-1} \binom{l}{i} \left(\frac{q}{1-q}\right)^i\right) K + q^l * Bl,\end{aligned}$$

but then β_l is essentially providing Bl with a fixed point structure for the functor $\overline{H}$ of Example 4.9 (with $r = 1$). As this functor is special, it must be the case that $Bl \cong l!(\frac{1}{1-q})^l K$. □

It should be clear how to modify the above for the case $r > 1$.

4.3. Differential equations reduced to functional equations

Following [11] we explain here how to produce solutions to differential equations using Proposition 4.5. We only need the case of one equation so we restrict to that case. For us, a differential equation in $\mathbf{LV}(\mathbf{k})$ is given by a functor $G : [\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ and an object K in $\mathbf{k}$. We think of this data as expressing the equation $\partial_t y = Gy$ with initial condition $y\emptyset = K$.

Lemma 4.14. *Let G preserve contacts. If F is defined by $FA = K + \int GA$ then it is constant at $\emptyset$ and raises contacts.*

Proof. Assume that ψ is a contact at n. Then

$$(F\psi)_{n+1} = id_K + \left(\int G\psi\right)_{n+1} = id_K + (G\psi)_n.$$

As G preserves contacts $(G\psi)_n$ is an iso. So $(F\psi)_{n+1}$ is an iso. □

So, if in the situation above it also happens that Alg_F has an initial object (e.g. if G preserves directed colimits) then F is special by Proposition 4.5. The relation with differential equations was described by Leroux and Viennot in Section 3 of [11]: there is an iso $K + \int GA \cong A$ if and only if $A\emptyset \cong K$ and $\partial A \cong GA$. So, in the notation of the previous lemma, the initial algebra for F is the solution to the combinatorial differential equation $\partial y = Gy$ with boundary condition $y\emptyset = K$.

Example 4.15 (*Leroux and Viennot [11] Example 4.1*). The essentially unique solution to the differential equation $\partial_t y = t^0 + y \cdot y$ with initial condition $y\emptyset = \emptyset$ in the category $\mathbf{LV}(\mathbf{Set}, \times, 1)$ can be described as *complete increasing binary trees* or as *alternating descending odd permutations*.

Example 4.16 (*Leroux and Viennot [11] Example 4.3*). The differential equation $\partial_t y = \frac{1}{1-y}$ with initial condition $y\emptyset = \emptyset$ has a solution which can be described as *increasing planar trees*.

Some of the main results of the present paper will rely on the following class of examples.

Example 4.17. Let $\mathbf{k}$ be a cocomplete category of coefficients and let K and L be non-initial objects of $\mathbf{k}$. We consider a differential equation in the category $\mathbf{LV}(\mathbf{k})$. If tensoring with K induces a cocontinuous functor $[\mathbb{L}, \mathbf{k}] \to [\mathbb{L}, \mathbf{k}]$ (as it does in our examples), the differential equation $\partial_t y = K \cdot y$ with initial condition L has a unique solution. We can then conclude that the solution $\mathbf{E}(Kt)L$ is *the* solution for this differential equation.

Remark 4.18. It seems relevant to mention that there exists also a theory of combinatorial differential equations for Joyal's *ordinary* species which has a completely different flavor from that by Leroux and Viennot. See for example [10] and [15].

5. Weyl categories

As recalled in the introduction, Stanley starts with a field K_0 of characteristic 0 and denotes the field of Laurent series with coefficients in K_0 by K. He then defines the K-vector space $\widehat{K}P$ of arbitrary linear combinations $\sum_{x \in P} c_x x$ with $c_x \in K$ and introduces exponential power series with coefficients in the ring $\mathrm{End}(\widehat{K}P)$ of endomorphisms of $\widehat{K}P$. His Theorem 2.5 is a statement about such exponential power series.

Our combinatorial interpretation of Theorem 2.2 requires the construction of a suitable category of coefficients that will play the role that $\mathrm{End}(\widehat{K}P)$ plays in Stanley's formulation. But key portions of the proofs rely only in simple

algebraic properties so we believe that it is convenient to capture these properties in a way that it is independent of their combinatorial manifestations. One way of doing this is through down-up algebras as in [2]. We will insist on requiring categorical coproducts.

Definition 5.1. Let r be a positive integer. A category of coefficients $(\mathscr{W}, \circ, I)$ is called an *r-Weyl category* if it is equipped with a choice of two objects D and U in $\mathscr{W}$ and morphisms $\sigma_0 : U \circ D \to D \circ U$ and $\sigma_1 : rI \to D \circ U$ such that the morphism $[\sigma_0, \sigma_1] : U \circ D + rI \to D \circ U$ is an isomorphism.

If $\mathscr{W}$ is an r-Weyl category then we will just use juxtaposition instead of $\circ$; in this way we can rewrite the isomorphism in Definition 5.1 as $DU \cong UD + rI$.

Remark 5.2. The terminology is obviously intended to make reference to Weyl algebras. Indeed, the complex associative algebra generated by elements d and u subject to the relation $du - ud = 1$ is the Weyl algebra sometimes denoted by A_1. As explained in [3], there is a natural action of A_1 on $\mathbb{C}[x]$ defined by $d \cdot P(x) = P'(x)$ and $u \cdot P(x) = xP(x)$ and this establishes an isomorphism of A_1 with the algebra of differential operators with polynomial coefficients (in one variable). Of course, the condition $du - ud = 1$ is just a manifestation of the Leibniz rule and as explained in the introduction we will think of D as $\partial/\partial U$.

Notice that the requirement (in Definition 5.1) of $+$ being categorical coproduct prevents rings from being examples of Weyl categories. But it is not surprising that for many identities valid in A_1 (not involving $-$) there are analogous isomorphisms in Weyl categories. There are probably deeper connections between Weyl categories and Weyl algebras but we will not attempt to pursue them here.

In this section we establish a number of basic facts about Weyl categories and we associate to every poset P a category of coefficients $\mathscr{W}_P$ equipped with a canonical choice of objects D and U. This choice of objects extends to an r-Weyl category structure on $\mathscr{W}_P$ if and only if P is r-differential. This result is, of course, analogous to Theorem 2.2 in [18].

Under a suitable completeness hypothesis on an r-Weyl category $\mathscr{W}$ we will be able to solve combinatorial and functional equations in $\mathbf{LV}(\mathscr{W})$ and prove a result analogous to Theorem 2.2 which, applied to the case $\mathscr{W} = \mathscr{W}_P$ will provide the combinatorial information obtained by Stanley.

Let $\mathscr{W}$ be an r-Weyl category. When seen as *constant* objects in $\mathbf{LV}(\mathscr{W})$, D and U provide this category with an r-Weyl structure. Whenever we consider a category of the form $\mathbf{LV}(\mathscr{W})$ we will assume that it is equipped with this Weyl structure.

5.1. The Weyl categories of operators

As we have already explained, our version of Stanley's theorem replaces the ring $\mathrm{End}(\widehat{K}P)$ with a suitably cocomplete Weyl category (Definition 5.1). But naturally, in order to derive concrete enumerative corollaries, concrete Weyl categories will have to be used. In this section we build such categories relying on well-known analogies between linear maps and cocontinuous functors.

Think of the category **Set** as K. If we let $|P|$ be the underlying set of the poset P then $[|P|, \mathbf{Set}]$ should be thought of as $\widehat{K}P$. Indeed, a functor $|P| \to \mathbf{Set}$ is essentially a family of sets indexed by $|P|$ and we will sometimes write $\sum_{x \in P} a_x$ with a_x in **Set** for an object in this category (and may even speak of *vectors*). Because of this we denote the category $[|P|, \mathbf{Set}]$ by $\widehat{P}$.

The natural next step is to consider the category of cocontinuous functors $\widehat{P} \to \widehat{P}$ as an alternative to the ring $\mathrm{End}(\widehat{K}P)$. This is essentially what we will do. But in order to have a category that is easier to visualize, we will rely on categorical results (using Kan extensions) which allow to describe such functors in more concrete terms. Readers unfamiliar with Kan extensions can simply take the monoidal categories $\mathscr{W}_P = ([|P \times P|, \mathbf{Set}], \circ, \delta)$ described below as combinatorial alternatives to $\mathrm{End}(\widehat{K}P)$.

For $A, B : |P \times P| \to \mathbf{Set}$ we define

$$(B \circ A)(x, y) = \sum_{t \in P} A(x, t) \times B(t, y)$$

and $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = \emptyset$ if $x \neq y$. With these definitions the proof of the following result is straightforward.

Lemma 5.3. *For any poset P, the structure $\mathscr{W}_P = ([|P \times P|, \mathbf{Set}], \circ, \delta)$ is a monoidally cocomplete (non-symmetric) monoidal category.*

We now introduce objects U and D in $\mathscr{W}_P$. The object U is determined by defining that $U(x, y)$ is the singleton 1 if y covers x and it is $\emptyset$ otherwise. The object D is defined by declaring $D(x, y)$ to be the singleton if x covers y and to be empty otherwise. In other words, $U(x, y) = D(y, x)$.

Lemma 5.4. *For any poset P, the following hold:*

(1) $DU(x, y) = x^+ \cap y^+$
(2) $UD(x, y) = x^- \cap y^-$

Proof. For the first item just calculate

$$DU(x, y) = \sum_{t \in P} U(x, t) \times D(t, y) \cong x^+ \cap y^+.$$

The second item is analogous. □

Before we state the next result let us make a short comment on axioms $(D2)$ and $(D3)$. As stated in [18] and reproduced in Section 1 these axioms state that certain pairs of finite sets have the same number of elements. Without being very explicit about it we are going to assume that the axioms actually provide concrete isomorphisms. For example, $(D3)$ should provide for each x in P an isomorphism $x^- + r \to x^+$. (Compare with the *R-correspondences* as in Definition 2.6.1 in [16].) With this in mind let us state a result analogous to Theorem 2.2 in [18].

Theorem 5.5. *Let P be a poset and r be a positive integer. Then P satisfies $(D2)$ and $(D3)$ if and only if the objects D and U extend to an r-Weyl structure on $\mathscr{W}_P$.*

Proof. We need to define an iso $DU(x, y) \to UD(x, y) + r\delta(x, y)$. From Lemma 5.4 we see that Axiom (D2) provides the iso in the case when $x \neq y$ and axiom (D3) provides the iso when $x = y$. □

In order to exemplify the combinatorial importance of the operators D and U Stanley highlights the following examples. (Although the lemma below is not stated as such, its content follows from Proposition 3.1 in [18] and the discussion following it.)

Lemma 5.6. *For D and U as defined above,*

(1) *$D^n(y, x)$ is the set of chains $x = x_0 < x_1 < \cdots < x_n = y$ in P such that x_i covers x_{i-1} for each $1 \leqslant i \leqslant n$.*
(2) *$(U + D)^n(y, x)$ is the set of sequences $x = x_0, x_1, \ldots, x_n = y$ such that for each $1 \leqslant i \leqslant n$, either x_i covers x_{i-1} or x_{i-1} covers x_i.*
(3) *$D^n U^n(x, x)$ is the set of closed walks of the form*

$$x < x_1 < \cdots x_{n-1} < x_n > y_{n-1} > \cdots > y_1 > x.$$

Let $\rho : P \to \mathbb{N}$ be a graded poset. Denote by $\alpha(n \to n + k)$ the set of chains $x_0 < x_1 < \cdots < x_k$ such that $\rho x_0 = n$ and for each $i < k$, x_{i+1} covers x_i. To discuss *closed* Hasse walks we denote by $\kappa(n \to n + k \to n)$ the set of Hasse walks $x_0 < x_1 < \cdots < x_k > x_{k+1} > \cdots > x_{2k}$ such that $\rho x_0 = n$, $x_0 = x_{2k}$ and each element is covered or is covered by the next in the obvious way. We will denote the set of closed Hasse walks that first go down and then up by $\kappa(n \to n - k \to n)$. The notation is borrowed from [18] although notice that Stanley uses $\alpha(n \to n + k)$ to denote the *number* of saturated chains while for us it is the *set* of such. We thought it inconvenient to introduce new notation to stress the difference. An analogous remark holds for κ. Finally notice that there is an obvious isomorphism $\kappa(n \to n + k \to n) \cong \kappa(n + k \to n \to n + k)$.

5.2. The combinatorial meaning of $f(U)$

Stanley's Theorem 2.2 involves ordinary power series $f(U)$ and $h(U)$ in the ring $K_0[\![U]\!]$. In this short section we explain how we will re-interpret such objects in the context of Weyl categories.

Let $\mathscr{W}$ be a monoidally cocomplete category and consider the functor $\mathbb{L} \to \mathscr{W}$ defined by $[k] \mapsto U^k$. It is clearly monoidal so, as the monoidal category $\mathscr{J}$ is the result Day's convolution, its universal property induces an essentially unique cocontinuous monoidal functor $\mathbf{u} : \mathscr{J} \to \mathscr{W}$ which intuitively assigns U to q. Explicitly $\mathbf{u}(\sum_{n\geqslant 0} a_n q^n) = \sum_{n\geqslant 0} a_n U^n$. So, for f in $\mathscr{J}$, $\mathbf{u}f$ will correspond to what Stanley writes as $f(U)$.

Certainly, Stanley's notation is more intuitive but ours is more efficient in some calculations. Moreover, we will sometimes find it useful to write $\mathbf{u}f$ as $\overline{f}$. In this way, one is able to make explicit the nature of an object in a formula without introducing new letters. This change in notation may appear artificial at first but we hope the reader will appreciate its advantages after using it in some calculations. In the statement of the main results we will recall the notation we introduced and relate it to Stanley's in order to ease the reading of the paper.

5.3. Residuated functors

Corollary 2.4 in [18] states that $Df(U) = rf'(U) + f(U)D$ holds for every $f(U)$ in $K[\![U]\!]$. Stanley explains that the effect of this result is that we can informally view D as the derivative $\frac{\partial}{\partial U}$. Of course, there will be an analogue in our setting. But before we state it let us introduce another relevant notion.

Definition 5.7. Let $\mathscr{W}$ be an r-Weyl category. A functor $\Psi : \mathscr{C} \to \mathscr{W}$ is called a *residuated* functor if it comes equipped with a functor $R : \mathscr{C} \to \mathscr{C}$ and natural transformations $\alpha_x : r\Psi(Rx) \to D(\Psi x)$ and $\beta_x : (\Psi x)D \to D(\Psi x)$ such that $[\alpha, \beta] : r\Psi(Rx) + (\Psi x)D \to D(\Psi x)$ is an isomorphism. The functor R is called the *residue* of Ψ. We also say that Ψ *is residuated by* R.

Usually we will avoid writing down α and β explicitly and simply work with the iso $D(\Psi x) \cong r\Psi(Rx) + (\Psi x)D$.

For example, let $\mathscr{W}$ be a monoidally cocomplete category and consider the functor $\mathbf{u} : \mathbb{L} \to \mathscr{W}$ defined in Section 5.2 in order to interpret objects in $K_0[\![U]\!]$. (Recall that for certain purposes we will denote $\mathbf{u}f$ by $\overline{f}$.)

Lemma 5.8. *The functor* $\mathbf{u} : \mathscr{J} \to \mathscr{W}$ *is residuated by* ∂_q. *In other words, for any* f *in* $\mathscr{J}$, $D\overline{f} \cong r\overline{\partial_q f} + \overline{f}D$.

Proof. Essentially the same proof of Corollary 2.4 in [18] except that we are working with isomorphisms instead of equality. But the fact that coproducts are commutative and that tensoring on either side preserves them allows the same proof to go through. □

The next two results allow us to build new residuated functors which, in turn, will play an important role in the proof of Theorem 6.1.

Lemma 5.9. *Let* $\Psi : \mathscr{C} \to \mathscr{W}$ *be residuated by* $R : \mathscr{C} \to \mathscr{C}$. *If* $\mathscr{C}$ *has finite coproducts and* Ψ *preserves them then the functor*

$$\mathscr{C} \xrightarrow{R_{+crt}} [\mathbb{L}, \mathscr{C}] \xrightarrow{\Psi_*} [\mathbb{L}, \mathscr{W}]$$

is also residuated by R.

Proof. We need an iso $r\Psi_*(R_{+crt}(Rx)) + \Psi_*(R_{+crt}x)D \to D\Psi_*(R_{+crt}x)$. That is: a family $r\Psi((R_{+crt}(Rx))[k]) + \Psi((R_{+crt}x)[k])D \to D\Psi((R_{+crt}x)[k])$ of isos indexed by $[k] \in \mathbb{L}$ and natural in x. But $(R_{+crt}(Rx))[k] = (cr)^k R(R^k x)$ so, using that Ψ preserves sums by hypothesis (and that D also does) then we need a family $(cr)^k r\Psi(R(R^k x)) + (cr)^k \Psi(R^k x)D \to (cr)^k D\Psi(R^k x)$. Hence, we have reduced the problem to finding a natural family of isomorphisms $r\Psi(R(R^k x)) + \Psi(R^k x)D \to D\Psi(R^k x)$. This is provided by the assumption that Ψ is residuated by R. □

Lemma 5.9 will be used in Section 5.4 to prove that our combinatorial interpretation of the assignment $h(U) \mapsto h(U + crt)$ is a residuated functor. The following lemma is almost immediate.

Lemma 5.10. *Let $\Psi : \mathscr{C} \to \mathscr{W}$ be residuated by R. Then $\Psi_* : [\mathbb{L}, \mathscr{C}] \to [\mathbb{L}, \mathscr{C}]$ is resituated by $R_* : [\mathbb{L}, \mathscr{C}] \to [\mathbb{L}, \mathscr{C}]$.*

We can use it to prove the following.

Lemma 5.11. *Let $\alpha : \mathscr{J} \to \mathscr{W}$ be a monoidal functor preserving finite coproducts and residuated by ∂_q. Then*

$$D(\mathbf{E}(\alpha_* F)) \cong r(\alpha_* \partial_{q_*} F) \cdot \mathbf{E}(\alpha_* F) + \mathbf{E}(\alpha_* F) D$$

for every F in $\mathbf{LV}(\mathscr{J})$.

Proof. Calculate:

$$\begin{aligned} D(\mathbf{E}(\alpha_* F)) &= D\alpha_*(\mathbf{E}F) && \text{Lemma 3.7} \\ &= r\alpha_* \partial_{q_*}(\mathbf{E}F) + (\alpha_*(\mathbf{E}F))D && \text{Lemma 5.10} \\ &= r\alpha_*(\partial_{q_*} F \cdot (\mathbf{E}F)) + \mathbf{E}(\alpha_* F)D && \text{Lemma 3.8 + Lemma 3.7} \\ &= r(\alpha_*(\partial_{q_*} F)) \cdot \mathbf{E}(\alpha_* F) + \mathbf{E}(\alpha_* F)D && \text{Lemma 3.7.} \quad \square \end{aligned}$$

We will mainly be interested in the case when $\alpha = \mathbf{u} = \overline{(\_)}$.

Corollary 5.12. *For any F in $\mathbf{LV}(\mathscr{J})$,*

$$D\left(\mathbf{E}\left(\int \mathbf{u}_* F\right)\right) \cong r\left(\int \mathbf{u}_* \partial_{q_*} F\right) \cdot \mathbf{E}\left(\int \mathbf{u}_* F\right) + \mathbf{E}(\mathbf{u}_* F) D.$$

Proof. Calculate:

$$\begin{aligned} D\left(\mathbf{E}\left(\textstyle\int (\mathbf{u}_* F)\right)\right) &= D\left(\mathbf{E}\left(\mathbf{u}_*\left(\textstyle\int F\right)\right)\right) && \text{Lemma 3.7} \\ &= r\left(\mathbf{u}_* \partial_{q_*} \textstyle\int F\right) \cdot \mathbf{E}\left(\mathbf{u}_* \textstyle\int F\right) + \mathbf{E}\left(\mathbf{u}_* \textstyle\int F\right) D && \text{Lemma 5.11} \\ &= r\left(\textstyle\int \mathbf{u}_* \partial_{q_*} F\right) \cdot \mathbf{E}\left(\textstyle\int \mathbf{u}_* F\right) + \mathbf{E}(\mathbf{u}_* F) D && \text{Lemma 3.7.} \quad \square \end{aligned}$$

Before we move on let us give another small application of Lemma 5.11. It is taken from inside the proof of Theorem 2.5(b) in [18] and will be used in Section 6.1.

Corollary 5.13. *Let T in $\mathbf{LV}(\mathscr{W})$ be defined by $T = c(r + U)t + c^2 r \frac{t^2}{2}$ then*

$$D\mathbf{E}T = crt \cdot \mathbf{E}T + (\mathbf{E}T)D.$$

Proof. Notice that T is

$$(cr + cU)t + c^2 r \frac{t^2}{2} = \mathbf{u}(cr + cq)t + \mathbf{u}c^2 r \frac{t^2}{2} = \mathbf{u}_*\left((cr + cq)t + c^2 r \frac{t^2}{2}\right).$$

Then, by Lemma 5.11, we need only to check that

$$\partial_{q_*}\left((cr + cq)t + c^2 r \frac{t^2}{2}\right) = \partial_q(cr + cq)t + (\partial_q c^2 r)\frac{t^2}{2} = ct,$$

so the result follows. $\square$

5.4. The combinatorial meaning of $h(U + crt)$

In Section 3.4 we observed that the exponential series $h(U + crt)$ appearing in Stanley's equality $\mathrm{e}^{(f(U)+cD)t} h(U) = \mathrm{e}^{\int_0^t f(U+crs)\,\mathrm{d}s} h(U + crt)\mathrm{e}^{cDt}$ is not constant. We then mentioned that the functor ∂_{q+crt} is a combinatorial analogue

of the assignment $h(q) \mapsto h(q + crt)$ and explained that it was going to be used to give a combinatorial interpretation of the assignment $h(U) \mapsto h(U + crt)$. In this section we explain how this is done.

First recall that in Section 5.2 we explained that we are going to replace statements of the form $h(U) \in K_0[\![U]\!]$ by an object $\mathbf{u}h = \overline{h}$ for an object h in $\mathscr{J}$. Now, for monoidally cocomplete $\mathscr{W}$, denote the composition

$$\mathscr{J} \xrightarrow{\partial_{q+crt}} [\mathbb{L}, \mathscr{J}] \xrightarrow{\mathbf{u}_*} [\mathbb{L}, \mathscr{W}]$$

by $\mathbf{v} : \mathscr{J} \to [\mathbb{L}, \mathscr{W}]$. In power series notation we have, for any h in $\mathscr{J}$, that $\mathbf{v}h = \sum_{k \geqslant 0} c^k r^k \mathbf{u}(\partial_q^k h)\frac{t^k}{k!} = \sum_{k \geqslant 0} c^k r^k \overline{\partial_q^k h}\frac{t^k}{k!}$. As in the case of $\mathbf{u}$ we leave r and c implicit as they will always be clear from the context. Also as in the case of $\mathbf{u}$ we introduce a nameless notation to facilitate some of the calculations. Indeed we sometimes write $\underline{h}$ instead of $\mathbf{v}h$. All told, the object $\underline{h}$ is what corresponds to $h(U + crt)$ in our setting.

We now prove key properties of the relation between D in $\mathscr{W}$ and the two functors $\mathbf{u}$ and $\mathbf{v}$. The proofs of these do not involve differential equations. For the rest of the section assume that $\mathscr{W}$ is monoidally cocomplete.

Corollary 5.14. *For any h in $\mathscr{J}$, $\partial_t(\mathbf{v}h) \cong cr\mathbf{v}(\partial_q h)$. That is: $\partial_t \underline{h} \cong cr\underline{\partial_q h}$.*

Proof. Use $\partial_t \mathbf{u}_* = \mathbf{u}_* \partial_t$ (Lemma 3.7) and Lemma 3.10 applied to $\psi = \partial_q$. □

Now recall the notion of residuated functor (Definition 5.7).

Corollary 5.15. *The functor $\mathbf{v}$ is residuated by ∂_q. In other words: for every h in $\mathscr{J}$ $D\underline{h} \cong r\underline{\partial_q h} + \underline{h}D$.*

Proof. Use Lemma 5.8 and Lemma 5.9 applied to $\Psi = \mathbf{u}$ and $R = \partial_q$. □

Item 3 in Proposition 5.16 below will be essential in the proof of the main Theorem while items 1 and 2 are used to prove item 3.

Proposition 5.16. *For any f in $\mathscr{J}$, there are in $\mathbf{LV}(\mathscr{W})$, isos as below*

(1) $\underline{f} \cong \overline{f} + cr \int \underline{\partial_q f}$

(2) $D\mathbf{E}(\int \underline{f}) \cong r(\int \underline{\partial_q f}) \cdot \mathbf{E}(\int \underline{f}) + \mathbf{E}(\int \underline{f})D$

(3) $(\overline{f} + cD) \cdot \mathbf{E}(\int \underline{f}) \cong \underline{f} \cdot \mathbf{E}(\int \underline{f}) + \mathbf{E}(\int \underline{f})(cD)$.

Notice that D appears above as a constant object in $\mathbf{LV}(\mathscr{W})$.

Proof. Item 1 follows from Lemma 3.9 applied to $\psi = \partial_q$. To prove item 2 apply Corollary 5.12 to $F = \partial_{q+crt}$. To prove item 3 we use the same trick that Stanley uses in the proof reproduced in the introduction (see Theorem 2.2); namely we apply item 1. Indeed, we calculate as below:

$$\begin{aligned}
\underline{f} \cdot \mathbf{E}\left(\int \underline{f}\right) + \mathbf{E}\left(\int \underline{f}\right)(cD) &= \left(\overline{f} + cr \int \underline{\partial_q f}\right) \cdot \mathbf{E}\left(\int \underline{f}\right) + c\mathbf{E}\left(\int \underline{f}\right) D \quad \text{item 1} \\
&= \overline{f}\mathbf{E}\left(\int \underline{f}\right) + \left(cr \int \underline{\partial_q f}\right) \cdot \mathbf{E}\left(\int \underline{f}\right) + c\mathbf{E}\left(\int \underline{f}\right) D \\
&= \overline{f}\mathbf{E}\left(\int \underline{f}\right) + c\left(\left(r \int \underline{\partial_q f}\right) \cdot \mathbf{E}\left(\int \underline{f}\right) + \mathbf{E}\left(\int \underline{f}\right) D\right) \\
&= \overline{f}\mathbf{E}\left(\int \underline{f}\right) + cD\mathbf{E}\left(\int \underline{f}\right) \qquad \text{item 2} \\
&= (\overline{f} + cD) \cdot \mathbf{E}\left(\int \underline{f}\right). \qquad \square
\end{aligned}$$

6. The main result

We now have all the concepts necessary to give our combinatorial interpretation of Stanley's Theorem 2.2. The main idea is that this result is essentially a statement about categories of the form $\mathbf{LV}(\mathscr{W}\hat{})$ for an r-Weyl category $\mathscr{W}$, and that a sufficient condition for it to hold is that $\mathscr{W}\hat{}$ be monoidally cocomplete. Stanley's enumerative corollaries are obtained by replacing $\mathscr{W}\hat{}$ with the category $\mathscr{W}_P$ described in Section 5.1.

The equality $\mathrm{e}^{(f(U)+cD)t}h(U) = \mathrm{e}^{\int_0^t f(U+crs)\,\mathrm{d}s}h(U+crt)\mathrm{e}^{cDt}$ stated in Stanley's theorem involves ordinary power series $f(U)$ and $h(U)$ in $K_0[\![U]\!]$ which determine $h(U+crt)$ and $f(U+crs)$ in $\mathrm{End}(\widehat{K}P)$.

In our interpretation, f and h are objects in $\mathscr{J}$ and, as explained in Section 5.2 these induce objects $\mathbf{u}f$ and $\mathbf{u}h$ in $\mathscr{W}\hat{}$ (which are analogous to $f(U)$ and $h(U)$) and which we denote by $\overline{f}$ and $\overline{h}$ to ease the main calculations. Also, f and h induce objects $\mathbf{v}f$ and $\mathbf{v}h$ in $\mathbf{LV}(\mathscr{W}\hat{})$ which, as explained in Section 5.4 are analogous to $f(U+crt)$ and $h(U+crt)$. (Again, we will use the nameless notation so that $\mathbf{v}f$ will be denoted by $\underline{f}$ and $\mathbf{v}h$ by $\underline{h}$.)

Theorem 6.1. *Let $\mathscr{W}$ be a monoidally cocomplete r-Weyl category. For any f, h in $\mathscr{J}$ and $c \in \mathbb{N}$ there exists an isomorphism*

$$\mathbf{E}((\overline{f}+cD)t)\cdot\overline{h} \cong \mathbf{E}\left(\int \underline{f}\right)\cdot\underline{h}\cdot\mathbf{E}(cDt)$$

in the category $\mathbf{LV}(\mathscr{W}\hat{})$.

Proof. We can first show that $\mathbf{E}((\overline{f}+cD)t) \cong \mathbf{E}\left(\int \underline{f}\right)\cdot\mathbf{E}(cDt)$ and then that $\mathbf{E}(cDt)\cdot\overline{h} \cong \underline{h}\cdot\mathbf{E}(cDt)$. In the first case we prove that both $\mathbf{E}((\overline{f}+cD)t)$ and $\mathbf{E}\left(\int \underline{f}\right)\cdot\mathbf{E}(cDt)$ are solutions for $\partial_t y = (\overline{f}+cD)\cdot y$ with initial condition $y\emptyset = I$. The first object is clearly a solution, so consider the second. The initial condition is trivially satisfied. Now, using the Leibniz rule, the formula for deriving exponentials, the distributive law and finally Proposition 5.16(3) we obtain:

$$\begin{aligned}
\partial_t\left(\mathbf{E}\left(\int \underline{f}\right)\cdot\mathbf{E}(cDt)\right) &= \partial_t\mathbf{E}\left(\int \underline{f}\right)\cdot\mathbf{E}(cDt) + \mathbf{E}\left(\int \underline{f}\right)\cdot\partial_t\mathbf{E}(cDt)\\
&= \underline{f}\cdot\mathbf{E}\left(\int \underline{f}\right)\cdot\mathbf{E}(cDt) + \mathbf{E}\left(\int \underline{f}\right)\cdot(cD)\cdot\mathbf{E}(cDt)\\
&= \left[\underline{f}\cdot\mathbf{E}\left(\int \underline{f}\right) + \mathbf{E}\left(\int \underline{f}\right)\cdot(cD)\right]\cdot\mathbf{E}(cDt)\\
&= (\overline{f}+cD)\cdot\mathbf{E}\left(\int \underline{f}\right)\cdot\mathbf{E}(cDt),
\end{aligned}$$

so the theory of combinatorial differential equations provides the required isomorphism.

To build the second isomorphism we show that both $\mathbf{E}(cDt)\cdot\overline{h}$ and $\underline{h}\cdot\mathbf{E}(cDt)$ satisfy the differential equation $\partial_t y = (cD)\cdot y$ with initial condition $y\emptyset = \overline{h}$. Clearly, $\partial_t(\mathbf{E}(cDt)\cdot\overline{h}) = (cD)\cdot\mathbf{E}(cDt)\cdot\overline{h}$. On the other hand, calculate:

$$\begin{aligned}
\partial_t(\underline{h}\cdot\mathbf{E}(cDt)) &= \partial_t\underline{h}\cdot\mathbf{E}(cDt) + \underline{h}\cdot\partial_t\mathbf{E}(cDt)\\
&= cr\underline{\partial_q h}\cdot\mathbf{E}(cDt) + \underline{h}(cD)\mathbf{E}(cDt) \quad \text{Corollary 5.14}\\
&= c(r\underline{\partial_q h} + \underline{h}D)\cdot\mathbf{E}(cDt)\\
&= c(D\underline{h})\cdot\mathbf{E}(cDt) \quad \text{Corollary 5.15}
\end{aligned}$$

and so, the theorem follows. □

Notice that while the proof of Theorem 6.1 follows Stanley's strategy, all the uses of *minus* or *division* have been removed.

6.1. The isomorphism $D\mathbf{P} \cong (U + r)\mathbf{P}$

Theorem 2.5 in [18] has a second part. That is, apart from the equality $e^{(f(U)+cD)t}h(U) = e^{\int_0^t f(U+crs)\,ds}h(U + crt)e^{cDt}$, Theorem 2.5 states that under certain conditions on $f(U)$ and $h(U)$, the following equation

$$e^{(f(U)+cD)t}h(U)\mathbf{P} = e^{(crt+c^2r\frac{t^2}{2}+cUt+\int_0^t f(U+crs)\,ds)}h(U + crt)\mathbf{P}$$

holds, where $\mathbf{P} = \sum_{x\in P} P$ in $\widehat{K}P$.

In this section we prove a result analogous to the one just stated. Of course, as in the case of Theorem 6.1 we need to reinterpret some of Stanley's ideas. In his approach, D and U are operators on the vector space $\widehat{K}P$ so he defines the object $\mathbf{P}$ in $\widehat{K}P$ and whose main property is that $D\mathbf{P} = (U + r)\mathbf{P}$ holds in $\widehat{K}P$. We have decided to abstract from this situation and reformulate this equation as an isomorphism taking place in the 'category of operators'. The relation between the two ideas will appear more explicit in Section 7.1.

Definition 6.2. Let $\mathscr{W}$ be an r-Weyl category. An object A in $\mathscr{W}$ is called *switching* if there exist maps $UA \to DA$ and $rA \to DA$ such that the induced $UA + rA \to DA$ is an isomorphism.

For example, if P is a differential poset and we denote the terminal object of $\mathscr{W}_P$ by $\mathbf{1}$ then $\mathbf{1}$ is switching. The terminal object is concretely defined by $\mathbf{1}(x, y) = 1$ the singleton set for every (x, y) so that $D\mathbf{1}(x, y) = \sum_p D(p, y)$ which is the set of elements in P that cover y. Analogously, $U\mathbf{1}(x, y)$ is the set of elements covered by y. So, as it is observed in Theorem 2.3 in [18], only $(D3)$ is used to prove that $\mathbf{1}$ is switching.In order to state the following result recall (from Section 5.2) that if $\mathscr{W}$ is monoidally cocomplete then we have a functor $\mathbf{u} : \mathscr{J} \to \mathscr{W}$ which intuitively replaces q by U. (Recall also that for notational convenience we sometimes write $\overline{f}$ instead of $\mathbf{u}f$.)

Lemma 6.3. *Let $\mathscr{W}$ be monoidally cocomplete and A be switching in $\mathscr{W}$. Then we have, in $\mathbf{LV}(\mathscr{W})$, the following isomorphisms*:

(1) $D\overline{f}A \cong r\overline{(\partial_q f)} + (U + r)\overline{f}A$, *for every f in $\mathscr{J}$*;
(2) $\mathbf{E}(cDt)A \cong \mathbf{E}(c(U + r)t + c^2r\frac{t^2}{2})A$.

Proof. The first item is a simple corollary of Lemma 5.8. (See also Corollary 2.4(b) in [18].) For the second item we follow the proof of Theorem 2.5(b) in [18], we show that both sides of the equation are solutions to the differential equation $\partial_t y = cDy$ with initial condition A. The left hand side of the equation clearly is a solution. So consider the right hand side. The initial condition is trivially satisfied. For the non-trivial case let $L = \mathbf{E}(c(U + r)t + c^2r\frac{t^2}{2})$. We then have that $\partial_t L \cong (c(U + r) + c^2rt) \cdot L \cong cL \cdot ((U + r) + crt)$. As we are assuming that A is switching, we have that

$$(U + r + crt)A = (U + r)A + crtA = DA + crtA = (crt + D)A.$$

Using Corollary 5.13 we can conclude that

$$\begin{aligned}\partial_t(LA) &= (\partial_t L)A = cL \cdot ((U + r) + crt)A \\ &= cL \cdot (crt + D)A = c(crtL + LD)A = cDLA. \qquad \square\end{aligned}$$

As illustration consider the following important particular case.

Example 6.4. Applying Lemma 6.3 to $c = 1$ and $A = \mathbf{1}$ in $\mathscr{W}_P$ (for differential P) we obtain that $\mathbf{E}(Dt)\mathbf{1} = \mathbf{E}(rt + r\frac{t^2}{2}) \cdot \mathbf{E}(Ut) \cdot \mathbf{1}$.

Using Lemma 6.3 and Theorem 6.1 together we conclude the following more general result which is analogous to the second part of Theorem 2.5 in [18].

Corollary 6.5. *Let $\mathscr{W}$ be a monoidally cocomplete r-Weyl category with a switching object A. For any f, h in $\mathscr{J}$ and $c \in \mathbb{N}$ there exists an isomorphism*

$$\mathbf{E}((\overline{f} + cD)t) \cdot \overline{h} \cdot A \cong \mathbf{E}\left(c(r+U)t + c^2 r\frac{t^2}{2} + \int \underline{f}\right) \cdot \underline{h} \cdot A$$

in the category $\mathbf{LV}(\mathscr{W})$.

7. Enumerative corollaries

In this section we discuss how to derive enumerative corollaries from Theorem 6.1. The main idea here is that if P is r-differential then there are different functors $\mathscr{W}_P \to \mathscr{J}$ which intuitively 'count' the combinatorial information contained in the objects of $\mathscr{W}_P$. In order to describe these functors it is convenient to recall a standard piece of category theory. Any functor $f : \mathscr{C} \to \mathscr{D}$ between essentially small categories induces a functor $f^* : [\mathscr{D}, \mathbf{Set}] \to [\mathscr{C}, \mathbf{Set}]$ which has both left and right adjoints (see Theorem I.9.4 in [13]). The functor f^* is defined by $(f^*H)C = H(fC)$ for each H in $[\mathscr{D}, \mathbf{Set}]$ and C in $\mathscr{C}$. The left adjoint to f^* is sometimes denoted by $f_!$ and it is defined, when $\mathscr{C}$ and $\mathscr{D}$ are groupoids (as in our case), by $(f_!T)D = \sum_{fC=D} TC$ for each T in $[\mathscr{C}, \mathbf{Set}]$ and D in $\mathscr{D}$. (We will not need the right adjoint to f^*.)

Consider for example a graded poset P with grading $\rho : P \to \mathbb{N}$. Of course, this monotone map induces a function $\rho : |P| \to \mathbb{N}$ between the underlying sets. In this way, we obtain a functor $\rho_! : [|P|, \mathbf{Set}] \to [\mathbb{N}, \mathbf{Set}]$. Now recall from Section 5.1 that we denote the category $[|P|, \mathbf{Set}]$ by $\widehat{P}$ and that we thought of it as analogous to the vector space $\widehat{K}P$ in Stanley's context. Moreover, since $\mathbb{N}$ is equivalent to $\mathbb{L}$, $[\mathbb{N}, \mathbf{Set}]$ is equivalent to $[\mathbb{L}, \mathbf{Set}]$ so that we choose to see $\rho_!$ as a functor $\rho_! : \widehat{P} \to \mathscr{J}$ that for each 'vector' F in $\widehat{P}$ (recall Section 5), assigns the object

$$\rho_! F = \sum_{n \geqslant 0} \left(\sum_{\rho x = n} Fx \right) q^n,$$

which intuitively counts the information in F according to the levels.

Another example is the second projection $\pi : P \times P \to P$ which assigns y to each pair (x, y) in $P \times P$. As before, this induces a function $\pi : |P \times P| \to |P|$ and so, a functor $\pi_! : \mathscr{W}_P \to \widehat{P}$ such that to each T in $\mathscr{W}_P$ assigns the 'vector'

$$\pi_! T = \sum_{x \in P} \left(\sum_{y \in P} T(y, x) \right),$$

in $\widehat{P}$.

For some fixed k consider the object D^k in $\mathscr{W}_P$. We then have that

$$\rho_! \pi_! D^k = \rho_! \left(\sum_{x \in P} \left(\sum_{y \in P} D^k(y, x) \right) \right) = \sum_{n \geqslant 0} \left(\sum_{\rho x = n} \sum_{y \in P} D^k(y, x) \right) q^n,$$

so Lemma 5.6 implies that $(\rho\pi)_! D^k = \rho_! \pi_! D^k = \sum_{n \geqslant 0} \alpha(n \to n + k) q^n$ is an object in $\mathscr{J}$ counting saturated chains.

Consider now the diagonal functor $\Delta : P \to P \times P$ defined by $\Delta p = (p, p)$. It induces a functor $\Delta^* : \mathscr{W}_P \to \widehat{P}$ defined by $(\Delta^* F)x = F(x, x)$. So, for example, $\Delta^*(D^k U^n)$ is the 'vector' $\sum_{x \in P} D^k U^n(x, x)$ which at 'coordinate x' counts the number of Hasse walks that start at x, go up n steps and then go down k steps finishing at x. It is then clear that $D^k U^n(x, x)$ is empty if $k \neq n$.

Now, the main enumerative results in [18] (e.g. Theorems 3.2 or 3.11) are stated in terms of exponential power series with coefficients of ordinary power series. In our context, these results will be expressed as isomorphisms in $\mathbf{LV}(\mathscr{J})$ which, in turn, will be obtained by pushing forward (in the sense of Section 3.3) certain isomorphisms in $\mathbf{LV}(\mathscr{W}_P)$.

But before we obtain the isomorphisms let us look at how the combinatorial objects in question appear in the image of a functor $\mathbf{LV}(\widehat{\mathscr{W}_P}) \to \mathbf{LV}(\mathscr{J})$.

Using the notation introduced in Section 3.3 the functor $(\rho\pi)_! : \widehat{\mathscr{W}_P} \to \mathscr{J}$ induces a functor $((\rho\pi)_!)_* : \mathbf{LV}(\widehat{\mathscr{W}_P}) \to \mathbf{LV}(\mathscr{J})$. Using the observation above we can calculate

$$((\rho\pi)_!)_*\mathbf{E}(Dt) = \sum_{k\geqslant 0}(\rho\pi)_! D^k \frac{t^k}{k!} = \sum_{k\geqslant 0}\left(\sum_{n\geqslant 0}\alpha(n \to n+k)q^n\right)\frac{t^k}{k!},$$

so, in particular, counting skew tableaux amounts to finding an explicit formula for $((\rho\pi)_!)_*\mathbf{E}(Dt)$ in the case when P is Young's lattice. One of the remarkable facts observed by Stanley is that in order to get such a formula the only thing we need to know about Young's lattice is that it is a differential poset. We will come back to this in Section 7.1.

As another example consider the object $\frac{1}{1-q}$ in $\mathscr{J}$ and denote $\mathbf{u}(\frac{1}{1-q})$ in $\widehat{\mathscr{W}_P}$ by $\frac{1}{1-U}$. Consider also the object in $\mathbf{LV}(\widehat{\mathscr{W}_P})$ given by $\mathbf{E}(Dt)\frac{1}{1-U}$ and denote it by $\frac{\mathbf{E}(Dt)}{1-U}$. Then apply $(\Delta^*)_* : \mathbf{LV}(\widehat{\mathscr{W}_P}) \to \widehat{P}$ to obtain

$$(\Delta^*)_*\frac{\mathbf{E}(Dt)}{1-U} = \sum_{k\geqslant 0}\Delta^*\left(D^k\sum_n U^n\right)\frac{t^k}{k!} = \sum_{k\geqslant 0}\left(\sum_n \Delta^*(D^kU^n)\right)\frac{t^k}{k!},$$

but we have just seen that $\Delta^*(D^kU^n)$ is the 'vector' $\sum_{x\in P}D^kU^k(x,x)$ and by Lemma 5.6 we have that $\rho_!(\Delta^*(D^kU^n)) = \sum_{n\geqslant 0}\kappa(n \to n+k \to n)q^n$ in $\mathscr{J}$ so

$$(\rho_!)_*\left((\Delta^*)_*\frac{\mathbf{E}(Dt)}{1-U}\right) = \sum_{k\geqslant 0}\left(\sum_{n\geqslant 0}\kappa(n \to n+k \to n)q^n\right)\frac{t^k}{k!}$$

is an object in $\mathbf{LV}(\mathscr{J})$ that counts closed Hasse walks.

7.1. Counting skew shapes

In this section we prove a combinatorial analogue of Theorem 3.2 in [18] which gives a formula for counting ascending chains in differential posets. As in the whole paper the idea is to be able to interpret the key ideas in [18] in our present context. The only observation we need in this case is the following.

Lemma 7.1. *If P is an r-differential poset then $\pi_!F \cong \Delta^*(F\mathbf{1})$ for every F in $\widehat{\mathscr{W}_P}$. Moreover, this lifts to $\mathbf{LV}(\widehat{\mathscr{W}_P})$ and, in particular, there exists an isomorphism $(\pi_!)_*\mathbf{E}(Dt) \cong (\Delta^*)_*(\mathbf{E}(Dt)\mathbf{1})$.*

Proof. Notice that $(\pi_!F)y = \sum_{x\in P}F(x,y) = (\Delta^*(F\mathbf{1}))y$. □

We can now prove the our analogue of Theorem 3.2 in [18].

Theorem 7.2. *If $\rho : P \to \mathbb{N}$ is a graded r-differential poset then there is an iso*

$$\sum_{k\geqslant 0}\left(\sum_{n\geqslant 0}\alpha(n \to n+k)q^n\right)\frac{t^k}{k!} \cong \mathbf{E}\left(\frac{r}{1-q}t + \frac{r}{1-q^2}\frac{t^2}{2}\right)F(P,q)$$

in $\mathbf{LV}(\mathscr{J})$.

Proof. Recall from the introduction to this section that the object in $\mathbf{LV}(\mathscr{J})$ counting skew shapes is $((\rho\pi)_!)_*\mathbf{E}(Dt) = (\rho_!)_*((\pi_!)_*\mathbf{E}(Dt))$. Lemma 7.1 then implies $(\rho_!)_*((\pi_!)_*\mathbf{E}(Dt)) = (\rho_!)_*((\Delta^*)_*(\mathbf{E}(Dt)\mathbf{1})$ and from Corollary 6.5

(or Example 6.4) we know that $\mathbf{E}(Dt)\mathbf{1}=\mathbf{E}(rt+r\frac{t^2}{2})\cdot\mathbf{E}(Ut)\cdot\mathbf{1}$. Applying Lemma 7.1 again we have that $(\pi_!)_*\mathbf{E}(Dt)=(\pi_!)_*(\mathbf{E}(rt+r\frac{t^2}{2})\cdot\mathbf{E}(Ut))$ and so it follows that $((\rho\pi)_!)_*\mathbf{E}(Dt)=\mathbf{E}(rt+r\frac{t^2}{2})\cdot((\rho\pi)_!)_*\mathbf{E}(Ut)$. But then

$$((\rho\pi)_!)_*\mathbf{E}(Ut)=\sum_k\left(\sum_n\alpha(n-k\to n)q^n\right)\frac{t^k}{k!}$$
$$=\sum_k q^k*\left(\sum_n\alpha(n\to n+k)q^n\right)\frac{t^k}{k!}=\mathbf{Q}(((\rho\pi)_!)_*\mathbf{E}(Dt))$$

so that $((\rho\pi)_!)_*\mathbf{E}(Dt)\cong\mathbf{E}(rt+r\frac{t^2}{2})\cdot\mathbf{Q}(((\rho\pi)_!)_*\mathbf{E}(Dt))$ which says that the object $((\rho\pi)_!)_*\mathbf{E}(Dt)$ is a fixed point for the functor described in Proposition 4.12 (with initial condition $F(P,q)$). The result follows. □

7.2. Counting closed walks

In this section we apply Theorem 6.1 in order to obtain an explicit formula to count closed Hasse walks in r-differential posets. This is analogous to Theorem 3.11 in [18].

Theorem 7.3. *If $\rho:P\to\mathbb{N}$ is a graded r-differential poset then there is an iso*

$$\sum_{k\geqslant 0}\sum_{n\geqslant 0}\kappa(n\to n+k\to n)q^n\frac{t^k}{k!}\cong\frac{1}{1-(r/(1-q))t}F(P,q)$$

in $\mathbf{LV}(\mathscr{J})$.

Proof. From the introduction to this section we know that the counting of closed Hasse walks is related to the object $\mathbf{E}(Dt)\frac{1}{1-U}=\mathbf{E}(Dt)\overline{\frac{1}{1-q}}$. We can apply Theorem 6.1 with $f=0$, $c=1$ and $h=\frac{1}{1-q}$ in order to get

$$\mathbf{E}(Dt)\frac{1}{1-U}=\underline{\frac{1}{1-q}}\cdot\mathbf{E}(Dt)=\left(\sum_k r^k\overline{\partial_q^k\frac{1}{1-q}}\frac{t^k}{k!}\right)\cdot\mathbf{E}(Dt)$$

and since $\partial_q^k\frac{1}{1-q}=\sum_{n\geqslant 0}\binom{n+k}{k}k!q^n$ we can calculate as follows:

$$\mathbf{E}(Dt)\frac{1}{1-U}=\left(\sum_k r^k\left(\sum_{n\geqslant 0}\binom{n+k}{k}k!U^n\right)\frac{t^k}{k!}\right)\cdot\mathbf{E}(Dt)$$
$$=\sum_k\sum_{i=0}^k\binom{k}{i}r^i\left(\sum_{n\geqslant 0}\binom{n+i}{i}i!U^n\right)D^{k-i}\frac{t^k}{k!}$$
$$=\sum_k\sum_{i=0}^k\binom{k}{i}r^i\left(\sum_{n\geqslant 0}\binom{n+i}{i}i!U^nD^{k-i}\right)\frac{t^k}{k!},$$

but since $\Delta^*\sum_{n\geqslant 0}\binom{n+i}{i}i!U^nD^{k-i}=\sum_{x\in P}\binom{k}{i}i!U^{k-i}D^{k-i}(x,x)$ we can conclude that

$$(\Delta^*)_*\left(\mathbf{E}(Dt)\frac{1}{1-U}\right)=\sum_k\sum_{i=0}^k\binom{k}{i}^2i!r^i\left(\sum_{x\in P}U^{k-i}D^{k-i}(x,x)\right)\frac{t^k}{k!}$$

and carry on as below:

$$\sum_{k\geqslant 0}\sum_{n\geqslant 0}\kappa(n\to n+k\to n)q^n\frac{t^k}{k!}=(\rho_!)_*\left((A^*)_*\frac{\mathbf{E}(Dt)}{1-U}\right)$$

$$=\sum_k\left(\sum_{i=0}^k\binom{k}{i}^2 i!r^i\left(\sum_{n\geqslant 0}\sum_{\rho x=n}U^{k-i}D^{k-i}(x,x)q^n\right)\right)\frac{t^k}{k!}$$

$$=\sum_k\left(\sum_{i=0}^k\binom{k}{i}^2 i!r^i\left(\sum_{n\geqslant 0}\kappa(n\to n-k+i\to n)q^n\right)\right)\frac{t^k}{k!}$$

$$=\frac{1}{1-rt}\otimes\sum_k\left(\sum_{n\geqslant k}\kappa(n\to n-k\to n)q^n\right)\frac{t^k}{k!},$$

but $\kappa(n\to n-k\to n)=\kappa(n-k\to n\to n-k)$ so that

$$\sum_{n\geqslant k}\kappa(n\to n-k\to n)q^n=\sum_{n\geqslant k}\kappa(n-k\to n\to n-k)q^n$$

$$=q^k*\sum_{n\geqslant 0}\kappa(n\to n+k\to n)q^n$$

and then altogether we have

$$\sum_{k\geqslant 0}\sum_{n\geqslant 0}\kappa(n\to n+k\to n)q^n\frac{t^k}{k!}=\frac{1}{1-rt}\otimes\sum_k q^k*\left(\sum_{n\geqslant 0}\kappa(n\to n+k\to n)q^n\right)\frac{t^k}{k!}$$

$$=\frac{1}{1-rt}\otimes\mathbf{Q}\left(\sum_k\sum_{n\geqslant 0}\kappa(n\to n+k\to n)q^n\frac{t^k}{k!}\right)$$

which, in other words, says that $\sum_{k\geqslant 0}\sum_{n\geqslant 0}\kappa(n\to n+k\to n)q^n\frac{t^k}{k!}$ is a solution to the functional equation studied in Proposition 4.13 (with initial condition $F(P,q)$). So the proof is finished. □

It seems relevant to remark that the proof of Theorem 3.11 in [18] is somewhat different. Stanley uses a bilinear pairing $\langle,\rangle:\widehat{K}P\times KP\to K$ which is continuous in the first coordinate and such that D and U are adjoint under this pairing. We have avoided this pairing in order to stress the use of special functors but we believe that it should be possible to translate the essential idea of this bilinear pairing into our context. In any case, the reader may find it instructive to compare both proofs.

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