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Bilateral shorted operators and parallel sums ☆

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Abstract

In this paper we study shorted operators relative to two different subspaces, for bounded operators on infinite dimensional Hilbert spaces. We define two notions of "complementability" in the sense of Ando for operators, and study the properties of the shorted operators when they can be defined. We use these facts in order to define and study the notions of parallel sum and subtraction, in this Hilbertian context.

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1. Introduction

This paper is devoted to generalize two operations, coming from electrical network theory: parallel sum of matrices and shorting of matrices. In [2], Anderson and Duffin defined, for positive (semidefinite) matrices A and B the parallel sum $A : B = A(A + B)^\dagger B$. The motivation for studying this operation, and its name, come from the following fact: if two resistive n-port networks, with impedance matrices A and B, are connected in parallel, then $A : B$ is the impedance matrix of the parallel connection. It should be mentioned that the impedance matrix of a resistive n-port

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network is a positive (semidefinite) $n \times n$ matrix. On the other side, in [1] Anderson defined, for a positive $n \times n$ matrix A and a subspace $\mathcal{S}$ of $\mathbb{C}^n$, the shorted matrix of A by $\mathcal{S}$. Just to give an idea about $A_{/\mathcal{S}}$, suppose that A has the block form $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where A_{11} is a $k \times k$ block and A_{22} is an $(n-k) \times (n-k)$ block. If $\mathcal{S}$ is the subspace spanned by the first k canonical vectors, then

$$A_{/\mathcal{S}} = \begin{pmatrix} A_{11} - A_{12}A_{22}^{\dagger}A_{21} & 0 \\ 0 & 0 \end{pmatrix},$$

where † denotes the Moore–Penrose inverse. (Some authors define $A_{/\mathcal{S}}$ as a linear transformation $\mathcal{S} \to \mathcal{S}$ avoiding the zeroes above.) The name shorted comes from the fact that it gives the joint impedance of a resistive n-port, some of whose parts have been short circuited. Here A is the impedance matrix of the original network and $A_{/\mathcal{S}}$ is the impedance matrix of the network after the short circuits. Both operations have been studied in Hilbert spaces context (see the historical notes below).

One of the goals of this paper is to extend the shorting operation to bounded linear operator between two different Hilbert spaces, given a closed subspace on each one. The solution we get, which we call the bilateral shorted operator, comes from a notion of weak complementability, which is a refinement of a finite dimensional notion due to Ando [6] and generalized by Carlson and Haynworth [12]. The bilateral shorted operator has been studied in finite dimensions by Mitra and Puri [33] (see also the papers by Goller [23] and Mitra and Prasad [31], who refined some results of [33]). However, their methods strongly depend on the existence of generalized inverses, so they can not be used for operators with non closed range (see [7,10]).

The second goal is to extend parallel summability for two bounded linear operators between different Hilbert spaces. It should be mentioned that Rao and Mitra [35], and Mitra and Prasad [31] have studied this extension in finite dimensional spaces. Again, generalized inverses are the main tool they use. In order to avoid generalized inverses, we frequently use what we call hereafter Douglas' theorem, an extremely useful result due to Douglas [19], which we describe after fixing some notations.

In these notes, $\mathcal{H}_1$ and $\mathcal{H}_2$ denote Hilbert spaces, $L(\mathcal{H}_1, \mathcal{H}_2)$ is the space of all bounded linear operators between $\mathcal{H}_1$ and $\mathcal{H}_2$, we write $L(\mathcal{H}_i) = L(\mathcal{H}_i, \mathcal{H}_i)$ and $L(\mathcal{H}_1)^+$ (resp. $L(\mathcal{H}_2)^+$) the cone of all positive operators on $\mathcal{H}_1$ (resp. $\mathcal{H}_2$). Recall that $C \in L(\mathcal{H})$ is called positive if $\langle Cx, x\rangle \geqslant 0$ for every $x \in \mathcal{H}$. For every $C \in L(\mathcal{H}_1, \mathcal{H}_2)$ its range is denoted by $R(C)$, its nullspace by $N(C)$. Given two self-adjoint operators $A, B \in L(\mathcal{H})$, $A \leqslant B$ means that $B - A \in L(\mathcal{H})^+$ (this is called the usual or Löwner order). A projection is an idempotent (bounded linear) operator. Given a closed subspace $\mathcal{S} \subseteq \mathcal{H}_1$, by $P_{\mathcal{S}} \in L(\mathcal{H}_1)$ is denoted the orthogonal projection onto $\mathcal{S}$. Douglas' theorem states that given $A \in L(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in L(\mathcal{H}_3, \mathcal{H}_2)$, the following conditions are equivalent:

1. $R(B) \subseteq R(A)$, 2. $\exists \lambda \geqslant 0 : BB^* \leqslant \lambda\, AA^*$ and 3. $\exists D \in L(\mathcal{H}_3, \mathcal{H}_1) : B = AD$.

With the additional condition $R(D) \subseteq N(A)^{\perp}$, D is unique and it is called the *reduced solution* of the equation $AX = B$; it holds that $\|D\|^2 = \inf\{\lambda \in \mathbb{R} : BB^* \leqslant \lambda\, AA^*\}$ and $N(D) = N(B)$.

We shall use the fact that each pair of closed subspaces $\mathcal{S} \subseteq \mathcal{H}_1$ and $\mathcal{T} \subseteq \mathcal{H}_2$ induces a representation of elements of $L(\mathcal{H}_1, \mathcal{H}_2)$ by 2×2 block matrices. In this sense, we identify each $A \in L(\mathcal{H}_1, \mathcal{H}_2)$ with a 2×2 matrix, let us say

$$A = \begin{matrix} \mathcal{T} \\ \mathcal{T}^{\perp} \end{matrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}, \tag{1}$$

where $A_{11} = P_{\mathcal{T}}A|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{T})$, $A_{12} = P_{\mathcal{T}}A|_{\mathcal{S}^{\perp}}$, $A_{21} = P_{\mathcal{T}^{\perp}}A|_{\mathcal{S}}$ and $A_{22} = P_{\mathcal{T}^{\perp}}A|_{\mathcal{S}^{\perp}}$.

Historical survey: In 1947, Krein [26] proved the existence of a maximum (with respect to the usual order) of the set $\mathscr{M}(A, S) = \{C \in L(\mathscr{H})^+ : C \leqslant A, R(C) \subseteq S\}$. Krein used this extremal operator in his theory of extensions of symmetric operators. See the paper by Smul'jian [40] for more results in similar directions. Many years later, Anderson [1] rediscovered, for finite dimensional spaces, the existence of the maximum which will be denoted $A_{/\mathscr{S}}$ and called the shorted operator of A by $\mathscr{S}$. Some time before, Anderson and Duffin [2] (see also [20]) had developed the binary matrix operation called parallel sum: if $A, B \in L(\mathbb{C}^n)^+$ the parallel sum $A : B$ is defined by the formula

$$A : B = A(A + B)^\dagger B.$$

Fillmore and Williams [21] defined the parallel sum of positive (bounded linear) operators on a Hilbert space $\mathscr{H}$ and extended many of Anderson–Duffin's results. It should be mentioned that their definition of parallel sum is based on certain Douglas reduced solutions. Anderson and Trapp [5] defined $A_{/\mathscr{S}}$ for a positive operator A on $\mathscr{H}$ and a closed subspace $\mathscr{S}$ of $\mathscr{H}$, and proved that $A_{/\mathscr{S}}$ can be defined by means of parallel sums, and conversely: if P is the orthogonal projection onto $\mathscr{S}$, then $A : nP$ converges to $A_{/\mathscr{S}}$ in the operator uniform norm; and for $A, B \in L(\mathscr{H})^+$, $A : B$ can be defined as the shorted operator of $\begin{pmatrix} A & A \\ A & A+B \end{pmatrix} \in L(\mathscr{H} \oplus \mathscr{H})^+$ by the subspace $\mathscr{H} \oplus \{0\}$. This is the approach we shall use here. The shorting of an operator is one of the manifestations of the Schur complement: if M is a square matrix with block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A and D are also square blocks and D is invertible, the classical Schur complement of D in M is $A - BD^{-1}C$ (see [11,16,36] for many results, applications and generalizations of this notion). Ando [6] proposed a generalization of Schur complements which is closer to the idea of the shorted operators. If A is a $n \times n$ complex matrix and $\mathscr{S}$ is a subspace of $\mathbb{C}^n$, A is called *$\mathscr{S}$-complementable* if there are matrices M_r and M_l such that $PM_r = M_r$, $M_l P = M_l$, $PAM_r = PA$ and $M_l AP = AP$. (Here P is the orthogonal projection onto $\mathscr{S}$). It holds that $AM_r = M_l AM_r = M_l A$ and AM_r does not depends on the particular choice of M_r and M_l; Ando calls $A_{\mathscr{S}} = AM_r$ the Schur compression and $A_{/\mathscr{S}} = A - A_{\mathscr{S}} = A - AM_r$ the Schur complement of A with respect to $\mathscr{S}$. He observes that, if A is a positive $n \times n$ matrix and $\mathscr{S}$ is the subspace generated by $n - k$ last canonical vectors, then $A_{/\mathscr{S}}$ has the block form

$$\begin{pmatrix} A - BD^\dagger C & 0 \\ 0 & 0 \end{pmatrix}$$

and therefore, his definition extends the classical Schur complement. Carlson and Haynworth [12] observe that a similar construction could be done starting with $A \in C^{n \times m}$ and subspaces $\mathscr{S} \in \mathbb{C}^n$ and $\mathscr{T} \in \mathbb{C}^m$. They defined and studied the notion of operators which are complementable with respect to a pair $(\mathscr{S}, \mathscr{T})$.

As Anderson and Duffin remarked in [2], the impedance matrix is positive only for resistive networks. In order to study networks with reactive elements, parallel summation and shorting must be extended to not necessarily positive matrices and operators. Rao and Mitra [35] defined and studied parallel sums of $m \times n$ matrices and Mitra [33] used their results to define a sort of bilateral shorted operator by two subspaces, one in $\mathbb{C}^n$ and the other in $\mathbb{C}^m$. A common feature in both extensions is the use of generalized inverses. It should be mentioned that these constructions can be applied to linear regression problems as in [33,34,31, Appendix], and in identification problems [28,29].

We summarize the contents of this paper. In Section 3 we study notion of *complementability* in infinite dimensional Hilbert spaces and we define the concept of *weakly complementability* (see Definition 3.5). We also prove in this section the basic properties of (weakly or not) complementable triples and we show some criteria for each kind of complementability. In Section 6, under some compatibility conditions between the operator A and the subspaces $\mathcal{S}$ and $\mathcal{T}$, we define a bilateral shorted operator $A_{/(\mathcal{S},\mathcal{T})} \in L(\mathcal{H}_1, \mathcal{H}_2)$, and we study the usual properties of a shorting operation. As Mitra [30] proved for finite dimensional spaces, we show that $A_{/(\mathcal{S},\mathcal{T})}$ is the maximum of a certain set for a situable order (the so called minus order) in $L(\mathcal{H}_1, \mathcal{H}_2)$. The rest of the paper is devoted the notions of parallel addition and subtraction of operators and their relationship with the shorted operator. The parallel addition is defined by means of the following device, due to Anderson and Trapp: given $A, B \in L(\mathcal{H}_1, \mathcal{H}_2)$, we say that A and B are *weakly parallel summable* (*resp. parallel summable*) if the triple $\begin{pmatrix} A & A \\ A & A+B \end{pmatrix} \in L(\mathcal{H}_1 \oplus \mathcal{H}_1, \mathcal{H}_2 \oplus \mathcal{H}_2)$, $\mathcal{H}_1 \oplus \{0\}$, $\mathcal{H}_2 \oplus \{0\}$ is weakly complementable (resp. complementable). In this case we define the *parallel sum* of A and B, denoted by $A : B \in L(\mathcal{H}_1, \mathcal{H}_2)$, as follows:

$$\begin{pmatrix} A:B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & A \\ A & A+B \end{pmatrix} \Big/_{(\mathcal{H}_1\oplus\{0\},\mathcal{H}_2\oplus\{0\})}.$$

We study the properties of this operator. Again, under the hypothesis of summability, all properties of the finite dimensional case are recovered in our context. In Section 5 we define the notion of parallel subtraction, we give some conditions which assures its existence and prove some of its properties. In Section 6, we extend to the bilateral case some well known formulae for the shorted operator in terms of parallel sums and subtractions showing that, as for positive operators, parallel and shorting operations can be defined one in terms of the other.

2. Preliminaries

We need the following two definitions of angles between subspaces in a Hilbert space; they are due, respectively, to Friedrichs and Dixmier (see [18,22], and the excellent survey by Deutsch [17]).

Definition 2.1. Given two closed subspaces $\mathcal{M}$ and $\mathcal{N}$, the *Friedrichs angle* between $\mathcal{M}$ and $\mathcal{N}$ is the angle in $[0, \pi/2]$ whose cosine is defined by

$$c[\mathcal{M}, \mathcal{N}] = \sup\left\{|\langle x, y\rangle| : x \in \mathcal{M} \ominus (\mathcal{M} \cap \mathcal{N}),\ y \in \mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N}) \text{ and } \|x\| = \|y\| = 1\right\}.$$

The *Dixmier angle* between $\mathcal{M}$ and $\mathcal{N}$ is the angle in $[0, \pi/2]$ whose cosine is defined by

$$c_0[\mathcal{M}, \mathcal{N}] = \sup\left\{|\langle x, y\rangle| : x \in \mathcal{M},\ y \in \mathcal{N} \text{ and } \|x\| = \|y\| = 1\right\}.$$

The next proposition collects the results on angles which are relevant to our work.

Proposition 2.2. 1. *Let $\mathcal{M}$ and $\mathcal{N}$ be to closed subspaces of $\mathcal{H}$. Then*

(a) $c[\mathcal{M}, \mathcal{N}] = c[\mathcal{M}^\perp, \mathcal{N}^\perp]$.
(b) $c[\mathcal{M}, \mathcal{N}] < 1$ *if and only if* $\mathcal{M} + \mathcal{N}$ *is closed.*
(c) $\mathcal{H} = \mathcal{M}^\perp + \mathcal{N}^\perp$ *if and only if* $c_0[\mathcal{M}, \mathcal{N}] < 1$.

2. (Bouldin [8], see also [25]) *Given* $B \in L(\mathcal{H}_1, \mathcal{H}_2)$ *and* $A \in L(\mathcal{H}_2, \mathcal{H}_3)$ *with closed range, then* $R(AB)$ *is closed if and only if* $c[R(B), N(A)] < 1$.

3. Complementable operators

In this section we study complementable operators. We recall different characterizations of this notion, their extensions to infinite dimensional Hilbert spaces, and the relationships among them. The next definition, due to Carlson and Haynsworth [12], is an extension of Ando's generalized Schur complement [6].

Definition 3.1. Given two projections $P_r \in L(\mathcal{H}_1)$ and $P_l \in L(\mathcal{H}_2)$, an operator $A \in L(\mathcal{H}_1, \mathcal{H}_2)$ is called (P_r, P_l)-complementable if there exist operators $M_r \in L(\mathcal{H}_1)$ and $M_l \in L(\mathcal{H}_2)$ such that

1. $(I - P_r)M_r = M_r$, $(I - P_l)AM_r = (I - P_l)A$,
2. $(I - P_l)M_l = M_l$ and $M_l A(I - P_r) = A(I - P_r)$.

We shall prove later that this notion only depends on the images of P_r and P_l. As in the finite dimensional case, we have the following alternative characterization of complementability. We use freely matrix decompositions like (1).

Proposition 3.2. *Let* $P_r \in L(\mathcal{H}_1)$ *and* $P_l \in L(\mathcal{H}_2)$ *be two projections whose ranges are* $\mathcal{S}$ *and* $\mathcal{T}$ *respectively. Given* $A \in L(\mathcal{H}_1, \mathcal{H}_2)$, *the following statements are equivalent*:

1. *A is* (P_r, P_l)*-complementable.*
2. $R(A_{21}) \subseteq R(A_{22})$ *and* $R(A_{12}^*) \subseteq R(A_{22}^*)$.
3. *There exist two projections* $\widehat{P} \in L(\mathcal{H}_1)$ *and* $\widehat{Q} \in L(\mathcal{H}_2)$ *such that*:

$$R(\widehat{P}^*) = \mathcal{S} \quad R(\widehat{Q}) = \mathcal{T} \quad R(A\widehat{P}) \subseteq \mathcal{T} \quad and \quad R((\widehat{Q}A)^*) \subseteq \mathcal{S}. \tag{2}$$

Proof. 1 ⇒ 2: By definition 3.1 it holds that $M_r = \begin{matrix} \mathcal{T} \\ \mathcal{T}^\perp \end{matrix} \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$, and $A_{21} = A_{22}C$.

Hence $R(A_{21}) \subseteq R(A_{22})$. Similar arguments show that $R(A_{12}^*) \subseteq R(A_{22}^*)$.

2 ⇒ 3: Let E and F be the reduced solutions of $A_{21} = A_{22}X$ and $A_{12}^* = A_{22}^*X$, respectively. Note that $E \in L(\mathcal{S}, \mathcal{S}^\perp)$ and $F \in L(\mathcal{T}^\perp, \mathcal{T})$. If

$$\widehat{P} = \begin{pmatrix} I & 0 \\ -E & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \in L(\mathcal{H}_1) \quad \text{and} \quad \widehat{Q} = \begin{pmatrix} I & -F \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{T} \\ \mathcal{T}^\perp \end{matrix} \in L(\mathcal{H}_2),$$

easy computations show that these projections satisfy Eq. (2).

3 ⇒ 1: Define $M_r = I - \widehat{P}$ and $M_l = I - \widehat{Q}^*$. Then $R(M_r) = \mathcal{S}^\perp$ and $R(M_l) = \mathcal{T}^\perp$, so conditions 1 and 3 of Definition 3.1 are satisfied. On the other hand

$$(I - Q)AM_r = (I - Q)A(I - \hat{P}) = (I - Q)A - (I - Q)A\hat{P} = (I - Q)A, \quad \text{and}$$
$$M_l A(I - P) = (I - \hat{Q}^*)A(I - P) = A(I - P) - \hat{Q}^* A(I - P) = A(I - P).$$

This shows that conditions 2 and 4 of Definition 3.1 also hold. □

The next characterization has been considered in [13] for self-adjoint operators in a Hilbert space. We prove an extension to our general setting.

Proposition 3.3. *Let $P_r \in L(\mathcal{H}_1)$ and $P_l \in L(\mathcal{H}_2)$ be two projections with ranges $\mathcal{S}$ and $\mathcal{T}$, respectively, and let $A \in L(\mathcal{H}_1, \mathcal{H}_2)$. Then the following statements are equivalent*:

1. *A is (P_r, P_l)-complementable.*
2. *$\mathcal{H}_1 = \mathcal{S}^\perp + A^{-1}(\mathcal{T})$ and $\mathcal{H}_2 = \mathcal{T}^\perp + A^{*-1}(\mathcal{S})$.*
3. *$c_0\left[\mathcal{S}, \overline{A^*(\mathcal{T}^\perp)}\right] < 1$ and $c_0\left[\mathcal{T}, \overline{A(\mathcal{S}^\perp)}\right] < 1$.*

Proof. $1 \iff 2$: Suppose that A is (P_r, P_l)-complementable. By Proposition 3.2, there exists a projection P such that $R(P^*) = \mathcal{S}$ and $R(AP) \subseteq \mathcal{T}$. Then, $N(P) = \mathcal{S}^\perp$ and $R(P) \subseteq A^{-1}(\mathcal{T})$. Hence $\mathcal{H}_1 = \mathcal{S}^\perp + A^{-1}(\mathcal{T})$.

Conversely, suppose that $\mathcal{H}_1 = \mathcal{S}^\perp + A^{-1}(\mathcal{T})$ and define $\mathcal{N} = \mathcal{S}^\perp \cap A^{-1}(\mathcal{T})$. Then $\mathcal{H}_1 = \mathcal{S}^\perp \oplus (A^{-1}(\mathcal{T}) \ominus \mathcal{N})$. Let $\tilde{P}$ be the oblique projection onto $A^{-1}(\mathcal{T}) \ominus \mathcal{N}$ parallel to $\mathcal{S}^\perp$. Then, $R(\tilde{P}^*) = N(\tilde{P})^\perp = R(I - \tilde{P})^\perp = \mathcal{S}$, and $R(A\tilde{P}) \subseteq \mathcal{T}$ because $R(\tilde{P}) = A^{-1}(\mathcal{T}) \ominus \mathcal{N}$. Similar arguments show that the existence of a projection Q such that $R(Q) = \mathcal{T}$ and $R((QA)^*) \subseteq \mathcal{S}$ is equivalent to the identity $\mathcal{H}_2 = \mathcal{T}^\perp + A^{*-1}(\mathcal{S})$.

$2 \iff 3$: It follows from Proposition 2.2 (item 3) and the equality $A^*(\mathcal{T}^\perp)^\perp = A^{-1}(\mathcal{T})$. □

Remark-Definition 3.4. Proposition 3.3, as well as Proposition 3.2, shows that the notion of (P_r, P_l)-complementable operators only depends on $R(P_l)$ and $R(P_r)$. Hence, from now on we shall say that an operator $A \in L(\mathcal{H}_1, \mathcal{H}_2)$ is $(\mathcal{S}, \mathcal{T})$-*complementable* instead of (P_r, P_l)-complementable.

In finite dimensional spaces, given a fixed subspace $\mathcal{S}$, every positive operator A is $(\mathcal{S}, \mathcal{S})$-complementable. Indeed, if $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$, the inclusion $R(A_{21}) \subseteq R(A_{22})$ always holds (see [40] for details). However, in infinite dimensional Hilbert spaces, only the inclusion $R(A_{21}) \subseteq R(A_{22}^{1/2})$ holds in general. As $R(A_{22}) = R(A_{22}^{1/2})$ if and only if A_{22} has closed range (which is the case in finite dimensional spaces), it is not difficult to find examples of positive operators which are not $(\mathcal{S}, \mathcal{S})$-complementable (e.g., see example 5.5 of [13]). For this reason we consider the following weaker notion of complementability:

Definition 3.5. Let $\mathcal{S} \subseteq \mathcal{H}_1$ and $\mathcal{T} \subseteq \mathcal{H}_2$ be closed subspaces. An operator $A \in L(\mathcal{H}_1, \mathcal{H}_2)$ is called $(\mathcal{S}, \mathcal{T})$-*weakly complementable* if

$$R(A_{21}) \subseteq R(|A_{22}^*|^{1/2}) \quad \text{and} \quad R(A_{12}^*) \subseteq R(|A_{22}|^{1/2}),$$

(according to the matrix decomposition of A given in Eq. (1)).

Remark 3.6. Observe that, by Douglas' theorem, $R(|A_{22}^*|) = R(A_{22})$ and $R(|A_{22}|) = R(A_{22}^*)$. Therefore this notion is, indeed, weaker than the previously defined notion of complementability. However, if $R(A_{22})$ is closed, then $R(|A_{22}^*|)$ is also closed and $R(A_{22}) = R(|A_{22}^*|) = R(|A_{22}^*|^{1/2})$. Thus, both notions of complementability coincide.

As an easy consequence of Douglas' theorem,we get the next alternative characterizations of $(\mathcal{S}, \mathcal{T})$-weakly complementable operators.

Proposition 3.7. *Given $A \in L(\mathcal{H}_1, \mathcal{H}_2)$, and closed subspaces $\mathcal{S} \subseteq \mathcal{H}_1$, $\mathcal{T} \subseteq \mathcal{H}_2$, then the following statements are equivalent*:

1. *A is $(\mathcal{S}, \mathcal{T})$-weakly complementable.*
2. *If $A_{22} = U|A_{22}|$ is the polar decomposition of A_{22}, then the equations $A_{21} = |A_{22}^*|^{1/2}UX$ and $A_{12}^* = |A_{22}|^{1/2}Y$ have solutions.*
3. $\sup\limits_{x\in\mathcal{S}} \frac{\|A_{21}x\|^2}{\langle|A_{22}^*|x,x\rangle} < \infty$ *and* $\sup\limits_{y\in\mathcal{T}} \frac{\|A_{12}^*y\|^2}{\langle|A_{22}|y,y\rangle} < \infty.$

4. Shorted operators

Recall that, in the classic case, i.e., if $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $\mathcal{S} = \mathcal{T}$ and $A \in L(\mathcal{H})^+$, Anderson and Trapp [5] proved that $A_{/\mathcal{S}} = \begin{pmatrix} A_{11} - C^*C & 0 \\ 0 & 0 \end{pmatrix}$, where C is the reduced solution of $A_{22}^{1/2}X = A_{21}$. Following this approach, we shall extend the notion of shorted operators to operators between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Throughout this section, $\mathcal{S} \subseteq \mathcal{H}_1$ and $\mathcal{T} \subseteq \mathcal{H}_2$ are closed subspaces and each operator $A \in L(\mathcal{H}_1, \mathcal{H}_2)$ is identified with a 2×2 matrix induced by these subspaces, as in (1).

Definition 4.1. Let $A \in L(\mathcal{H}_1, \mathcal{H}_2)$ be $(\mathcal{S}, \mathcal{T})$-weakly complementable, and let F and E be the reduced solutions of the equations $A_{21} = |A_{22}^*|^{1/2}UX$ and $A_{12}^* = |A_{22}|^{1/2}X$, respectively, where U is the partial isometry of the polar decomposition of A_{22}. The *bilateral shorted operator* of A to the subspaces $\mathcal{S}$ and $\mathcal{T}$ is

$$A_{/\mathcal{S},\mathcal{T}} = \begin{pmatrix} A_{11} - F^*E & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark 4.2. If A_{22} has closed range, then $A_{/(\mathcal{S},\mathcal{T})} = \begin{pmatrix} A_{11} - A_{12}A_{22}^{\dagger}A_{21} & 0 \\ 0 & 0 \end{pmatrix}$. On the other hand, if $\mathcal{S} = \mathcal{T}$ and $A = A^*$ our definition extends the notion of shorted operator given in [27].

In the following proposition we collect some basic properties of shorted operators. The proof is straightforward.

Proposition 4.3. *Let $A \in L(\mathcal{H}_1, \mathcal{H}_2)$ be $(\mathcal{S}, \mathcal{T})$-weakly complementable. Then*

1. *for every $\alpha \in \mathbb{C}$, αA is $(\mathcal{S}, \mathcal{T})$-weakly complementable, and $(\alpha A)_{/(\mathcal{S},\mathcal{T})} = \alpha(A_{/(\mathcal{S},\mathcal{T})})$,*
2. *A^* is $(\mathcal{T}, \mathcal{S})$-weakly complementable, and $(A_{/(\mathcal{S},\mathcal{T})})^* = (A^*)_{/(\mathcal{T},\mathcal{S})}$,*
3. *$A_{/(\mathcal{S},\mathcal{T})}$ is $(\mathcal{S}, \mathcal{T})$-weakly complementable and $(A_{/(\mathcal{S},\mathcal{T})})_{/(\mathcal{S},\mathcal{T})} = A_{/(\mathcal{S},\mathcal{T})}$,*
4. *if $A = A^*$ and $\mathcal{S} = \mathcal{T}$, then $A_{/(\mathcal{S},\mathcal{S})}$ is self-adjoint.*

The next Proposition is similar to Theorem 1 by Butler and Morley in [9]. For the reader's convenience, we include a proof adapted to our setting. First we need a lemma and some notations: We write $x_n \xrightarrow[n\to\infty]{\text{w}} x$ to denote that the sequence $\{x_n\}$ in a Hilbert space $\mathcal{H}$ converges *weakly* to x. By Alaoglu's theorem, the closed unit ball of a $\mathcal{H}$ is weakly compact. This implies that any bounded sequence $\{x_n\}$ in $\mathscr{H}$ admits weakly convergent subsequences. Recall that, if $A \in L(\mathscr{H}_1, \mathscr{H}_2)$ and $x_n \xrightarrow[n\to\infty]{\text{w}} x$ in $\mathscr{H}_1$, then $Ax_n \xrightarrow[n\to\infty]{\text{w}} Ax$ in $\mathscr{H}_2$ (see [39, Thm. 3.17]).

Lemma 4.4. *Let $A, B \in L(\mathscr{H}_1, \mathscr{H}_2)$ be such that $R(A^*) \subseteq R(|B|^{1/2})$. Suppose that there exists a sequence $\{y_n\}$ in $\mathscr{H}_1$, $d \in \mathscr{H}_2$ and a positive number M satisfying*

$$Ay_n \xrightarrow[n\to\infty]{w} d, \quad By_n \xrightarrow[n\to\infty]{w} 0, \quad \textit{and} \quad \langle |B|y_n, y_n\rangle \leqslant M.$$

Then $d = 0$.

Proof. Denote $a_n = |B|^{1/2}y_n$, $n \in \mathbb{N}$. Note that $\|a_n\|^2 = \langle |B|y_n, y_n\rangle \leqslant M$. By the previous remarks, we know that there exists $z \in \mathscr{H}_2$ and a subsequence of $\{a_n\}$, which we still call $\{a_n\}$, such that $a_n \xrightarrow[n\to\infty]{w} z$. Let $B = U|B|$ be the polar decomposition of B. As $By_n \xrightarrow[n\to\infty]{\text{w}} 0$ and $B = (U|B|^{1/2})|B|^{1/2}$, we can deduce that $z \in N(|B|^{1/2})$. Let C be the reduced solution of $A^* = |B|^{1/2}X$. Then $R(C) \subseteq N(|B|^{1/2})^{\perp}$, so that $N(|B^{1/2}|) \subseteq N(C^*)$ and $z \in N(C^*)$. Therefore, the facts

$$|B|^{1/2}y_n \xrightarrow[n\to\infty]{\text{w}} z \quad \text{and} \quad Ay_n = C^*|B|^{1/2}y_n \xrightarrow[n\to\infty]{\text{w}} d$$

imply that $d = C^*z = 0$. $\square$

Proposition 4.5. *Let $A \in L(\mathscr{H}_1, \mathscr{H}_2)$ be $(\mathscr{S}, \mathscr{T})$-weakly complementable. Then, given $x \in \mathscr{S}$ there exist a sequence $\{y_n\} \subseteq \mathscr{S}^{\perp}$ and a positive number M such that*

$$A\begin{pmatrix} x \\ y_n \end{pmatrix} \xrightarrow[n\to\infty]{} A_{/(\mathscr{S},\mathscr{T})}\begin{pmatrix} x \\ 0 \end{pmatrix}, \quad \textit{and} \quad \langle |A_{22}|y_n, y_n\rangle \leqslant M, \quad n \in \mathbb{N}.$$

Conversely, if there exists a sequence $\{z_n\}$ in $\mathscr{S}^{\perp}$, $d \in \mathscr{T}$, and a positive number M such that

$$A\begin{pmatrix} x \\ z_n \end{pmatrix} \xrightarrow[n\to\infty]{} \begin{pmatrix} d \\ 0 \end{pmatrix}, \quad \textit{and} \quad \langle |A_{22}|z_n, z_n\rangle \leqslant M, \tag{3}$$

then $\begin{pmatrix} d \\ 0 \end{pmatrix} = A_{/(\mathscr{S},\mathscr{T})}\begin{pmatrix} x \\ 0 \end{pmatrix}$.

Proof. Let E and F be the reduced solutions of $A_{21} = |A_{22}^*|^{1/2}UX$ and $A_{12}^* = |A_{22}|^{1/2}X$, respectively. As $R(E) \subseteq \overline{R(U^*|A_{22}^*|^{1/2})} = \overline{R(|A_{22}|^{1/2})}$, given $x \in \mathscr{H}_1$ there is a sequence $\{y_n\}$ such that $|A_{22}|^{1/2}y_n \xrightarrow[n\to\infty]{} -Ex$. Then

$$\begin{aligned} A_{21}x + A_{22}y_n &= A_{21}x + U|A_{22}|^{1/2}|A_{22}|^{1/2}y_n \\ &= A_{21}x + |A_{22}^*|^{1/2}U(|A_{22}|^{1/2}y_n) \xrightarrow[n\to\infty]{} 0, \quad \text{and} \\ A_{11}x + A_{12}y_n &= A_{11}x + F^*|A_{22}|^{1/2}y_n \\ &= A_{21}x + F^*(|A_{22}|^{1/2}y_n) \xrightarrow[n\to\infty]{} A_{/(\mathscr{S},\mathscr{T})}(x). \end{aligned}$$

Finally, since the sequence $\{|A_{22}|^{1/2}y_n\}$ converges, then $\sup_{n\in\mathbb{N}}\langle |A_{22}|y_n, y_n\rangle < \infty$. Conversely, suppose that there exists another sequence $\{z_n\}$ in $\mathscr{S}^{\perp}$ which satisfies (3). If

$w_n = y_n - z_n$, then $\langle |A_{22}|w_n, w_n\rangle \leqslant K$. On the other hand, $A_{11}x + A_{12}y_n \xrightarrow[n\to\infty]{} d$ and $A_{11}x + A_{12}z_n \xrightarrow[n\to\infty]{} A_{/(\mathscr{S},\mathscr{T})}$. Therefore, $A_{12}w_n \xrightarrow[n\to\infty]{} d - A_{/(\mathscr{S},\mathscr{T})}(x)$. In a similar way, we obtain that $A_{22}w_n \xrightarrow[n\to\infty]{} 0$. Therefore, by Lemma 4.4, we get that $d = A_{/(\mathscr{S},\mathscr{T})}(x)$. □

Corollary 4.6. *Let $A \in L(\mathscr{H}_1, \mathscr{H}_2)$ be $(\mathscr{S}, \mathscr{T})$-weakly complementable. Then*

$$R(A) \cap \mathscr{T} \subseteq R(A_{/(\mathscr{S},\mathscr{T})}) \subseteq \overline{R(A)} \cap \mathscr{T}, \tag{4}$$

$$R(A^*) \cap \mathscr{S} \subseteq R((A_{/(\mathscr{S},\mathscr{T})})^*) \subseteq \overline{R(A^*)} \cap \mathscr{S}. \tag{5}$$

In particular, $R(A_{/(\mathscr{S},\mathscr{T})}) = R(A) \cap \mathscr{T}$ and $R((A_{/(\mathscr{S},\mathscr{T})})^*) = R(A^*) \cap \mathscr{S}$ if $R(A)$ is closed.

Proof. Firstly, we shall prove that $R(A_{/(\mathscr{S},\mathscr{T})}) \subseteq \overline{R(A)} \cap \mathscr{T}$. Clearly, by definition, $R(A_{/(\mathscr{S},\mathscr{T})}) \subseteq \mathscr{T}$. On the other hand, given $x \in \mathscr{H}_1$, by Proposition 4.5, there exists a sequence $\{y_n\}$ in $\mathscr{S}^\perp$ such that $A\binom{Px}{y_n} \xrightarrow[n\to\infty]{} A_{/(\mathscr{S},\mathscr{T})}\binom{x}{0}$. Thus $R(A_{/(\mathscr{S},\mathscr{T})}) \subseteq \overline{R(A)}$.

In order to prove the first inclusion in (4), take $x \in R(A) \cap \mathscr{T}$, and let $z \in \mathscr{H}_1$ such that $Az = x$. If P is the orthogonal projection onto $\mathscr{S}$, then $A\binom{Pz}{z - Pz} = \binom{x}{0}$, and, by Proposition 4.5, we get $A_{/(\mathscr{S},\mathscr{T})}(Pz) = x$. The other inclusions follow in the same way. □

Next, we shall study the shorting operation on $(\mathscr{S}, \mathscr{T})$-complementable operators.

Proposition 4.7. *Let $A \in L(\mathscr{H}_1, \mathscr{H}_2)$ be $(\mathscr{S}, \mathscr{T})$-complementable. For every $x \in \mathscr{S}$ there exists $y \in \mathscr{S}^\perp$ such that*

$$A\begin{pmatrix} x \\ y \end{pmatrix} = A_{/(\mathscr{S},\mathscr{T})}\begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Moreover, there exist projections $P \in L(\mathscr{H}_1)$ and $Q \in L(\mathscr{H}_2)$ such that

$$R(P^*) = \mathscr{S}, \quad R(Q) = \mathscr{T} \quad and \quad QA = AP = A_{/(\mathscr{S},\mathscr{T})}. \tag{6}$$

Proof. By Proposition 3.2, there exists a projector $P \in L(\mathscr{H}_1)$ such that $R(P^*) = \mathscr{S}$ and $R(AP) \subseteq \mathscr{T}$. The matrix decomposition of P with respect to $\mathscr{S}$ is $\begin{pmatrix} I & 0 \\ E & 0 \end{pmatrix}$, where I is the identity operator of $\mathscr{S}$ and $E \in L(\mathscr{S}, \mathscr{S}^\perp)$. If $x \in \mathscr{S}$ and $y = Ex$, then $A\binom{x}{y} = AP\binom{x}{0} \in \mathscr{T}$. If $z_n = y$ for every $n \in \mathbb{N}$, the sequence $\{z_n\}$ satisfies (3). Hence, by Proposition 4.5, $A\binom{x}{y} = A_{/(\mathscr{S},\mathscr{T})}\binom{x}{0}$. Therefore $AP = A_{/(\mathscr{S},\mathscr{T})}$. In a similar way it can be proved that there exists $Q \in L(\mathscr{H}_2)$ with $R(Q) = \mathscr{T}$ such that $QA = A_{/(\mathscr{S},\mathscr{T})}$. □

Remark 4.8. Note that we actually prove that if there exists a projection P such that $R(P^*) = \mathscr{S}$ and $R(AP) \subseteq \mathscr{T}$, then, by Proposition 4.5, $AP = A_{/(\mathscr{S},\mathscr{T})}$. This result, for positive operators, appeared in [13], where the role of P is played by a so-called A-self-adjoint projection, i.e., a projection which is self-adjoint with respect to the sesquilinear form $\langle x, y\rangle_A = \langle Ax, y\rangle$. The reader is referred to [15,24,14] for more information about A-self-adjoint projections.

Corollary 4.9. *Let $A \in L(\mathscr{H}_1, \mathscr{H}_2)$ be $(\mathscr{S}, \mathscr{T})$-complementable. Then,*

$$R(A_{/(\mathscr{S},\mathscr{T})}) = R(A) \cap \mathscr{T} \quad and \quad N(A_{/(\mathscr{S},\mathscr{T})}) = \mathscr{S}^\perp + N(A).$$

Proof. By Corollary 4.6, it holds that $R(A) \cap \mathscr{T} \subseteq R(A_{/(\mathscr{S},\mathscr{T})})$ and

$$\mathscr{S}^{\perp} + N(A) \subseteq (\mathscr{S} \cap \overline{R(A^*)})^{\perp} \subseteq R({A_{/(\mathscr{S},\mathscr{T})}}^*)^{\perp} = N(A_{/(\mathscr{S},\mathscr{T})}).$$

On the other hand, by Proposition 4.7, there exist two projections $P \in L(\mathscr{H}_1)$ and $Q \in L(\mathscr{H}_2)$ which satisfy Eq. (6). Hence, $R(A_{/(\mathscr{S},\mathscr{T})}) = R(AP) \subseteq R(A)$, and

$$N(A_{/(\mathscr{S},\mathscr{T})}) = N(AP) = N(P) \oplus (R(P) \cap N(A)) \subseteq \mathscr{S}^{\perp} + N(A),$$

because $N(P) = R(P^*)^{\perp} = \mathscr{S}^{\perp}$. □

Remark 4.10. If $A \in L(\mathscr{H}_1, \mathscr{H}_2)$ be $(\mathscr{S}, \mathscr{T})$-complementable, then, by Corollary 4.9, the subspaces $\mathscr{S}^{\perp} + N(A)$, $\mathscr{S} + N(A)^{\perp}$, $\mathscr{T}^{\perp} + R(A)^{\perp}$ and $\mathscr{T} + \overline{R(A)}$ must be closed. Moreover, if $R(A)$ is closed then, by Proposition 2.2, $A(\mathscr{S}^{\perp})$, $A^*(\mathscr{T}^{\perp})$, and $R(A_{22})$ are also closed. Hence, in this case, generalized inverse methods can be used. Nevertheless, by using the approach developed in this work, one can get almost all known properties of the Schur complements in finite dimensional spaces, for complementable operators in general Hilbert spaces, including those operators whose ranges are not closed.

The minus partial order. In [30], Mitra proved (for matrices in $\mathbb{C}^{m\times n}$) that $A_{/(\mathscr{S},\mathscr{T})}$ is the unique maximum of the set

$$\mathscr{M}^{-}(A, \mathscr{S}, \mathscr{T}) = \left\{C \in \mathbb{C}^{m\times n} : C \leqslant^{-} A, \quad R(C) \subseteq \mathscr{T} \quad \text{and} \quad R(C^*) \subseteq \mathscr{S}\right\},$$

where the partial ordering is the so called minus order: $C \leqslant^{-} A$ if

$$R(C) \cap R(A - C) = \{0\} \quad \text{and} \quad R(C^*) \cap R(A^* - C^*) = \{0\}.$$

A similar result can be obtained in our setting with suitable changes. Firstly, we need to extend the minus order to infinite dimensional Hilbert spaces:

Definition 4.11. Given $A, B \in L(\mathscr{H}_1, \mathscr{H}_2)$, we write $A \leqslant^{-} B$ if:

$$\text{(a) } c_0\left[\overline{R(A)}, \overline{R(B - A)}\right] < 1 \quad \text{and} \quad \text{(b) } c_0\left[\overline{R(A^*)}, \overline{R(B^* - A^*)}\right] < 1.$$

Remark 4.12. In the finite dimensional case, condition (a) is equivalent to $R(A) \cap R(B - A) = \{0\}$ and condition (b) is equivalent to $R(A^*) \cap R(B^* - A^*) = \{0\}$. So, Definition 4.11 extends the (finite dimensional) minus order. Also notice that $A \leqslant^{-} B$ if and only if $A^* \leqslant^{-} B^*$, by the symmetry of conditions (a) and (b).

The next proposition provides equivalent conditions to condition (a) in Definition 4.11, which are simpler to handle. A similar result for condition (b) can be obtained by taking adjoints.

Proposition 4.13. *Given $A, B \in L(\mathscr{H}_1, \mathscr{H}_2)$, the following statements are equivalent:*

1. $c_0\left[\overline{R(A)}, \overline{R(B - A)}\right] < 1.$
2. *There exists a projection $Q \in L(\mathscr{H}_2)$ such that $R(Q) = \overline{R(A)}$ and $A = QB$.*
3. *There exists a projection $Q \in L(\mathscr{H}_2)$ such that $A = QB$.*

Proof. 1 $\Longrightarrow$ 2: Let $\mathscr{L} = \overline{R(A)} \oplus \overline{R(B - A)}$, which is closed by Proposition 2.2. Let $Q \in L(\mathscr{H}_2)$ be the projection with $R(Q) = \overline{R(A)}$ and $N(Q) = \overline{R(B - A)} \oplus \mathscr{L}^{\perp}$. Then, $QB = Q((B - A) + A) = QA = A$.

$2 \Longrightarrow 3$: It is apparent.

$3 \Longrightarrow 1$: Since $A = QB$ and $B - A = (I - Q)B$, it holds that $R(A) \subseteq R(Q)$ and $R(B - A) \subseteq R(I - Q) = N(Q)$. Hence, $c_0\left[\overline{R(A)}, \overline{R(B - A)}\right] \leqslant c_0[R(Q), N(Q)] < 1$. □

Corollary 4.14. *Let $A, B \in L(\mathscr{H}_1, \mathscr{H}_2)$.*

1. *If $A \leqslant^- B$, then $R(A) \subseteq R(B)$ and $R(A^*) \subseteq R(B^*)$.*
2. *The relation $\leqslant^-$ is a partial order (i.e. it is reflexive, antisymmetric and transitive).*
3. *If $A \leqslant^- B$ and B is a projection, then A is also a projection.*

Proof. The first two statements follow easily from Proposition 4.13. If $A \leqslant^- B$ and $B^2 = B$, by Proposition 4.13 applied to A and B (resp A^* and B^*) there exist projections P and Q such that $R(P^*) = \overline{R(A^*)}$, $R(Q) = \overline{R(A)}$ and $A = QB = BP$. Then $A^2 = (QB)(BP) = QBP = A$. □

Theorem 4.15. *Let $A \in L(\mathscr{H}_1, \mathscr{H}_2)$ be $(\mathscr{S}, \mathscr{T})$-complementable, and let*

$$\mathscr{M}^-(A, \mathscr{S}, \mathscr{T}) = \left\{C \in L(\mathscr{H}_1, \mathscr{H}_2) : C \leqslant^- A, R(C) \subseteq \mathscr{T} \text{ and } R(C^*) \subseteq \mathscr{S}\right\}.$$

Then, $A_{/(\mathscr{S},\mathscr{T})} = \max_{\leqslant^-} \mathscr{M}^-(A, \mathscr{S}, \mathscr{T})$.

Proof. By Propositions 4.7 and 4.13, we know that $A_{/(\mathscr{S},\mathscr{T})} \leqslant^- A$. On the other hand, by Corollary 4.9, $R(A_{/(\mathscr{S},\mathscr{T})}) \subseteq \mathscr{T}$ and $R((A_{/(\mathscr{S},\mathscr{T})})^*) \subseteq \mathscr{S}$. Hence, $A_{/(\mathscr{S},\mathscr{T})} \in \mathscr{M}^-(A, \mathscr{S}, \mathscr{T})$. On the other hand, given $C \in \mathscr{M}^-(A, \mathscr{S}, \mathscr{T})$, there exists a projection $E \in L(\mathscr{H}_2)$ such that $C = EA$. Let $P \in L(\mathscr{H}_1)$ be a projection as in Proposition 4.7 such that $R(P^*) = \mathscr{S}$ and $A_{/(\mathscr{S},\mathscr{T})} = AP$. The inclusion $R(C^*) \subseteq \mathscr{S}$ implies that $P^*C^* = C^*$. Therefore

$$C = CP = EAP = EA_{/(\mathscr{S},\mathscr{T})}.$$

In a similar way, there exists a projection F such that $C^* = F(A_{/(\mathscr{S},\mathscr{T})})^*$. So, by Proposition 4.13, $C \leqslant^- A_{/(\mathscr{S},\mathscr{T})}$. □

Corollary 4.16. *Let $A \in L(\mathscr{H})$ be a projection. If $\mathscr{S}, \mathscr{T} \subseteq \mathscr{H}$ are closed subspaces such that A is $(\mathscr{S}, \mathscr{T})$-complementable, then $N(A) + \mathscr{S}^\perp$ is closed,*

$$\mathscr{H} = (R(A) \cap \mathscr{T}) \oplus (N(A) + \mathscr{S}^\perp),$$

and $A_{/(\mathscr{S},\mathscr{T})}$ is the projection given by this decomposition.

Proof. By Theorem 4.15, $A_{/(\mathscr{S},\mathscr{T})} \leqslant^- A$. Hence it must be a projection by Corollary 4.14. The rest of the statement follows from Corollary 4.9. □

Next, we shall study the effect of shorting a shorted operator. The following proposition was proved for self-adjoint operators by Ando (see [16]).

Corollary 4.17. *Let $A \in L(\mathscr{H}_1, \mathscr{H}_2)$, and consider closed subspaces $\mathscr{S}, \widehat{\mathscr{S}}$ of $\mathscr{H}_1$ and $\mathscr{T}, \widehat{\mathscr{T}}$ of $\mathscr{H}_2$. Then, it holds that*

$$(A_{/(\mathscr{S},\mathscr{T})})_{/(\widehat{\mathscr{S}},\widehat{\mathscr{T}})} = A_{/(\mathscr{S}\cap\widehat{\mathscr{S}},\mathscr{T}\cap\widehat{\mathscr{T}})} \tag{7}$$

if every operator is complementable with respect to the corresponding pair of subspaces.

Proof. Straightforward calculations show that $\mathcal{M}^-(A_{/(\mathcal{S},\mathcal{T})}, \widehat{\mathcal{S}}, \widehat{\mathcal{T}}) = \mathcal{M}^-(A, \mathcal{S} \cap \widehat{\mathcal{S}}, \mathcal{T} \cap \widehat{\mathcal{T}})$. Then apply Theorem 4.15. □

Remark 4.18. Actually, the last result holds with weaker hypothesis; in fact, it is only needed that any two of the three shorted operators exist. The reader is referred to [6] for the proofs of these facts. Ando's proof, valid for a single subspace ($\mathcal{S} = \mathcal{T}$), can be easily extended to our setting.

5. Parallel sum and parallel subtraction

The device of parallel sum of matrices has been developed by Anderson and Duffin in [2]. The extension to general Hilbert spaces is due to Anderson and Trapp in [5] (see also [31] and [32]). The key idea was to define parallel sum through shorted operators. In this section, we shall define parallel sum between operators following the ideas of Anderson and Trapp (see, in particular, [5] Section 6). Even in the scalar case, not every two operators are summable. So, we need to define the concept of summable operators.

Definition 5.1. Let $A, B \in L(\mathcal{H}_1, \mathcal{H}_2)$. We say that A and B are *weakly parallel summable* if the next range inclusions hold:

1. $R(A) \subseteq R(|A^* + B^*|^{1/2})$ and $R(B) \subseteq R(|A^* + B^*|^{1/2})$.
2. $R(A^*) \subseteq R(|A + B|^{1/2})$ and $R(B^*) \subseteq R(|A + B|^{1/2})$.

In this case, the *parallel sum* of A and B, denoted by $A : B \in L(\mathcal{H}_1, \mathcal{H}_2)$, is

$$\begin{pmatrix} A : B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & A \\ A & A + B \end{pmatrix} \Big/_{(\mathcal{H}_1 \oplus \{0\}, \mathcal{H}_2 \oplus \{0\})}.$$

Remark 5.2. Note that the pair (A, B) is weakly summable if and only if the operator matrix $\begin{pmatrix} A & A \\ A & A+B \end{pmatrix}$ is $(\mathcal{H}_1 \oplus \{0\}, \mathcal{H}_2 \oplus \{0\})$-weakly complementable. Hence, the parallel sum is well defined.

Proposition 5.3. *Let $A, B \in L(\mathcal{H}_1, \mathcal{H}_2)$ be weakly parallel summable operators and let E_A, E_B, F_A and F_B be, respectively, the reduced solutions of the equations*

$$A = |A^* + B^*|^{1/2} U X, \quad B = |A^* + B^*|^{1/2} U X, \tag{8}$$
$$A^* = |A + B|^{1/2} X, \quad B^* = |A + B|^{1/2} X, \tag{9}$$

where U is the partial isometry of the polar decomposition of $A + B$. Then:

$$A : B = F_A^* E_B = F_B^* E_A. \tag{10}$$

Proof. Note that $|A^* + B^*|^{1/2} U = U |A + B|^{1/2}$. Then, adding in (8) and in (9), we get

$$|A + B|^{1/2} = E_A + E_B, \quad \text{and} \quad |A^* + B^*|^{1/2} U = F_A^* + F_B^*,$$

by the uniqueness of the reduced solution. By its definition, $A : B = A - F_A^* E_A$. Then

$$A : B = A - F_A^* E_A = F_A^*(|A + B|^{1/2} - E_A) = F_A^* E_B.$$

The other equality follows in a similar way. □

Corollary 5.4. *Let* $A, B \in L(\mathcal{H}_1, \mathcal{H}_2)$ *be weakly parallel summable. Then* $A : B = B : A$.

Corollary 5.5. *Let* $A, B \in L(\mathcal{H}_1, \mathcal{H}_2)$ *be weakly parallel summable and suppose that the operator* $A + B$ *has closed range. Then* $A : B = A - A(A + B)^\dagger A = A(A + B)^\dagger B$.

Using Proposition 4.5 we obtain the following analogous result with respect to parallel sum.

Proposition 5.6. *Let* $A, B \in L(\mathcal{H}_1, \mathcal{H}_2)$ *be weakly parallel summable and* $x \in \mathcal{H}_1$. *Then there exists a sequence* $\{y_n\}$ *and* $M > 0$ *such that*

$$A(x + y_n) \underset{n\to\infty}{\longrightarrow} A : B(x), \quad B(y_n) \underset{n\to\infty}{\longrightarrow} -A : B(x)$$

and $\langle |A + B| y_n, y_n\rangle \leqslant M$. *Conversely, if there exist* $d \in \mathcal{H}_2$, *a sequence* $\{y_n\}$ *in* $\mathcal{H}_1$ *and a real number M such that*

$$A(x + y_n) \underset{n\to\infty}{\longrightarrow} d, \quad B(y_n) \underset{n\to\infty}{\longrightarrow} -d, \quad \text{and} \quad \langle |A + B| y_n, y_n\rangle \leqslant M,$$

then $A : B(x) = d$.

Corollary 5.7. *Let* $A, B \in L(\mathcal{H}_1, \mathcal{H}_2)$ *be weakly parallel summable. Then*

$$R(A) \cap R(B) \subseteq R(A : B) \subseteq \overline{R(A)} \cap \overline{R(B)}$$

Proof. Given $x \in R(A) \cap R(B)$, let $y, z \in \mathcal{H}_1$ such that $Ay = Bz = x$. Then $A((y + z) - z) = x = B(-z)$. In consequence, taking $w = y + z$ and $y_n = -z$ for every $n \in \mathbb{N}$, by Proposition 5.6 we have that $A : B(w) = x$, which prove the first inclusion. The second inclusion follows immediately from Proposition 5.6. □

Parallel summable operators

Let $A, B \in L(\mathcal{H}_1, \mathcal{H}_2)$. As we have already pointed out in Remark 5.2, the operator pair (A, B) is weakly summable if and only if the block matrix

$$M = \begin{pmatrix} A & A \\ A & A + B \end{pmatrix},$$

is $(\mathcal{H}_1 \oplus \{0\}, \mathcal{H}_2 \oplus \{0\})$-weakly complementable. From this point of view, it is natural consider pairs of operators (A, B) such that M is $(\mathcal{H}_1 \oplus \{0\}, \mathcal{H}_2 \oplus \{0\})$-complementable. In this section we shall study such pairs of operators.

Definition 5.8. Let $A, B \in L(\mathcal{H}_1, \mathcal{H}_2)$. We say that A and B are *parallel summable* if

$$R(A) \subseteq R(A + B) \quad \text{and} \quad R(A^*) \subseteq R(A^* + B^*).$$

Note that these conditions imply that $R(B) \subseteq R(A + B)$ and $R(B^*) \subseteq R(A^* + B^*)$.

Remark 5.9. This notion is indeed stronger than weakly summability. For example, take $A, D \in L(\mathcal{H})^+$ such that $A \leqslant D$ but $R(A) \not\subseteq R(D)$. Denote $B = D - A \in L(\mathcal{H})^+$. By Douglas' theorem, $R(A) \subseteq R(A^{1/2}) \subseteq R(D^{1/2}) = R((A + B)^{1/2})$. Similarly, since $B \leqslant D$, then also $R(B) \subseteq$

$R(D^{1/2}) = R((A+B)^{1/2})$. However, by hypothesis, the pair (A, B) can not be parallel summable, because $R(A) \nsubseteq R(A+B) = R(D)$.

Both notion coincides, for instance, if $R(A+B)$ is closed. In fact, in this case $R(A+B) = R(|(A+B)^*|) = R(|(A+B)^*|^{1/2})$ and $R((A+B)^*) = R(|A+B|) = R(|A+B|^{1/2})$.

Clearly, for parallel summable operators, some of the already proved properties can be improved. Let us mention, for instance, the following ones.

Proposition 5.10. *Let $A, B \in L(\mathscr{H}_1, \mathscr{H}_2)$ be parallel summable and $x \in \mathscr{H}_1$. Then, there exists $y \in \mathscr{H}_1$ such that $A(x+y) = A : B(x)$ and $By = -A : B(x)$. Moreover, there are projections $P \in L(\mathscr{H}_1 \oplus \mathscr{H}_1)$, $Q \in L(\mathscr{H}_2 \oplus \mathscr{H}_2)$ such that $R(P^*) = \mathscr{H}_1 \oplus \{0\}$, $R(Q) = \mathscr{H}_2 \oplus \{0\}$ and*

$$Q\begin{pmatrix} A & A \\ A & A+B \end{pmatrix} = \begin{pmatrix} A & A \\ A & A+B \end{pmatrix} P = \begin{pmatrix} A:B & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. It follows immediately from Proposition 4.7. □

Corollary 5.11. *If $A, B \in L(\mathscr{H}_1, \mathscr{H}_2)$ are parallel summable, then $R(A:B) = R(A) \cap R(B)$.*

Parallel subtraction

Given two operators $A, C \in L(\mathscr{H}_1, \mathscr{H}_2)$, it seems natural to study the existence of a solution of the equation $A : X = C$, that is, if there exists an operator $B \in L(\mathscr{H}_1, \mathscr{H}_2)$ parallel summable with A such that $A : B = C$. For positive operators this question has been studied, for example, in [3,38,4,37]. Clearly, equation $A : X = C$ may have no solutions for some pair of operators (A, C). Indeed, Corollary 5.11 implies that, if equation $A : X = C$ has a solution, then $R(C) \subseteq R(A)$ and $R(C^*) \subseteq R(A^*)$, or, equivalently, $R(C-A) \subseteq R(A)$ and $R((C-A)^*) \subseteq R(A^*)$.

In this section, we shall prove that, if $R(C-A) = R(A)$ and $R((C-A)^*) = R(A^*)$, then there exists a solution of equation $A : X = C$. Moreover, we shall find a distinguished solution, the *parallel subtraction of the operators* C and A. Given $A \in L(\mathscr{H}_1, \mathscr{H}_2)$, let $\mathscr{D}_A$ be the set of operators defined by

$$\mathscr{D}_A := \{C \in L(\mathscr{H}_1, \mathscr{H}_2) : R(C-A) = R(A) \quad \text{and} \quad R((C-A)^*) = R(A^*)\}.$$

Proposition 5.12. *Let $A \in L(\mathscr{H}_1, \mathscr{H}_2)$. Then the map $C \leftrightarrow C : (-A)$ is a bijection between the sets $\mathscr{D}_A$ and $\mathscr{D}_{-A}$ with inverse $D \mapsto D : A$.*

Proof. By the definition of summability, it is clear that $-A$ and C are summable, for every $C \in \mathscr{D}_A$. Let E be the reduced solution of $C - A = AX$ and let Q be a projection onto $\mathscr{H}_2 \oplus \{0\}$ such that $Q\begin{pmatrix} -A & -A \\ -A & C-A \end{pmatrix} = \begin{pmatrix} C:(-A) & 0 \\ 0 & 0 \end{pmatrix}$.

Since $\begin{pmatrix} C:(-A)+A & 0 \\ 0 & 0 \end{pmatrix} = Q\begin{pmatrix} 0 & -A \\ -A & C-A \end{pmatrix}$ and $\begin{pmatrix} 0 & -A \\ -A & C-A \end{pmatrix}\begin{pmatrix} -E & 0 \\ -I & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, we get that $\begin{pmatrix} C:(-A)+A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -E & 0 \\ -I & 0 \end{pmatrix} = Q\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$.

This implies that $R(A) \subseteq R(C:(-A)+A)$. As the other inclusion always holds, we get $R(A) = R(C:(-A)+A)$. In a similar way we can prove that $R(A^*) = R((C:(-A)+A)^*)$. Thus, the mapping $\Phi : \mathscr{D}_A \to \mathscr{D}_{-A}$ given by $\Phi(C) = C : (-A)$ is well defined. To prove that $\Phi^{-1}(D) = D : A$, take $C \in \mathscr{D}_A$ and $x \in \mathscr{H}_1$. Then there exists $y, z \in \mathscr{H}_1$ such that

$$C:(-A)(x+y)=(C:(-A)):A(x),\quad Ay=-(C:(-A)):A(x),$$

$$C(x+y+z)=C:(-A)(x+y),\quad \text{and}\quad Az=-(C:(-A))(x+y).$$

So, $A(y+z)=0$ which implies that $C(y+z)=0$. Hence

$$Cx=C(x+y+z)=C:(-A)(x+y)=(C:(-A)):A(x),$$

and the proof is complete. □

Corollary 5.13. *Let $A\in L(\mathscr{H}_1,\mathscr{H}_2)$. For every $C\in\mathscr{D}_A$, the equation*

$$A:X=C$$

has a solution. Moreover, $C:(-A)$ is the unique solution X which also satisfies

$$R(A+X)=R(A)\quad\textit{and}\quad R((A+X)^*)=R(A^*).$$

Definition 5.14. Given $A\in L(\mathscr{H}_1,\mathscr{H}_2)$, and $C\in\mathscr{D}_A$, the *parallel subtraction* between the operators A and C, denoted by $C\div A$, is defined as the unique solution of equation $A:X=C$ guaranteed by Proposition 5.13.

Remark 5.15. Note that, according to our definition, it holds that $C\div A=C:(-A)$; in particular, several properties of parallel sum are inherited by parallel subtraction.

6. Shorted formulas using parallel sum

In this section we shall prove some formulas for shorted operator using parallel sums and subtractions. Throughout this section $\mathscr{S}$ and $\mathscr{T}$ will be two fixed closed subspaces of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. The following lemma was proved in [38] for pairs of positive operators.

Lemma 6.1. *Let $A\in L(\mathscr{H}_1,\mathscr{H}_2)$ be $(\mathscr{S},\mathscr{T})$-complementable. Let $B\in L(\mathscr{H}_1,\mathscr{H}_2)$ be such that (A,B) and $(A_{/(\mathscr{S},\mathscr{T})},B)$ are parallel summable and $A:B$ is $(\mathscr{S},\mathscr{T})$-complementable. Then*

$$A_{/(\mathscr{S},\mathscr{T})}:B=(A:B)_{/(\mathscr{S},\mathscr{T})}.$$

Proof. Let $x\in\mathscr{S}$. By Proposition 5.10, there exists $y\in\mathscr{H}_1$ such that

$$A_{/(\mathscr{S},\mathscr{T})}(x+y)=A_{/(\mathscr{S},\mathscr{T})}:B(x),\quad\text{and}\quad By=-A_{/(\mathscr{S},\mathscr{T})}:B(x).$$

Let $z\in\mathscr{S}^\perp$ such that $A(x+y+z)=A_{/(\mathscr{S},\mathscr{T})}(x+y)$. Then, $A(x+y+z)=A_{/(\mathscr{S},\mathscr{T})}:B(x)$ and $By=-A_{/(\mathscr{S},\mathscr{T})}:B(x)$. So, $A:B\binom{x}{z}=A_{/(\mathscr{S},\mathscr{T})}:B(x)$, which implies, by Proposition 4.5, that $(A:B)_{/(\mathscr{S},\mathscr{T})}x=A_{/(\mathscr{S},\mathscr{T})}:B(x)$. □

The next technical result will be useful throughout this section.

Proposition 6.2. *Let $A\in L(\mathscr{H}_1,\mathscr{H}_2)$ be $(\mathscr{S},\mathscr{T})$-complementable. If $B\in L(\mathscr{H}_1,\mathscr{H}_2)$ satisfies $R(B)=\mathscr{T}$ and $R(B^*)=\mathscr{S}$; then there exists $n_0\in\mathbb{N}$ such that, for every $n\geqslant n_0$, A and nB are parallel summable.*

We need the following lemma.

Lemma 6.3. *Let* $A, B \in L(\mathscr{H}_1, \mathscr{H}_2)$ *such that* $R(A) \subseteq \mathscr{T}$, $R(A^*) \subseteq \mathscr{S}$, $R(B) = \mathscr{T}$ *and* $R(B^*) = \mathscr{S}$. *Then there exists* $n_0 \in \mathbb{N}$ *such that, for every* $n \geqslant n_0$, $R(A + nB) = \mathscr{T}$ *and* $R((A + nB)^*) = \mathscr{S}$.

Proof. It suffices to prove that $\mathscr{T} \subseteq R(A + nB)$ and $\mathscr{S} \subseteq R((A + nB)^*)$, because the reverse inclusions hold by hypothesis. Since $R(B^*) = \mathscr{S}$, Douglas' theorem assures that there exists $\alpha > 0$ such that $B^*B \geqslant \alpha P_{\mathscr{S}}$. Then

$$|A + nB|^2 = A^*A + n^2B^*B + n(A^*B + B^*A) \geqslant (\alpha n^2 - n\|(A^*B + B^*A)\|)P_{\mathscr{S}}.$$

Take $n_1 \in \mathbb{N}$ such that $\alpha n^2 > n\|(A^*B + B^*A)\|$ for every $n \geqslant n_1$. By Douglas' theorem, $\mathscr{S} \subseteq R(|A + nB|) = R((A + nB)^*)$, for $n \geqslant n_1$. In a similar way, we can prove that there exists $n_2 \in \mathbb{N}$ such that for every $n \geqslant n_2$, $\mathscr{T} \subseteq R((A + nB))$ holds. Hence, the statement is proved by taking $n_0 = \max\{n_1, n_2\}$. □

Proof of Proposition 6.2. Take, as in Proposition 4.7, a projection $P \in L(\mathscr{H}_1)$ such that $AP = A_{/(\mathscr{S},\mathscr{T})}$ and $R(P^*) = \mathscr{S}$. Since $N(B) = \mathscr{S}^\perp = R(I - P)$, it holds that $B(I - P) = 0$ and $BP = B$. By Lemma 6.3 there exists $n_1 \in \mathbb{N}$ such that $R((A_{/(\mathscr{S},\mathscr{T})} + nB)^*) = \mathscr{S}$, for every $n \geqslant n_1$. Fix $n \geqslant n_1$. Given $x \in \mathscr{H}_1$, there exists $y \in \mathscr{S}$ such that $A_{/(\mathscr{S},\mathscr{T})}x = (A_{/(\mathscr{S},\mathscr{T})} + nB)y$. If $z = Py + (I - P)x \in \mathscr{H}_1$, then

$$\begin{aligned} Ax &= A(Px + (I - P)x) = A_{/(\mathscr{S},\mathscr{T})}x + A(I - P)x \\ &= (A_{/(\mathscr{S},\mathscr{T})} + nB)y + (A + nB)(I - P)x \\ &= (A + nB)Py + (A + nB)(I - P)x = (A + nB)z. \end{aligned}$$

This shows that $R(A) \subseteq R(A + nB)$. Following the same lines, it can be shown that there exists $n_2 \in \mathbb{N}$ such that $R(A^*) \subseteq R(A + nB)^*$ for $n \geqslant n_2$. Thus, A and nB are parallel summable for $n \geqslant \max\{n_1, n_2\}$. □

Parallel sum may be defined in terms of shorted operators and the next Proposition shows a converse relation.

Proposition 6.4. *Let* $A \in L(\mathscr{H}_1, \mathscr{H}_2)$ *be* $(\mathscr{S}, \mathscr{T})$*-complementable. If* $B \in L(\mathscr{H}_1, \mathscr{H}_2)$ *satisfies* $R(B) = \mathscr{T}$ *and* $R(B^*) = \mathscr{S}$, *then there exists* $n_0 \in \mathbb{N}$ *such that*:

1. *The pair* (A, nB) *is summable for every* $n \geqslant n_0$, *and*
2. $A_{/(\mathscr{S},\mathscr{T})} = \lim_{n\to\infty} A : (nB)$ (*in the norm topology*).

Firstly, we shall prove Proposition 6.4 in the following particular case.

Lemma 6.5. *Let* $A, B \in L(\mathscr{H}_1, \mathscr{H}_2)$ *be such that* $R(A) \subseteq \mathscr{T}$, $R(A^*) \subseteq \mathscr{S}$, $R(B) = \mathscr{T}$ *and* $R(B^*) = \mathscr{S}$. *Then*, $A : (nB) \xrightarrow[n\to\infty]{\|\cdot\|} A$.

Proof. Lemma 6.3 implies that there exists $n_0 \geqslant 1$ such that, for every $n \geqslant n_0$, A and nB are parallel summable. Fix $n \geqslant n_0$. By definition, $A : (nB) = A - F_n^*E_n$, where F_n and E_n are, respectively, the reduced solution of $A^* = |A + nB|^{1/2}X$ and $A = |(A + nB)^*|^{1/2}U_nX$, and U_n

is the partial isometry of the polar decomposition of $A + nB$. We shall show that $\|E_n\| \underset{n\to\infty}{\longrightarrow} 0$ (resp. $\|F_n\| \underset{n\to\infty}{\longrightarrow} 0$), which clearly implies the desired norm convergence. By Douglas' theorem,

$$\|E_n\| = \inf\{\lambda \in \mathbb{R} : A^*A \leqslant \lambda|A + nB|\}, \quad n \in \mathbb{N}, \tag{11}$$

and there exist $\alpha, \beta > 0$ such that $A^*A \leqslant \beta P_{\mathscr{S}}$ and $B^*B \geqslant \alpha P_{\mathscr{S}}$. Then $(A^*A)^2 \leqslant \beta^2 P_{\mathscr{S}}$, and

$$\begin{aligned}|A + nB|^2 &= A^*A + n^2B^*B + n(A^*B + B^*A)\\ &\geqslant (\alpha n^2 - n\|(A^*B + B^*A)\|)P_{\mathscr{S}} \leqslant \frac{\alpha n^2 - n\|(A^*B + B^*A)\|}{\beta^2}(A^*A)^2.\end{aligned}$$

Recall that Löwner's theorem states that for every $r \in (0, 1]$ $f(x) = x^r$ is operator monotone, i.e. if $0 \leqslant A \leqslant B$, then $A^r \leqslant B^r$. Therefore, if n is large enough,

$$A^*A \leqslant \frac{\beta}{(\alpha n^2 - n\|(A^*B + B^*A)\|)^{1/2}}|A + nB|.$$

Hence, (11) implies that $\|E_n\| \underset{n\to\infty}{\longrightarrow} 0$. Analogously, we get that $\|F_n\| \underset{n\to\infty}{\longrightarrow} 0$. □

Proof of Proposition 6.4. By Proposition 6.2, there exists n_0 such that for every $n \geqslant n_0$ the pairs (A, nB) and $(A_{/(\mathscr{S},\mathscr{T})}, nB)$ are parallel summable. Since the hypothesis of Lemma 6.1 are satisfied, for every $n \geqslant n_0$, it holds that

$$A : nB = (A : nB)_{/(\mathscr{S},\mathscr{T})} = A_{/(\mathscr{S},\mathscr{T})} : nB.$$

Then, by Lemma 6.5 with $A_{/(\mathscr{S},\mathscr{T})}$ playing the role of A, we get $A : nB \underset{n\to\infty}{\longrightarrow} A_{/(\mathscr{S},\mathscr{T})}$. □

Our last result relates parallel sum, parallel subtraction and shorted operators.

Proposition 6.6. *Let $A \in L(\mathscr{H}_1, \mathscr{H}_2)$ be $(\mathscr{S}, \mathscr{T})$-complementable, and let $L \in L(\mathscr{H}_1, \mathscr{H}_2)$ be such that $R(L) = \mathscr{T}$ and $R(L^*) = \mathscr{S}$. Then, there exists $n \in \mathbb{N}$ such that*

1. *A and nL are summable.*
2. $A_{/(\mathscr{S},\mathscr{T})} \in \mathscr{D}_{-nL}$.
3. $(A : nL) \div nL = A_{/(\mathscr{S},\mathscr{T})}$.

Proof. The first two assertions follows from Proposition 6.2 and Lemma 6.3, respectively. Since $R((A : nL) \div nL) \subseteq \mathscr{T}$ and $R(((A : nL) \div nL)^*) \subseteq \mathscr{S}$ then, by Lemma 6.1,

$$(A : nL) \div nL = ((A : nL) \div nL)_{/(\mathscr{S},\mathscr{T})} = (A_{/(\mathscr{S},\mathscr{T})} : nL) \div nL.$$

Finally, by Proposition 5.12, $(A_{/(\mathscr{S},\mathscr{T})} : nL) \div nL = (A_{/(\mathscr{S},\mathscr{T})} \div nL) : (-nL) = A_{/(\mathscr{S},\mathscr{T})}$, and the proof is complete. □

Remark 6.7. Proposition 6.4 was proved for positive operators by Anderson and Trapp in [5] and by Pekarev and Smul'jian in [38]. It was also considered by Mitra and Puri who proved formula $A_{/(\mathscr{S},\mathscr{T})} = \lim_{n\to\infty} A : (nB)$ of Proposition 6.4 for rectangular matrices (see [33]).

However, their proof can not be extended to infinite dimensional Hilbert spaces because it involves generalized inverses which, in our setting, only exist for closed range operators. Finally, the reader will find a generalization of Proposition 6.6 for positive operators in [38].

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