

# A new derivation of the conformally flat stationary cyclic non-circular spacetimes

Eloy Ayón-Beato<sup>1,2</sup>, Cuauhtemoc Campuzano<sup>1,3</sup>, Alberto García<sup>4,5</sup>

<sup>1</sup> Departamento de Física, Facultad de Física e Inteligencia Artificial, Universidad Veracruzana, 91000 Xalapa, Veracruz, México

<sup>2</sup> Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.

<sup>3</sup> Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de la Plata, C.C. 67, (1900) La Plata, Argentina

CEFIMAS-SCA, Ave. Santa Fe 1145, C1059ABF, Buenos Aires, Argentina

<sup>4</sup> Department of Physics, University of California, Davis, CA 95616, USA.

<sup>5</sup> Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN,

Apdo. Postal 14-740, 07000, México D.F., México.

E-mail: [ccvargas@fis.cinvestav.mx](mailto:ccvargas@fis.cinvestav.mx)

**Abstract.** We present an alternative way to derive the conformally flat stationary cyclic non-circular spacetimes. We show that there is no room for stationary axisymmetric non-circular axisymmetric spacetimes. We reproduce the well know results for this sort of spacetimes recently reported in [1].

## 1. Introduction

One of the most outstanding problems in general relativity is the search of interior solutions for describing isolated rotating bodies and the corresponding exterior solutions to the Einstein equations. The description of the rotating masses by means of a perfect fluid energy-momentum tensor coupled to gravitational fields has been studied extensively. These solutions must be matched with the exterior vacuum solutions. From the geometrical point of view, the searching of solutions usually is achieved in stationary circular axisymmetric spacetimes (Lewis-Papapetrou metric). Formally speaking it is expected to have more general stationary spacetimes, as was proposed in [2]. For the spacetime with two isometries, one of them being timelike,  $\eta = \partial_t$ , and the other one spacelike,  $\xi = \partial_\phi$ , there are four subclasses of spacetimes where the non-circular, circular, cyclic and axial properties are included. Although the cyclic and axial symmetry properties for these spacetimes, were established by Carter [3], there is a misuse by some authors of the concept of axisymmetry for spacetimes that exhibits a Lewis-Papapetrou structure. It was pointed out in [4] (and references there in) that not all spacetimes with these structure have an axis of symmetry.

From the astrophysical point of view, the stationary non-circular spacetimes are interesting because these spaces describe toroidal magnetic fields—magnetic field in the direction of the rotational Killing vector—which results from a non-circular electromagnetic stress-energy tensor. The toroidal magnetic fields are found in a “baby” rotating neutron star [5], in contracts with the circular case that allows

poloidal magnetic fields only [5, 6, 7]. However, there is likelihood of exist non-negligible toroidal magnetic fields in nature. In addition, a meridional flow may also exist in the interior of a neutron star, which violates the circularity of the spacetime [5, 6, 7]. The circularity conditions are compatible with the absence of the momentum currents in the meridional orthogonal planes to both  $\eta$  and  $\xi$ . In the case of a fluid, this means that there is no convective motion but only circular motion around the axis of symmetry. The convective motion have no direct influence on the exterior solutions, then the uniqueness and existence for asymptotically flat vacuum exterior fields give the interior sources applied in the non-circular case. This fact is relevant when the match problem is attacked as was shown in [8].

## 2. Stationary non-circular spacetimes

The circular case is defined by the Frobenius integrability conditions,  $\xi_{[a;b}\xi_c\eta_{d]} = 0$  and  $\eta_{[a;b}\eta_c\xi_{d]} = 0$ ; whereas for the non-circular case this conditions are not satisfied. This fact is reflected in the structure of the metric as follows:  $g_{it} = 0$  and  $g_{i\phi} = 0$  whilst  $g_{it} \neq 0$  and  $g_{i\phi} \neq 0$ , respectively. As is shown by Petrov [9], making use of the allowed transformations, exist a coordinate system where the components  $g_{i\phi} = 0$  and  $g_{it} \neq 0$ . Without loss of generality we describe a stationary cyclic symmetric non-circular spacetime by the line element:

$$ds^2 = e^{-2Q}(dx^2 + dy^2) + 2n dxdt + 2m dydt - e^{-2G} \frac{(dt + ad\phi)(dt - bd\phi)}{(a+b)}, \quad (1)$$

where the metric functions  $Q, G, n, m, a$  and  $b$  depend on  $(x, y)$ . Defining  $P := Q - G$ ,  $N := e^{2G}n$  and  $M := e^{2G}m$ , the line element (1) reads as

$$ds^2 = e^{-2G} \left[ e^{-2P}(dx^2 + dy^2) + 2N dxdt + 2M dt dy - \frac{(dt + ad\phi)(dt - bd\phi)}{(a+b)} \right]; \quad (2)$$

this is a slight generalization of the metric used in [10]. For the metric (2) the circularity conditions are written

$$\begin{aligned} \xi_{[a;b}\xi_c\eta_{d]} &= \frac{1}{4} e^{-6G(x,y)} \left[ -2 \frac{M}{(a+b)^2} \left( \frac{db}{dx} a + \frac{da}{dx} b \right) + \frac{dN}{dy} - \frac{dM}{dx} \right] \epsilon_{abcd} \\ \eta_{[a;b}\eta_c\xi_{d]} &= \frac{1}{2} e^{-6G(x,y)} \frac{M}{(a+b)^2} \left( \frac{da}{dx} b^2 - \frac{db}{dx} a^2 \right) \epsilon_{abcd}. \end{aligned} \quad (3)$$

Where  $\epsilon_{abcd}$  is the volumen four-form. Note that for this spacetime, when the functions  $M$  and  $N$  vanishes, we recover the spaces studied in [10].

## 3. Conformally flat spacetimes

In the non-circular class of spacetimes the computing are lengthy and hardly to manage. Thereby the use of the symbolic computing is useful in the algebraic manipulation. We will show the existence of conformal flatness in this kind of spacetimes when some functional dependence are imposed under the metric.

The structure of the metric for the new branch is the same as (2). Now the functions inside of the square bracket in (2), depend on  $x$ . Below we shall provide the meaning of this anzat related to the vanishing of some parameters. For such limit the stationary circular cyclic symmetric spacetime, reported in [10], are obtained. Under

this condition with the four independent components of the Weyl tensor  $C_{t\phi xy}$ ,  $C_{txty}$ ,  $C_{t\phi ty}$  and  $C_{tx\phi x}$ , we are able to arrange a system of coupled nonlinear differential equations. Obviously this system is satisfied identically modulo the vanishing of the Weyl tensor. In order to obtain conformally flat spacetimes we have:

$$[(a+b)e^{-2P} + 2ab(N^2 + M^2)][(a+b)(\ddot{a} + 2\dot{a}\dot{P}) - 2\dot{a}^2] - 2[(N^2 + M^2)(b^2\dot{a} + a^2\dot{b}) + ab(a+b)(N^2 + M^2)]\dot{a} = 0, \quad (4)$$

$$[(a+b)e^{-2P} + 2ab(N^2 + M^2)][(a+b)(\ddot{b} + 2\dot{b}\dot{P}) - 2\dot{b}^2] - 2[(N^2 + M^2)(a^2\dot{b} + b^2\dot{a}) + ab(a+b)(N^2 + M^2)]\dot{b} = 0, \quad (5)$$

$$[(a+b)e^{-2P} + 4ab(N^2 + M^2)][(a+b)^2\ddot{P} + 2\dot{a}\dot{b}] - 2ab(a+b)^2(N^2 + M^2)\dot{P} - 2(a+b)(N^2 + M^2)[2ab(a+b)\dot{P} + (b^2\dot{a} + a^2\dot{b})]\dot{P} = 0, \quad (6)$$

$$-2(a+b)(N^2 + M^2)(b^2\dot{a} + a^2\dot{b})\dot{M} - 4ab(a+b)^2[(N^2 + M^2)\dot{P} + (N^2 + M^2)]\dot{M} + [(a+b)e^{-2P} + 4ab(N^2 + M^2)][(a+b)^2(\ddot{M} + 2\dot{P}\dot{M}) - 4ab\dot{M}] = 0, \quad (7)$$

where the dot stands for  $\dot{f} := df/dx$

Our integration strategy starts subtracting the equations (4) and (5), and multiplying the resulting equation by  $(\dot{a} + \dot{b})/[\dot{a}\dot{b}(b - \dot{a})]$ . This procedure results in the first integral,

$$\dot{a}\dot{b} = \epsilon k^2 (a+b)^{-2} (\dot{a} - \dot{b})^2, \quad (8)$$

where  $k^2$  is a integration constant and the parameter  $\epsilon$  takes the values  $\pm 1$ .

We turn now to divide the Eq. (4) by  $\dot{a}$  and adding the result to Eq.(5) divided by  $\dot{b}$ ; the resulting equation times  $e^{2P}\dot{a}\dot{b}$  is an exact differential equation such that

$$\frac{d}{dx} \left\{ \frac{\dot{a}\dot{b}e^{2P}}{(a+b)[(a+b)e^{-2P} + 4ab(M^2 + N^2)]} - K \right\} = 0, \quad (9)$$

where  $K$  is a integration constant.

As is shown in [10] the equation (8) is integrated by introducing the functions  $X = X(x)$  and  $Y = Y(x)$  given by  $a + b = 2kY$ ,  $a - b = 2kX$ . The insertion of these functions in (8) leads us to the general integral for  $Y$  in terms of  $X$ ,

$$\nu(X - X_0) = \sqrt{Y^2 + \epsilon}, \rightarrow Y^2 = (X - X_0)^2 - \epsilon, \quad (10)$$

where  $X_0$  is an integration constant.

Since we express  $Y$  as a function of  $X$  which suggest a change of variable  $x \rightarrow X$  such that  $d/dx = \dot{X}d/dX$ ,  $\dot{f} = \dot{X}f_{,X} =: \dot{X}f'$  and  $\ddot{f} = \dot{X}^2 f'' + \ddot{X}f'$ . Making this change of variable into the equations (6) and (7), and substituting  $a(X)$ ,  $b(X)$  and the expressions for  $\dot{X}$  and  $\ddot{X}$ , are straightforwardly obtained from equations (4) and (9). Then we get,

$$4k^2 Y^2 P'' - 8k^2 Y^2 P'^2 + 8k^2 (X - X_0) P' + 2k^2 \frac{\epsilon}{Y^2} = 0, \quad (11)$$

$$4k^2 Y^2 M'' + 8k^2 (X - X_0) M' + 4k^2 \frac{\epsilon}{Y^2} M = 0, \quad (12)$$

where, in order to avoid singularities we need to assume that  $[2kYe^{-2P} + 4k^2(Y^2 - X^2)(M^2 + N^2)] \neq 0$ . This set of equations has the integrals,  $P(X) = -\frac{1}{2} \ln \left( \frac{C_1 X + C_0}{Y} \right)$  and  $M(X) = \frac{\alpha X + \beta}{Y}$  ( see [10] for details).

In the case of the metric (2), the change of variable  $x \rightarrow X$  as well as the substitution of the expressions for  $P$ ,  $M$ ,  $a$  and  $b$  in terms of  $X$ , under the trivial translation  $X \rightarrow X + X_0$ , and redefining the function  $Q$  as  $Q \rightarrow Q + 1/4 \ln(X^2 - \epsilon)$  one arrive to

$$ds^2 = e^{-2G(X,y)} \left\{ \frac{C_0 + C_1 X}{\dot{X}^2} dX^2 + (C_0 + C_1 X) dy^2 + 2 \frac{\sqrt{X^2 - \epsilon}}{\dot{X}} N dX dt + 2(\alpha X + \beta) dy dt - \frac{k}{2} (2X X_0 + X_0^2 + \epsilon) d\sigma^2 - 2(X + X_0) d\sigma dt - \frac{dt^2}{2k} \right\}, \quad (13)$$

where  $G(X, y)$  and  $N(X)$  are arbitrary functions. From (9),

$$\dot{X}^2 = \frac{K}{\epsilon} [8(X^2 - \epsilon)(2X X_0 + X_0^2 + \epsilon)(C_0 + C_1 X)kN(X)^2 + 8(C_0 + C_1 X)(\alpha X + \beta)^2(2X X_0 + X_0^2 + \epsilon)k - 4(X^2 - \epsilon)(C_0 + C_1 X)^2]. \quad (14)$$

On the other hand the circularity conditions (3) for the metric (13) reads as

$$\xi_{[a;b} \xi_c \eta_{d]} = \frac{1}{4} e^{-6G(X,y)} [-\alpha(2X X_0 + 3X_0^2 - \epsilon) + 4\beta(X + X_0)] \epsilon_{abcd}, \quad (15)$$

$$\eta_{[a;b} \eta_c \xi_{d]} = \frac{1}{2} e^{-6G(X,y)} k(\alpha X + \beta)(X_0^2 - \epsilon) \epsilon_{abcd}. \quad (16)$$

#### 4. Stationary non-circular axial and cyclic spacetimes: conformal flatness

Thus the metric (13) describes a conformally flat stationary non-circular spacetime. A necessary condition for a spacetime to have an axis of symmetry, is that the vector field  $\xi$  vanishes along it. The coefficients in front of the terms  $dt d\sigma$  and  $d\sigma^2$  in the metric vanish when are evaluated on the axis of symmetry. In our case that happens when  $X = -X_0$  and  $X_0^2 - \epsilon = 0$ . Hence, in order to have real solutions  $\epsilon$  must be equal to one. From the expression for  $\dot{X}$  (14), one see that it vanishes along the axis of symmetry. However, although the form of the function  $N$  will be restricted one can avoid a metric singularity. Furthermore, we note that the evaluation of Eq.(15) and Eq.(16), vanish on the axis of symmetry, that is, the values compatible with the existence of axis coincide with the values at which the Frobenius conditions are fulfilled. In consequence the circularity conditions of spacetime is established. In conclusion, there is no room for stationary axisymmetric spacetimes with non-circular contributions. Moreover, since no axis of symmetry was found, the spacetime is stationary cyclic symmetric non-circular and conformally flat and it is described by the metric (13). The parameters in the metric (13), that coming from the integration processes have some relevance. The parameter  $\epsilon = \pm 1$ , tell us when the pure Killing sector is orthogonal. It is fulfilled only for  $\epsilon = 1$  and coincides with the value in which the metric possesses an axis. For  $\epsilon = -1$  no orthogonality is established for the Killing sector. The sign in the parameter  $K$  just say to us how the vectors are projected over directions of the Killing vectors, and the parameter  $\nu$  could be absorbed by this parameter. Now the sign in the parameter  $k$ , just gives to us the interchange between of coordinates  $\sigma$  and  $\tau$ , as we shown in [10].

## 5. Discussion

The spacetimes described by the line element (13) are the conformally flat stationary non-circular cyclic symmetric non-circular spacetimes. In the limit  $N(X) = 0$ ,  $\alpha = \beta = 0$  we recover the spacetimes reported in [10]. For the kind of spacetime reported here one asks the following: can the above spacetimes describe a realistic physical situation? If the answer is positive, which kind of matter can be the sources to these spacetimes? These questions are the subject of future researches, since it is expected that such solutions will describe models for rotating isolated bodies.

## Acknowledgments

We thank E. Rojas for fruitful discussions. This work has been partially supported by CONACyT Grants 38495E, 48779F, 34222E, 43532F, CONICYT/CONACyT Grant 2001-5-02-159, and by grants 1040921, 7040190, 1051064, and 1060831 from FONDECYT. Partial support from MEXUS-Sabbatical Fellowship at UC Davis, CA, and the assistance of S. Carlip is acknowledged by AAG. The generous support of Empresas CMPC to the Centro de Estudios Científicos (CECS) is also acknowledged. CECS is a Millennium Science Institute and is funded in part by grants from Fundación Andes and the Tinker Foundation.

## References

- [1] E. Ayon-Beato, C. Campuzano and A. A. García, Phys. Rev. D **74**, 024014 (2006) [arXiv:gr-qc/0506004].
- [2] A. A. García and C. Campuzano, arXiv:gr-qc/0310054 (2003).
- [3] B. Carter, Commun. math. Phys. **17**, 233 (1970).
- [4] A. Barnes and J. M. M. Senovilla, arXiv:gr-qc/0305091.
- [5] E. Gourgoulhon and S. Bonazzola, Phys. Rev. D **48**, 2635 (1993).
- [6] K. Ioka and M. Sasaki, Phys. Rev. D **67**, 124026 (2003).
- [7] K. Ioka and M. Sasaki, Astrophys. J. **600**, 296 (2004).
- [8] R. Vera, Class. Quant. Grav. **20**, 2785 (2003) [arXiv:gr-qc/0305108].
- [9] A. Z. Petrov, *Einstein Spaces* (Pergamon Press, Oxford 1969).
- [10] A. A. García and C. Campuzano, Phys. Rev. D **66**, 124018 (2002) [Erratum-ibid. D **68**, 049901 (2003)] [arXiv:gr-qc/0205080].